Statistical Mechanics
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Lecture – 19
Binominal Distribution

Good morning to all of you. So, today we will continue the discussion to another very important distribution statistical mechanics, which is called as the Binominal Distribution.

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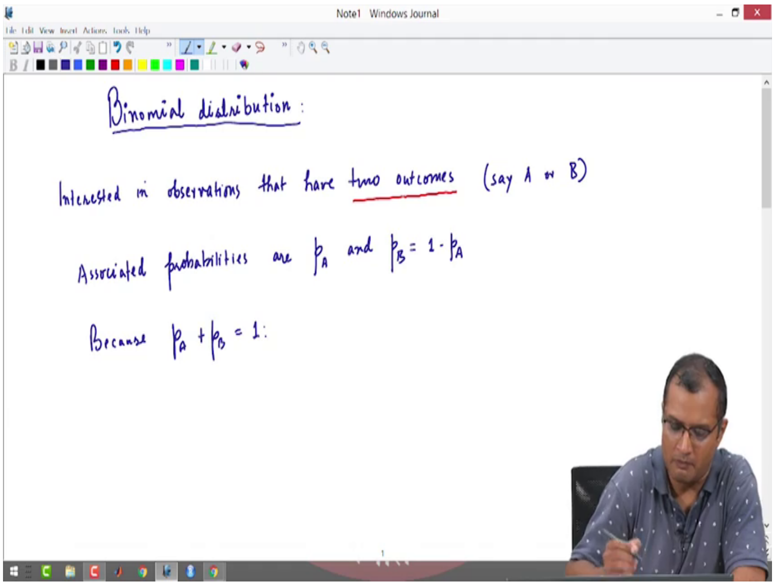

So, we will be focusing our attention to binominal distribution and I will give you an example where in fact, there are several examples where this distribution is applicable. And, I will discuss towards the end of this lecture very important example for binominal distribution, which is random walk and to that and will be discussing random walk in one dimension because, that is the simplest case, which has all the features of a random walk, except that it is the easiest to handle.

So, we will discuss the one dimensional random walk as an example of the binominal discussion right. So, let us define, what is a binominal distribution. Now the word bi here signifies that you have 2 out comes ok. So, I am interested in events, which have two out comes only or interested in let me use a more accurate word interested in observations that have only two outcomes. Let us say let us pick the example of a coin toss ok. So, I am go to underline it because, that is important criterion for a binominal distribution only two out comes ok.

If there are more than two outcomes then you have ah multi nominal distribution, but that is not the subject matter of this lecture, we are only interested in events or observations, where the outcome can only be 2, one of the two. So, let us say the outcomes are A or B, A or B here, could refer head or tails in a coin toss. Now associated with these events, there are probabilities which I am go to call as to the probability of getting an A are p A and p B. So, if I toss a coin which is unbiased, then the probability of getting a head, let us say p of A then the probability of getting a tail is p of B or since there are only out come for a coin toss, I can say that the probability of B is 1 minus p of A because, the probabilities have to be well behaved ok because, I want the probabilities to be well behaved.

So, this is like a conservation of norm for the probability ok.

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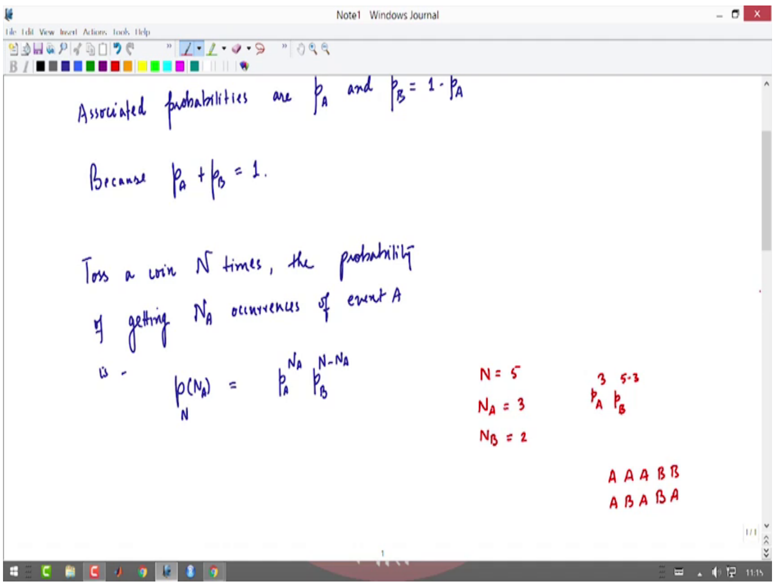

Now, let us continue to the example of coin toss. So, if I toss a coin N times. So, let us say toss a coin N times, this is like in the languages start make we are doing experiment N times each sign we get an observation and suppose you are interested in the probability of particular ah realization with in N tosses, you saw the heads N A times ok.

So, the probability of getting N A occurrences of the event A, you know it is given as I am go in to call it as p of N A. So, the probability of getting the occurrence A N A number of times in a total of N tosses or N experiments will is given as where you had N A times the event A and a probability of each event is P A so, but you got it N A time. So, it has to raised to the power of A these are independent events.

And naturally when you had A occurring N A times, you had the B occurring N B times which is N minus N N A ok. So, the probability is N minus N A p B raised to N minus N A times p A N raised to N A, but as you can see. So, this is would this would be for example, I am go to pick a very simple example. So, you let us take N to be you know 5 and you say that I have tossed the coin N times 5 times and I got the heads 3 times and I got the tails 2 times ok.

So, what is a probability of getting 3 heads and 2 tails, if I toss it 5 times? Well, you can say that I the toss probability is p of A, which is 1.5, I know it is 1.5 raised to the power 3 and I need to for each time, I mean out of a total 5, if there were 3 heads there were 2 tails. So, the probability of 3 heads and 2 tails, the joint probability is p B to the power of 5 minus 3, which is 2 ok.

And let me draw one realization of this. So, I got a heads, a heads and I got a tail twice ok, but then you can also have a head tail then a head tail then head ok.

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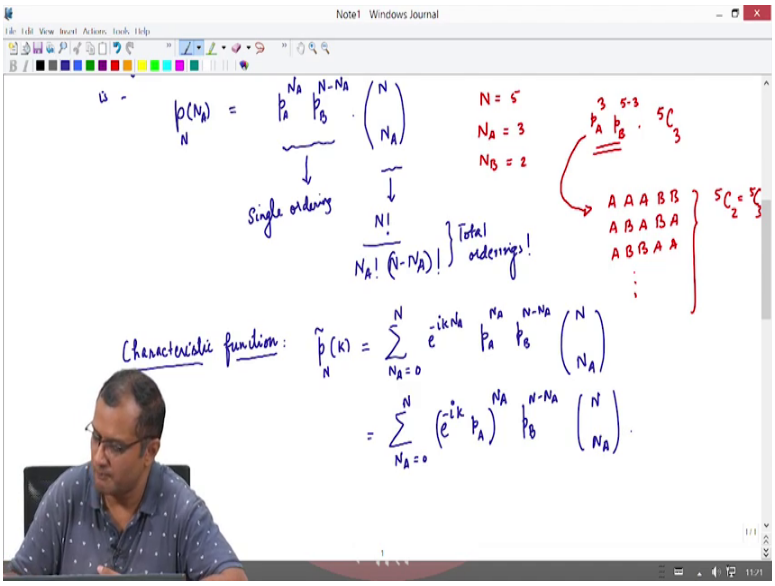

You could also have heads then 2 tails and then 2 heads and so on. So, what you have written here is basically the probability of one ordering, but I know there are several ordering and precisely the number of orderings is known to me, which is 5 C 2 or 5 C 3.

So definitely, I must multiply this probability, which is for a single occurrence with the number of orderings I can have. So, I need to multiplied with 5 C 3, it is also if I multiply with 5 C 2 because, the binominal combination is symmetric in it is two variables. The fact is this probability that I have written here is for a single ordering and 1 has to account for all the orderings possible and precisely for that I am going to multiply it with N C N A, this is a abbreviation used for the combination I can get. So, this is this stands for N factorial upon N A factorial into N minus N A factorial ok. So, this factor here that I have written as a combination is the number of orderings, you can write down for any outcomes and N B out comes ok.

What I written here as a probability, is a probability of a single outcome or a single ordering and the multiplication factor here is the total number of orderings that you can this is the total orderings, you can have for a given number of A outcomes ok. So, that is the meaning of a binominal probability fine. So now, I want to develop some very important mathematics here that will greatly simplify your task for computing moments and cumulants of a binominal distribution.

And this is going to be very important, when you apply binominal distribution to problems in random walks that are profound in statistical mechanics ok. So, clearly with that goal in mind, I am going to some build some very interesting mathematics here that will help you, apply the concepts of a binominal distribution two problems in random walk ok.

So, let me straight away construct the generator of moments, which we call as the by this lecture, you already know that the generator of moments in our course is a characteristic function ok. So, I am going to construct the characteristic function as simple Fourier transform of my probability distribution, which is the binominal distribution. Now this is the distribution which as discrete number of you know out comes, the random variable N A is discrete. So, have to discrete Fourier transform here ok.

So, let us do that so, the Fourier transform of our PDF is now p of K out of N trials. So, this is the characteristic function that I am interested after N trials or N measurement or

N experiments. Now this is given as nothing, but the discrete Fourier transform from the integration variable or the summent variable is now N A and this will run from 0 because, that is the lowest value for A, you cannot have less than 0, 0 is the minimum value a N A can take.

So, you this would be saying all out comes at B, and the maximum value for N A has to be N, it is like saying all out comes of the N measurements were A. So, N A will go from 0 to N e to the power i k N A because, N A is my integration or summent variable times my binominal distribution, which is p of A to the power N A, p of B to the power of N minus N B N A into the combination N C N A ok. Now let us club these two factors together exponential and P A and you shall see that this begins to look like a formula that you know already e raised to minus i k into p of A or clubbed together and they are raised to a common power N A, times p B to the power N minus N A, times the combination N C N A and as I said, this is a well known expansion of the sum p A plus p B ok.

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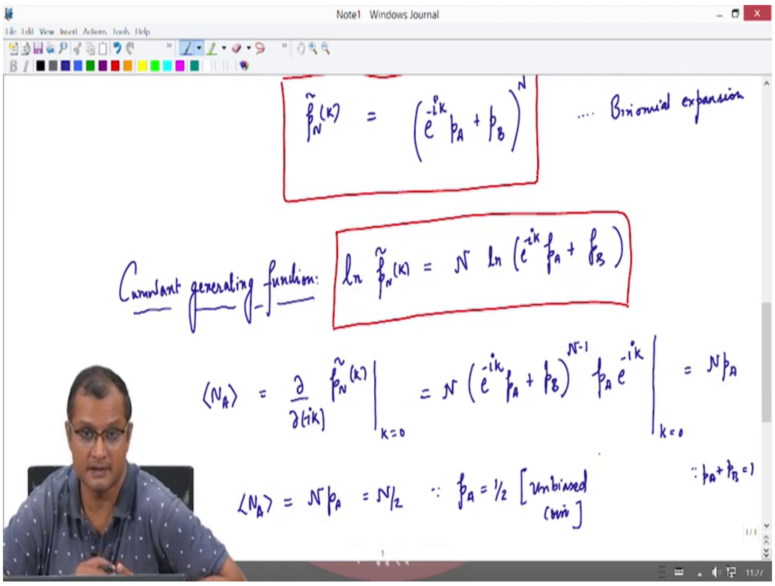

So, this is binominal expansion of the sum of e to the power minus i k into p A plus p B raised to the power N ok.

So, this is the binominal expansion that we know from elementary mathematics. So, you now have a characteristic function ready and you can do a lot of things with it. So for example, you can now straight away go and compute the moments of a binominal moments of the random variable N A even also compute the a cumalants the first cumulant being the mean itself, the second cumulant being the variance of the random variable, the third cumulant being the conus of the random variable and the fourth moment will be kurtosis of the random variable, all these cumulants and the moments can be computed once, you have the characteristic function.

So before, we dwell into all that, let us box this important result and then we will proceed towards the computation of the cumulant generate of function ok. So, this is my characteristic function ready for use ok. So, you can compute the cumulant generating function by simply taking a logarithm of the this ok. So, the cumulate generating function can be computed by adjusting logarithm of the moment generating function or the characteristic function, which is nothing, but N times if you take the logarithm of ln e raised to minus i k p of A plus p of B ok.

So, you have now the cumulant generating function also available with you, you can do a lot of interesting stuff with this ok. So now, you have a very powerful handle to compute moments of the random variable, in fact actually do a very interesting physics problem, which is random walk problem ok. So, we will take up random walk problem shortly and demonstrate the use of this powerful formulas ok. So, I just want mention something here, that if you look at your characteristic function, it is very easy that if you want to compute the let us say the mean of N A.

So for example, if you toss a coin N times and you want to know how many times, what is the average number of heads that you can get? Well the answer is straight away in front of you because, if you take you know this is coin has only two sides heads and tails. So after N trials, you will get approximately half the number of heads that is known to you, but you can also see that it is straight forward it is also obtained from your formula here.

If you just use the definition of means, well the average of N A is nothing, but the first derivative with respect to i k of your moment generating functions, after N trials and this has to be computed k equal to 0 and the saying that you do not you can convince yourself that this machinery is working ok you know for coin toss, this is very simple, well let us just apply this machinery to very simple problem for a proof of principle, we can take the machinery for more complex problems later ok. So, you can just take the derivative and it will become N times e to the power minus i k p of A plus p of B to the power N minus

1 and there is an e raised to minus i k inside, which will give me p A times e raised to minus i k ok. And definitely, I have to compute this derivative it origin ok. So, you can see that this is nothing, but put k equals to 0, here what you get is nothing, but N times p A because p A plus p B is 1 and that is what you want ok.

So, your average number of heads after N trails is nothing, but N trials is nothing, but the probability of getting a head in a single toss. Now if probability of getting head is half, this should be nothing, but N by 2 because the coin is unbiased for a unbiased coin ok. So, this can this should convince you that this machinery is working and you are not you are getting what you expect the answer is expected.

Now we have to take this machinery of moment generating functions to simple, but very important problem in physics, which is the random walk problem and that we provided here as a worked example to illustrate the concepts of the preceding discussion ok. So, I am go to take the example of random walk.

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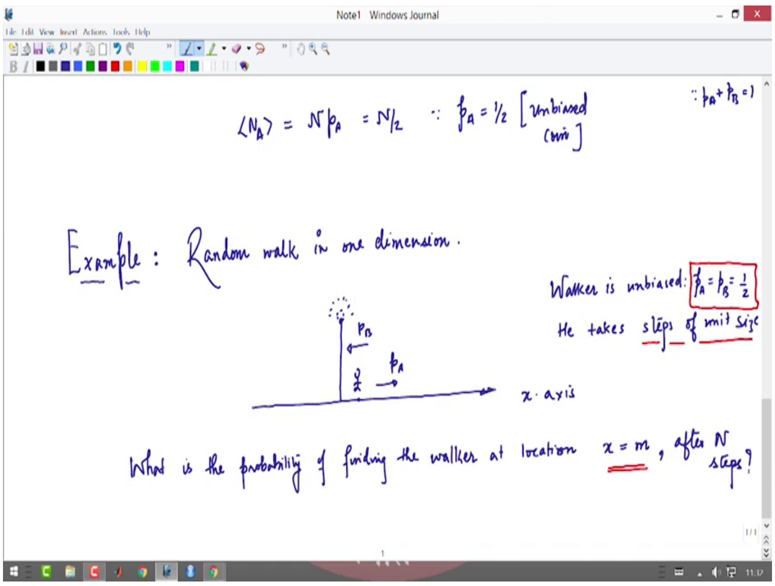

This should be a worked example, which will demonstrate application of the concept in the random walk ok. So, I am go to take random walk in one dimension, dimension is one only for simplicity it is not it is easy to discuss everything in one dimension without much difficulty and it explains the concept.

So, you can generalize this to N dimensions without only difference would be with calculations become tedious. Right so let us look at a random walk in one dimension. So, let us say you have you can always you know say that the walker our 1 d walker, he is walking on a line, let us say this line is the x axis and there is a lamp post here ok, where he starts his walk. Now a walker can go right with a probability p of A and the walker can go left with the probability p of B.

Let us say that the walker is unbiased naturally this has to be a drunkered walker, he has drunk he has consumed. So, much alcohol that he does not know where to go, he is unbiased. So, which means he sufficiently drunk for his probability forward step and backward step to be equal ok, he is taking steps in the forward direction and backward direction with equal probability. Now suppose, I ask you a following question what is the probability? And one more thing, a walker is unbiased and he takes steps of unit size ok.

Now, this is going to be very ideal drunker, whose step size is going to be fixed, but let us say that is an approximation, I am going to you know sort of work with. So, he takes steps of unit size.

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Yeah.

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Sorry, this is one half it has to be one half; obviously, otherwise it is it is garbage. So, let me correct immediate, myself thanks for the correction, there is a typo here the sum of probabilities has to be 1. So, let me just quickly correct myself, I want to write p A plus p B as 1, I ended up writing p A equals p B equals to 1.

So, what I mean is basically that the probabilities are equal and they add up to 1, which means they all have to be half ok. So, let us underline these two important assumptions, because we are going to use them in our discussion. So, this is one important thing that I have assumed that the probabilities are equal in the forward and the backward direction and they are half. Another thing that I have assumed is that the steps are of unit size ok. These are the two assumptions I am going to work with.

Now, the question that I have raised here is there is. So, what is the probability of finding the walker at location x equals to m? Ok. So, what is the probability that the walker after N steps is at location x equals to m? Ok. So, so I am going to just write down, after N steps question is simple, he is taken a large number of steps, let us say 1000 and or 500 or 10000 or whatever.

And after these many steps, what is the probability that he is at some point m on the x axis? Ok.

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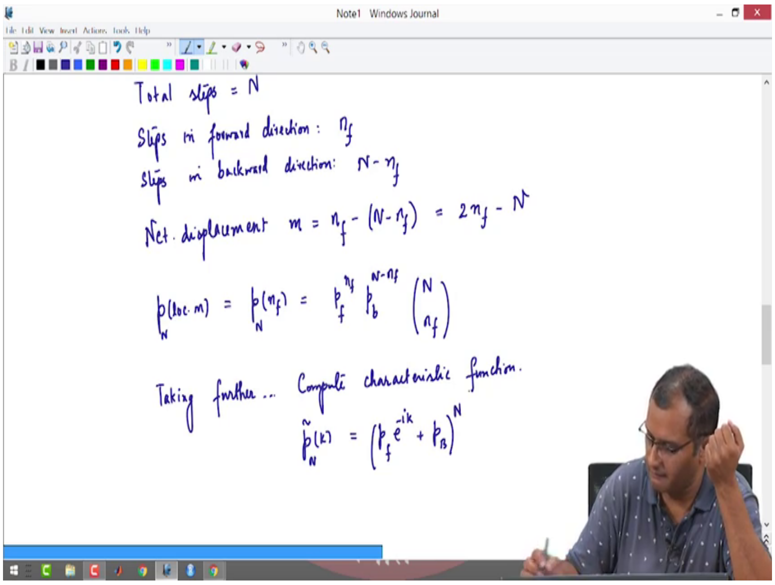

So, naturally you can tackle this problem by saying that total number of steps is N and let us say steps in the forward direction is let us say, this is n f ok. So, steps in the backward direction actually should be N minus n f, because the steps are the total number steps is the big N and so, your net displacement assuming the walker starts from origin ok. So, as I said the walker starts from the lamp post he starts from the lamp post so, his initial position is lamp post itself.

Let us say as the lamp post is at origin and so, the net displacement of this walker, which is our variable m has to be number of steps that the walker takes in the forward direction minus the number of steps, the walker has taken in the backward direction. Now this is already computed. So, we know that the backward direction steps are N minus nf ok, which gives us the value as twice of n f minus N the begin fine.

So, what is being asked is basically the probability of. So, the probability that the walker is at the location m after N steps ok. So, this is basically nothing, but if I take the average value or you know. So, what is being asked is basically, what is the let me just correct myself. So, what is asked is basically, we need to compute the average m. So, what is the ok. So, we are trying to find out the average probability of finding the walker at the location x equals to m fine. So, let us start with that and then we will prove the average m, we will find out the average m that is also fine.

So, this probability of finding the walker at location m, probability of finding the walker at location m at location m is nothing, but the probability of finding the walker, who has taken the nf forward steps in N steps in the total of N steps, because you have taken nf forward steps naturally as taken N minus nf backward steps. So, it is it suffices to just take the probability of n f here and that would be nothing, but as we have already seen the probability of forward step to the power nf times the probability of the backward step to the power N minus n f and the combination, which is N c nf that is the probability of finding the walker at location m with nf forward steps ok.

So now, you can just take the, that is the answer to your first question let us take it further. So, taking further you can compute the characteristic function which is nothing, but the Fourier transform this distribution and that as you have already done turns out to be p forward to the power e raised to minus i k plus p backward to the power N ok. This is the expression here ok. So, let us plug the values of forward and backward, we know this as half.

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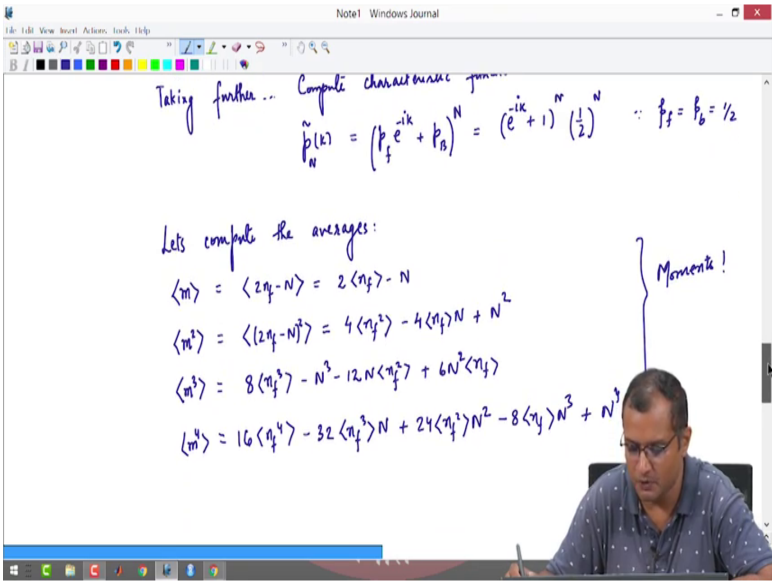

So, what you have is basically e raised to minus i k you can take the half outside plus 1 to the power N into 1 by 2 to the power N ok. So, if you plug pf and pB as pf and forward and backward as half; you can take half outside and what you will get is just e raised to minus ik plus 1 to the power N in to half to the power N. So, you can now compute what are these.

So, suppose you are interested now in computing the averages suppose you want to compute ok. So, you want to compute the average displacement or you want to compute the square of the displacement average or you want to compute the you know the average of cube over displacement these are all averages that one might 1 might be willing to compute then the calculation becomes very simple.

So, we know that mod m is nothing, but mod of twice nf minus N because m itself is given here. So, this is your m let us underline it. So, this is your m. So, if you want average of this, this is just twice of average nf minus N ok. So, this would be just twice of nf minus N because N is a constant the big N is big the big N is constant. So, average of big N is nothing, but N itself right.

So, now can also compute m square as twice n f minus N the whole square which is if you open the brackets what you get is basically 4 times nf square minus 4 nf into N plus N square ok. And similarly you can compute the other values for moments as I am just writing it down from my notes you can consulted, but you can compute them on your own and verify that this is correct ok. So, let me just write down the values directly you can compute them by yourself ok.

So, this is very straight forward. So, I am just directly writing down the result and for m 4 this is 16 nf to the power 4 minus 32 nf cube N plus 24 nf square mod N square minus 8 nf into N cube plus N to the power 4 let me just quickly check if everything is right and then we will proceed forward yes. So, this is all correct ok. So, these are the expressions for moments all the moments I here and we can compute all these moments very simply ok. So, let us first compute to compute if you look at the right hand side of all these moments all that is required is basically just the moments of nf ok.

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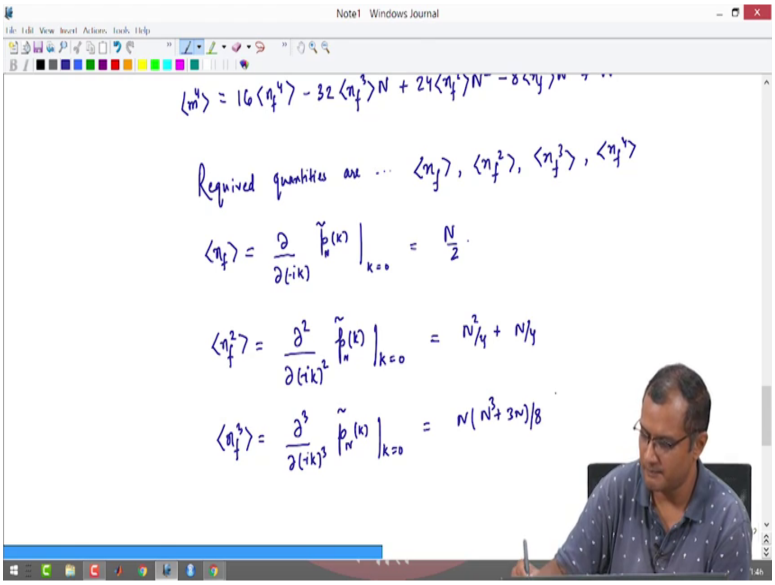

So, while required quantities are just the just the moments of nf all you need is basically these moments and the 4th moment. And that will give you the moments of m. So, let us compute the moments of n small nf and which is nothing, but as you already know is and that you already know this is N times p of a which is N by 2 this we have already computed, but if you want compute you can compute again from this expression.

So, you already have the value for pn of k and from here if you want to compute you can compute it directly. So, you can take the derivative with respect to N and get the answer. So, so, that would be the answer for nf then you can compute the value for nf square which would be the second derivative of P of k at origin and using the value of P of k from the top you can compute this to be N square by 4 plus N by 4 just work it out and for nf cube you can compute this as the 3rd derivative and this will come out to be N into N square plus 3 N by 8.

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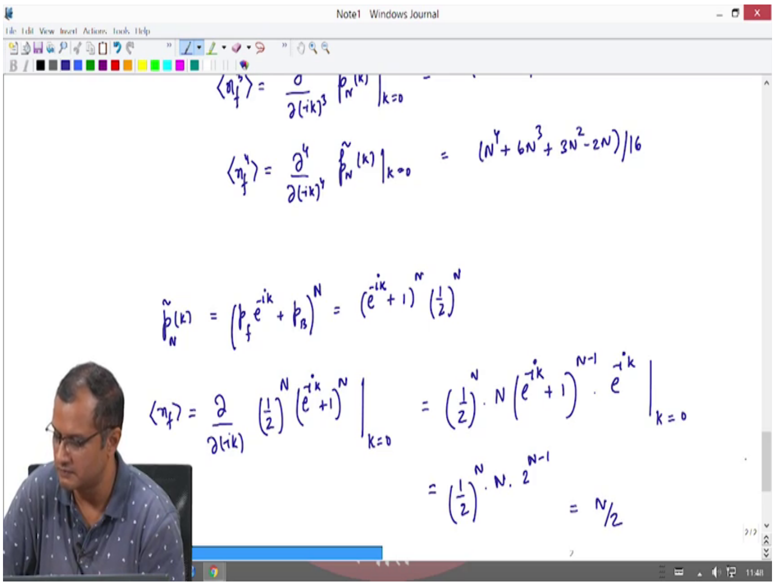

And the 4th moment will come out to be N 4 plus 6 N cube plus 3 N square minus 2 N by 16 you have to do this calculation on your own is a very straight forward calculations and you will get the answer very easily. So, if you want I can do 1 or two of them for your for your help. So, I am write down the this let me just copy paste this expression. So, that so, this is the expression for the cumulants generating function the moment generating function. So, from there you can easily see the first derivative comes out to be N times p A. So, let us take the first derivative which is nf ok. So, I am just take the derivative of 1 by 2 to the power N e raised to minus i k plus 1 to the power N.

And this derivative has to be taken at k equal to 0. So, 1 by 2 to the power N can come out here this is just a constant what you have inside is just N times e to the power minus i k plus 1 to the power N minus 1 and what you have inside is just e raised to minus i k that is it and that is taken at k equals to 0. So, what you have here is at k equal to 0 this simply becomes 1 by 2 to the power N into N in to this become 2 raised to N minus 1 because this is 1 this is 1 this is 2 2 raised to N minus 1 and that is nothing, but N by 2. Fine if you want we can do 1 more just to.

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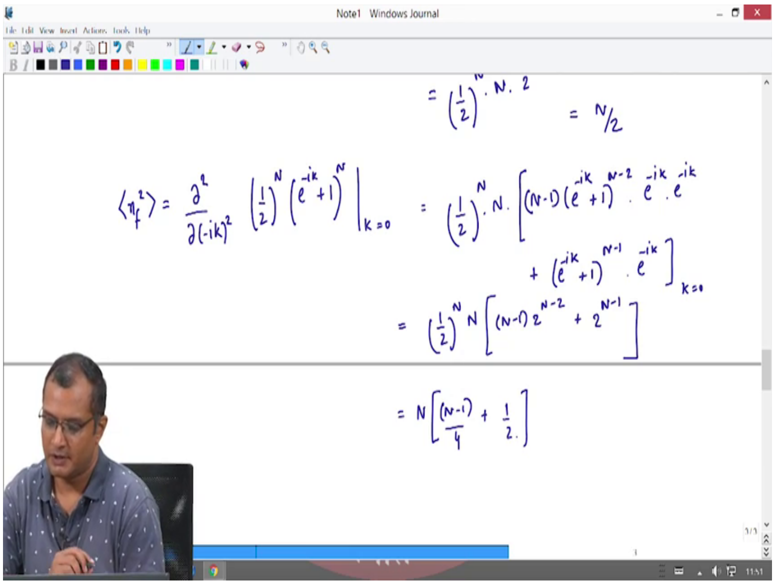

Sort of give you an idea I am I will do the second cumulants also and leave the 3rd and 4th as assignments ok. So, the second cumulants is second derivative sorry I keeps in cumulants this is the second moment of the moment generating function which I am going to just write from the formula above. So, which is nothing, but the 1 by 2 to the power N into e raised i k plus 1 to the power N as k equal to 0.

So, this 1 one half to the power N will come out and the first derivative is already written in front of you. So, am I going to take the first derivative as this on the top and take one more derivative here. So, this would give me N minus 1 to the power a multiplied by e raised to minus i k plus 1 power N minus 2 into e raised to minus i k, but these also need is to minus ik outside plus e raised to minus i k plus 1 to the power N minus 1 it e raised to minis i k and all this is taken at k equal to 0 fine.

So, this will give you if you put k equals to 0 what you have outside the remains 1 half to the power N in to N, let us see what happens to the inside. You have N minus 1 into 2 raised to N minus 2 plus 2 raised to N minus 1 that is it and that becomes if you if you. So, this could be if you take 2 raised to N inside if you take 2 raised to N minus yeah. So, what you do is basically you take 2 raised to N inside. So, what you will have is N minus 1 to the by 4 plus 1 by 2 ok. So, I taking the 1 upon 2 raised to N inside.

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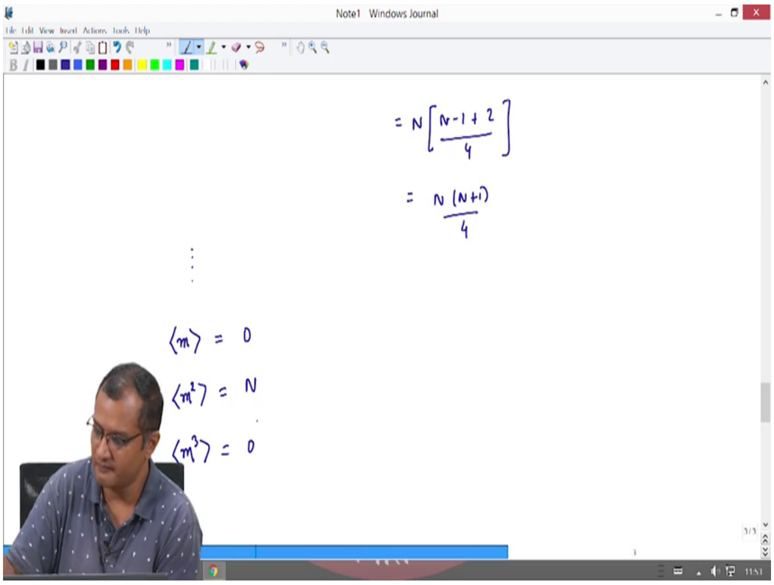

So, this will give me N into N minus 1 plus 2 upon 4 which is N into N plus 1 by 4 ok. So, that is what we had for the second cumulants N square plus N by 4 fine. So, I am leaving the other 2 cumulants other 2 moments to be calculated by you the which is the straight forward derivative only thing is become tedious. So, I am going to leave this as an assignment. Eventually when you plug all these moments of nf the forward displacements into the expression for m which is here ok. So, let me box this. So, after you have find all the moments of nf you are suppose to plug it here in the right hand side and then you will get the values for all these m so, you will get all these ms.

This 2nd moments 3rd moment and the 4th moment ok. So, the moments of the net displacement are then straight forwardly calculated. So, and this is an important. So, the first moment of the unbiased random walk has to be 0 it will come out to be 0, the second moment will scale with system size and that as given as just N, the 3rd moment will also come out to be 0 all odd moments will come out to be 0.

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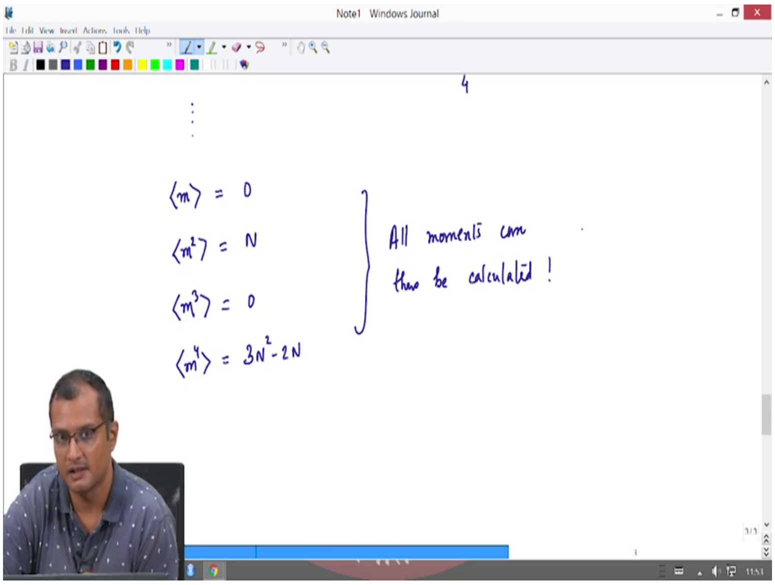

The 4th moment will come out to be 3 N square minus 2 N this is the e very power full method. So, all moments can thus be calculated and I am going to argue that they are calculated very easily if you know the techniques of the characteristic functions ok.

So, this calculation just involves taking the appropriate derivatives of the characteristic function and simply computing the moments from this relationship that we have derived. So, the averages of the net displacement of a random walker which is taking unbiased steps in one dimension are straight forwardly calculated and this is a direct application of the characteristic functions that we have discussed in our course so far.

So, I will end the lecture here and when next time in the need we will start off with the discussion on Poisson distribution, which is another very important distribution in statistical mechanics. So, these are the 3 major distributions and I will cover some worked examples before you meet your first assignment set in this course ok.

So, thank you very much I will close the lecture here.