Statistical Mechanics
Prof. Ashwin Joy
Department of Physics
Indian Institute of Technology, Madras

Lecture – 33
Free electrons (Fermi Gas) in a Metal (Continuation)

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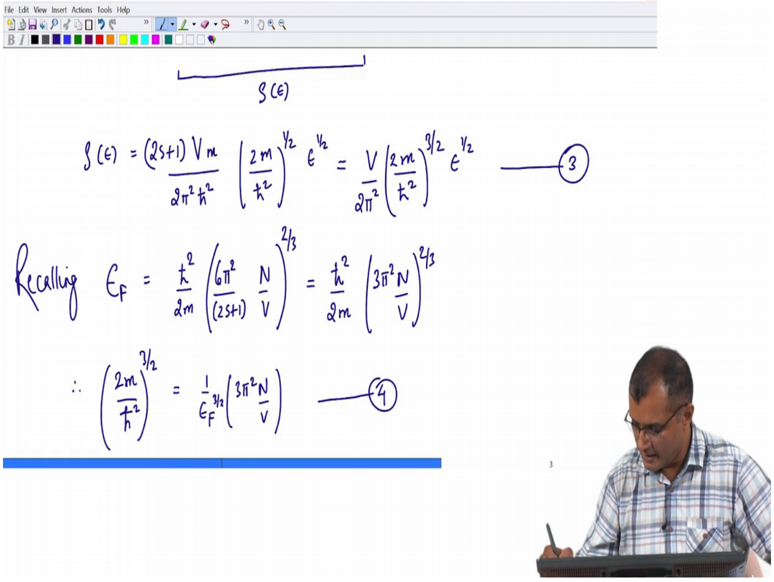

$$\underbrace{\qquad\qquad\qquad}_{\rho(\epsilon)}$$

$$\rho(\epsilon) = \frac{(2s+1)\,V m}{2\pi^2\hbar^2}\left(\frac{2m}{\hbar^2}\right)^{1/2}\epsilon^{1/2} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} \longrightarrow ③$$

$$\text{Recalling}\quad \epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{(2s+1)}\frac{N}{V}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$\therefore \left(\frac{2m}{\hbar^2}\right)^{3/2} = \frac{1}{\epsilon_F^{3/2}}\left(\frac{3\pi^2 N}{V}\right) \longrightarrow ④$$

So, now you can see that, I can recast this definition of the density of states into more usable representation. So, what I am going to do here is rewrite this density of states, as a you can take S the spin of electrons to be half and that will make the 2 S plus 1 as 2. So, I can write the density of states, as a V into 2 m by h cut square, raise to 3 half and a divided by 2 pi square, into E raise to 1 half because, there is a m by h cut square also, which I have taken inside the square root.

So, it has become raised to 3 half ok. So, and here also I can write down the free energy the harp the Fermi energy as. So, I am going to just write it somewhat closer and writing down the Fermi energy as h cut square by 2 m, into 6 pi square; 6 pi square will simply become 2 pi 3 pi square because, 2 S plus 1 is 3.

So, let us use that and this becomes just 3 pi square, into N by V, whole to the power, 2 by 3 all right. So, it looks like more usable to me at least and one more trick here, we can substitute for, what we will do is a from the energy Fermi energy expression, I can write for 2 m by h cut square, as 1 upon EF into 3 pi square N by V, to the power of two third.

And if I raise this to the power 3 by 2 then, I can say that this is nothing but, 1 upon Fermi energy raised to 3 by 2, into 3 by pi and 3 pi square N by V. Now, you can simply plug this; you can simply plug this value of 2 m by h cut square raise to 3 by 2 here, in the density of states. So, let us call the density of states expression as a equation 3 and let us call this as equation 4.

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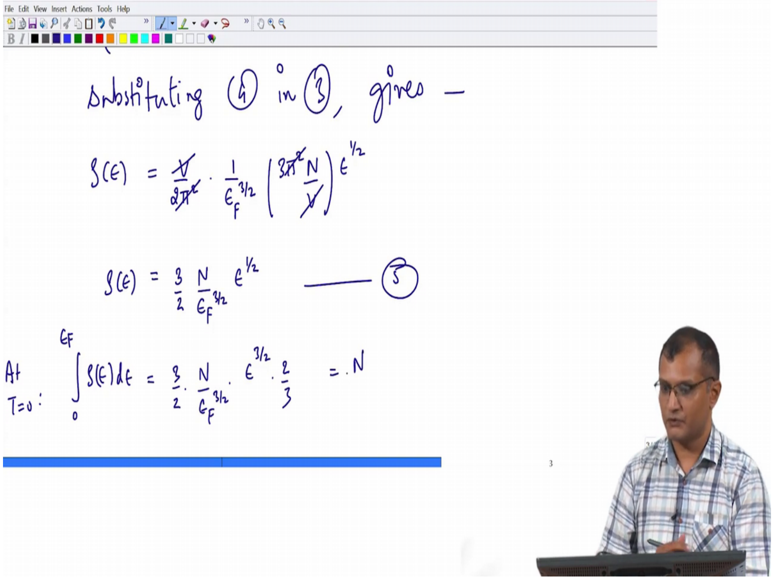

Substituting ④ in ③, gives –

$$\rho(\epsilon) = \frac{\cancel{V}}{\cancel{2\pi^2}} \cdot \frac{1}{\epsilon_F^{3/2}} \left(\frac{3\cancel{\pi^2} N}{\cancel{V}} \right) \epsilon^{1/2}$$

$$\rho(\epsilon) = \frac{3}{2} \frac{N}{\epsilon_F^{3/2}} \epsilon^{1/2} \quad \text{———} \quad ⑤$$

At T=0 :

$$\int_0^{\epsilon_F} \rho(\epsilon) d\epsilon = \frac{3}{2} \cdot \frac{N}{\epsilon_F^{3/2}} \cdot \epsilon^{3/2} \cdot \frac{2}{3} = N$$

And substituting 4 in 3 gives us, I can write down my density of states, as V upon 2 pi square and into 1 upon EF to the power 3 half, into 3 pi square N by V, into E raise to half. So, you can see a lot of terms can be knocked off, you can knock off the 2 volumes, you can knock off the pi square and what you have with the at the end of the day is just 3 by 2, N over EF to the power 3 half, into square root of E and I think this is a more convenient use expression for the density of states right.

So, you can easily say this is the right expression because, if I integrate this rho E over the entire you know, you know at T equal to 0, if I integrate this density of states, I will just get N. So, it is very easy, you can just check, this will give me nothing but, 3 by 2 into N upon EF to the power of 3 by 2 and when square root of E is integrated, it gives you E raise to 3 by 2 into 2 by 3. So, this is nothing but N.

So, my expression of the density of states is correct fine.

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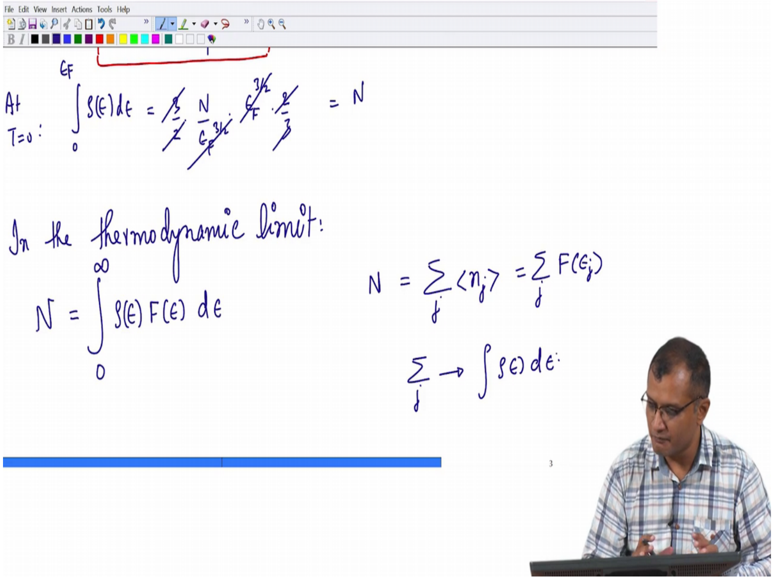

At $T=0$: $\int_0^{\epsilon_F} \rho(\epsilon)\,d\epsilon = \cancel{\frac{3}{2}} \frac{N}{\cancel{\epsilon_F^{3/2}}} \cdot \cancel{\epsilon_F^{3/2}} \cdot \cancel{\frac{2}{3}} = N$

In the thermodynamic limit:

$$N = \int_0^{\infty} \rho(\epsilon) F(\epsilon)\, d\epsilon$$

$$N = \sum_j \langle n_j \rangle = \sum_j F(\epsilon_j)$$

$$\sum_j \to \int \rho(\epsilon)\, d\epsilon$$

So, now I have my density of states, I am going to compute some very important quantities, thermodynamic quantities. So, this is a very important expression for me ok. So, the first important quantity that I am going to in fact, the 2 quantities that we are going to be computing are, we know that in the limit N tending to infinity and V tending to infinity which is the thermodynamic limit, I am going to be computing the you know an expression for chemical potential but, for that I will start with the expression for N.

Now, N is nothing but, a integrating 0 to infinity, the density of states, Now, I know this is the energy density of states, into my Fermi function d E. Let us understand the integrand and the limits here, this is not just arbitrary integral, it has direct consequence from summing over discrete degrees of you know, discrete energy of states and discrete states of energy and what I have done is that if you recall we had simply the summation over all occupation numbers or summation over all the Fermi functions.

So, my integral has become basically, this my summation has basically become an integral, which is a overall density of states and whatever function that I come wanted to compute ok.

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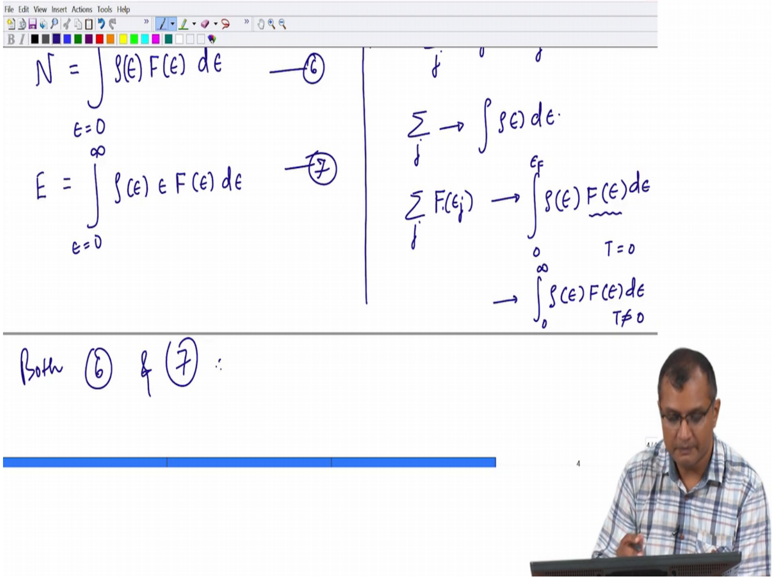

$$N = \int_{\epsilon=0} \rho(\epsilon)\, F(\epsilon)\, d\epsilon \quad \text{—} \textcircled{6}$$

$$E = \int_{\epsilon=0}^{\infty} \rho(\epsilon)\, \epsilon\, F(\epsilon)\, d\epsilon \quad \text{—} \textcircled{7}$$

$$\sum_{j} \rightarrow \int \rho(\epsilon)\, d\epsilon$$

$$\sum_{j} F(\epsilon_j) \rightarrow \int_{0}^{\epsilon_F} \rho(\epsilon)\, \underset{\sim}{F(\epsilon)}\, d\epsilon \qquad T=0$$

$$\rightarrow \int_{0}^{\infty} \rho(\epsilon)\, F(\epsilon)\, d\epsilon \qquad T \neq 0$$

Both ⑥ & ⑦ :

That is my summation over j F Ej, will simply become integral rho E, F E, d E. Now, if I am at a T equal to 0 then, I know that all energy levels, above the Fermi level, will be not occupied, will be vacant. So, this Fermi function is basically 0 above the Fermi energy.

So, for the purpose of integration, the upper limit will be just the Fermi energy, if you are doing the integration at T equal to 0 but, if you allow the temperature to become non zero then, in principle there is no upper limit on the energy because, it can go to infinity of course, the probability density of finding in a state of infinite energy occupied will be 0 but, the mathematical upper limit of the integral will be not a Fermi energy anymore, it will be infinity, at a non zero temperatures.

So, do not worry about infinity, the Fermi function will take care of it ok. So, since we are going to be interested in temperatures, that are nonzero. So, I am going to write it as a thermodynamic limit and non zero temperatures, I am going to be interested in integrals of this type, for total particles and for total energy, we will write down the integral as a our density of states, into energy, into a Fermi function, integrated over all possible values of energy, in this case the energies will go from 0 to infinity ok.

Now, if you look at these 2 integrals, both these integrals for E and N, let us give them some numbers equation 6 and equation 7, both the 6 and 7 are of the type you know.

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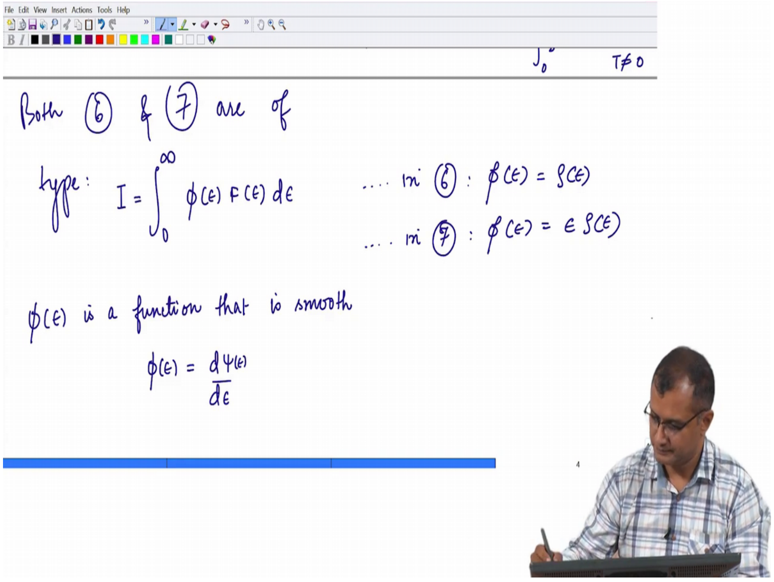

$\int_0$ $T \neq 0$

Both ⑥ & ⑦ are of

type: $I = \int_0^\infty \phi(\epsilon)\, F(\epsilon)\, d\epsilon$ .... in ⑥ : $\phi(\epsilon) = g(\epsilon)$

.... in ⑦ : $\phi(\epsilon) = \epsilon\, g(\epsilon)$

$\phi(\epsilon)$ is a function that is smooth

$$\phi(\epsilon) = \frac{d\psi(\epsilon)}{d\epsilon}$$

They are of the type, an integral 0 to infinity, some function phi of E, into F of E d E, you know for example, for in equation 6, we have taken the function phi as nothing but, the density of states. And in 7, we have taken the function phi as, E times the density of states. So, both the 6 and 7 are of this type. So, I am going to call this integral as integral I and we can write down, we can compute you know before we do that, let us observe a few things about this function phi, we know that the function phi is a function of energy.

So, phi is a function and I am going to say it is a function that is smooth and which means it can be written as a derivative of you know, some function. So, I can write down phi as a derivative of some function ok.

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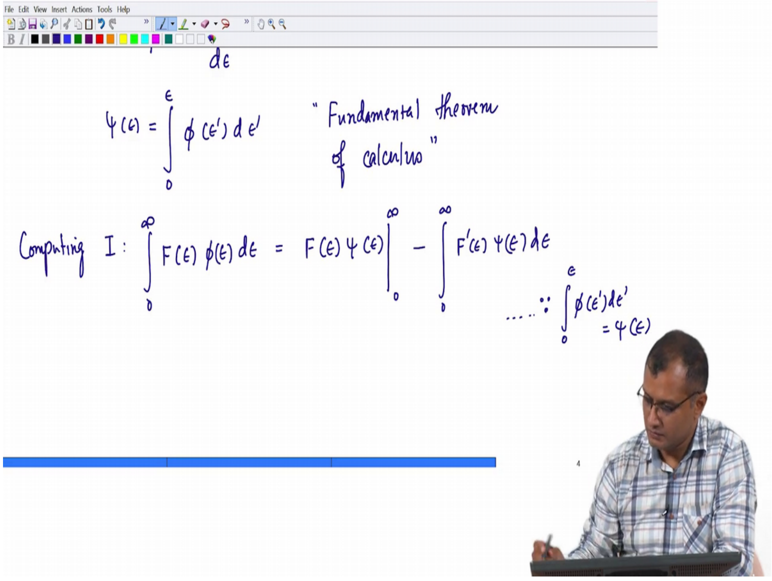

$d\epsilon$

$$\psi(\epsilon) = \int_0^{\epsilon} \phi(\epsilon')\, d\epsilon'$$

"Fundamental theorem of calculus"

Computing I: $\int_0^{\infty} F(\epsilon)\,\phi(\epsilon)\, d\epsilon = F(\epsilon)\,\psi(\epsilon)\Big|_0^{\infty} - \int_0^{\infty} F'(\epsilon)\,\psi(\epsilon)\, d\epsilon$

$\ldots\ldots \because \int_0^{\epsilon} \phi(\epsilon')\, d\epsilon' = \psi(\epsilon)$

So, in this way, I can write down the function psi as, you know an integral of a this function phi of E ok. So, if you look at this by this is just by the fundamental theorem of calculus. So, now, I can do the following, I can compute phi, compute the integral I. So, I can write down integral I as 0 to infinity, I am going to write down the Fermi function first and then write the function phi of E.

So, if you do it by parts, then it becomes simply the Fermi function into integration of phi which is nothing but, my function psi and the limits here are 0 to infinity, minus integration of the derivative of the Fermi function, into psi. Please note that our integration of phi, has already been taken to be psi, that is why fine. So, then I can simply, I can simply write down, you just this is enough to write down. Now, this will be a function of E prime. So, that is what I have done.

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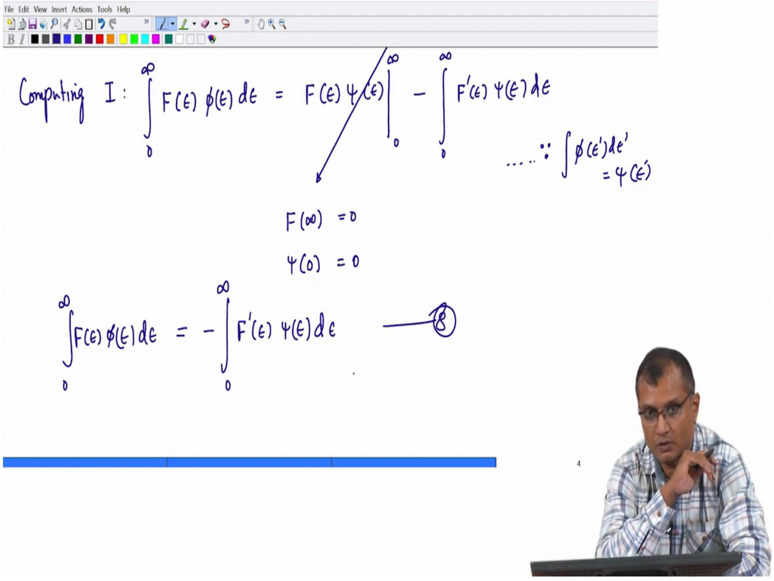

Computing I: $\int_0^\infty F(\epsilon)\,\phi(\epsilon)\,d\epsilon = F(\epsilon)\,\psi(\epsilon)\Big|_0^\infty - \int_0^\infty F'(\epsilon)\,\psi(\epsilon)\,d\epsilon$

$\ldots\ldots \because \int \phi(\epsilon')\,d\epsilon' = \psi(\epsilon)$

$F(\infty) = 0$

$\psi(0) = 0$

$$\int_0^\infty F(\epsilon)\,\phi(\epsilon)\,d\epsilon = -\int_0^\infty F'(\epsilon)\,\psi(\epsilon)\,d\epsilon \quad \text{—— ⑧}$$

And you can see that, the first term here goes to 0 simply because, at infinity the we do not know what is going on with the function psi but, I is no for sure that my Fermi function is 0 to infinity. So, this is 0 at infinity and at the 0 the Fermi function is finite but, psi of 0 is 0 and that is very clean very very easy to see from the fundamental theorem of calculus that I have written here, the way I have defined psi, its evaluate epsilon equal to 0 is nothing but, the integral phi, where the definite integral has the same upper and lower limit.

So, then psi of epsilon, at epsilon equal to 0 is 0. So, that pushes the first term to 0, leaving me with the second term alone, which is a negative integral of F prime, psi E, d E. So, now, we can our left hand side is basically, the integral 0 to infinity, F E phi E d E. So, let us call this as some equation, which is equation number 8, couple of things to note here.

First of all the behaviour of our F prime, I want to solve this integral, this integral can be solved for a general class of the functions phi provided we use the fact that our F prime is a very sharply behaving, you see the Fermi function, at T equal to 0.

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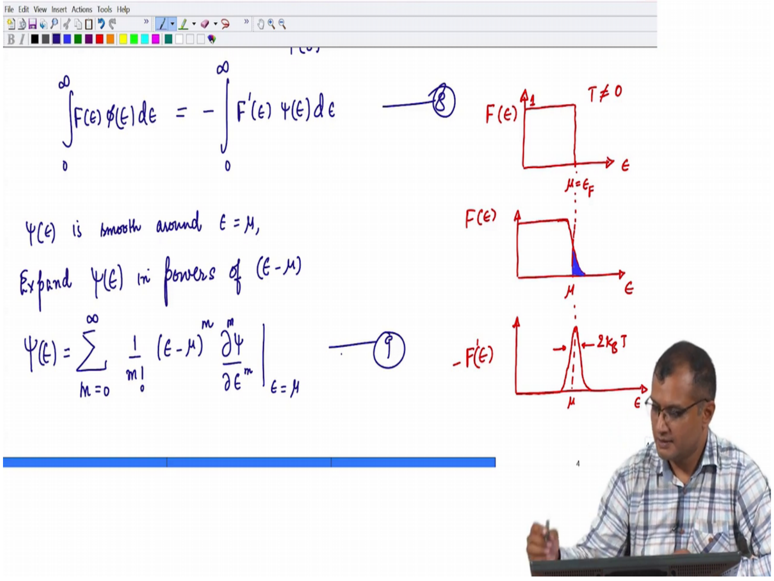

$$\int_0^\infty F(\epsilon)\,\phi(\epsilon)\,d\epsilon = -\int_0^\infty F'(\epsilon)\,\psi(\epsilon)\,d\epsilon \quad \text{——} ⑧$$

$\psi(\epsilon)$ is smooth around $\epsilon = \mu$,

Expand $\psi(\epsilon)$ in powers of $(\epsilon - \mu)$

$$\psi(\epsilon) = \sum_{m=0}^{\infty} \frac{1}{m!} (\epsilon - \mu)^m \left.\frac{\partial^m \psi}{\partial \epsilon^m}\right|_{\epsilon = \mu} \quad \text{——} ⑨$$

So, my T equal to 0, this chemical potential becomes E F and this is sharply falling to 0 and it is 1 everywhere, as a function of E ok. So, at E equals to EF then, I know my Fermi function falls to 0 but, it all nonzero temperatures, I know that my Fermi function, allows for some excitation of electrons above the Fermi surface. So, a small fraction of electrons of the order T by T F are excited above the. So, I am going to say that, this is my excitation.

So, these are precisely the particles that are excited and this would be at a some energy scale which is mu, this is not exactly at the Fermi energy because, this scale has moved slightly, good temperature but, I know its about that is the chemical potential for my problem. So, I know for sure that my, if I take the derivative of Fermi energy then, it will be very sharply peaked around this mu.

And particularly I will take the negative derivative of the Fermi function then, I know for all these values where Fermi function is constant, there shall be no derivative or no variation and very close to mu, I will have a derivative nonzero, you peaking it exactly at mu and again because it has to come down to 0, as the Fermi function it becomes 0 and becomes remains 0 all the way to infinity.

So, this derivative has to come to 0 and you can see this is a sharply peaked function, where the width is of the order of twice k T, about k T over k times T F is the fraction of electrons excited above the Fermi surface. So, the width of this is somewhat twice k T and now using this fact and the fact that function phi of E is smooth, around my energy E equals to mu, I can expand, psi of E in powers of the displacement around the mu.

So, I can expand it in powers of E minus mu. So, I can write down psi of E, as a summation over it is a like a Taylor series. So, summation m going from 0 to infinity, 1 upon m factorial, this is Taylor expansion of psi around equals to mu, you can write down E minus mu to the power m, into del psi by del E, the m th derivative at E equals to mu, this is the Taylor expansion.

Now, you can plug this. So, called smooth behaviour of psi, in equation 8 and try to solve it, is the last step that is required and the smoothness of psi helps us to solve the equation 8.

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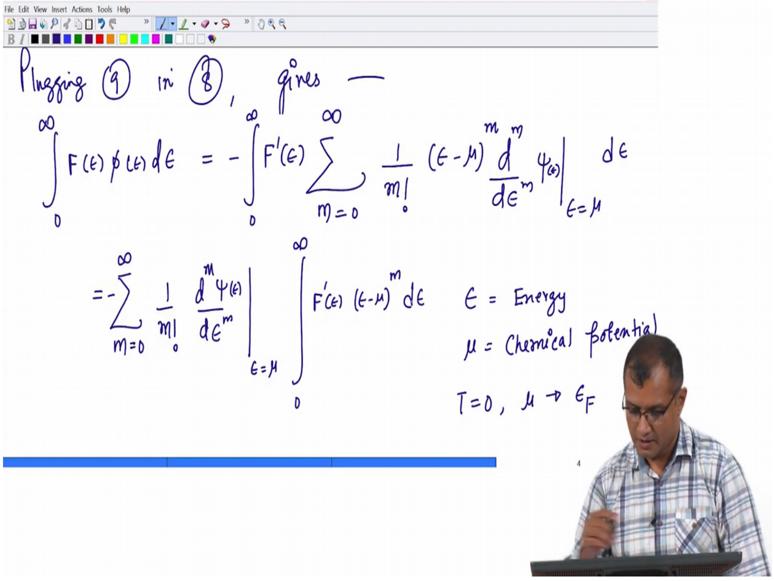

Plugging ⑨ in ⑧, gives —

$$\int_0^\infty F(\epsilon)\,\phi(\epsilon)\,d\epsilon = -\int_0^\infty F'(\epsilon) \sum_{m=0}^{\infty} \frac{1}{m!} (\epsilon-\mu)^m \left.\frac{d^m}{d\epsilon^m}\psi(\epsilon)\right|_{\epsilon=\mu} d\epsilon$$

$$= -\sum_{m=0}^{\infty} \frac{1}{m!} \left.\frac{d^m \psi(\epsilon)}{d\epsilon^m}\right|_{\epsilon=\mu} \int_0^\infty F'(\epsilon)\,(\epsilon-\mu)^m\,d\epsilon$$

ϵ = Energy

μ = Chemical potential

$T=0$, $\mu \to \epsilon_F$

So, you can see that plugging 9 in equation 8, gives us integral that we are chasing 0 to infinity, F of E into phi of E, as a negative of the integral 0 to infinity, F prime E into summation m going from 0 to infinity.

1 upon m prime m factorial, E minus mu to the power m and since psi is just a function of E, I am going to write down the partial as the ordinary derivative, psi is not a function of any other variable. So, purely respecting that, I am going to write it as, m th ordinary derivative, of psi at E equals to mu and this point I would like to remind here, this mu is

the chemical potential, the distinction of chemical potential from the Fermi energy is only at nonzero temperatures.

Like I have already showed, E is our energy, that can take values between 0 to infinity, mu is our chemical potential, which sets the scale of the energy, in some sense. So, at T equal to 0, the chemical potential becomes the Fermi surface energy or the value of the Fermi energy. So, the scale of the problem, which is mu becomes the Fermi energy at T equal to 0, that is the only distinction. So, at any non 0 temperatures, this scale remains the chemical potential, as reflected in our Fermi Dirac statistics.

If you go back, as you can see here, this is a scale of the energy ok, which is the chemical potential. Now, let us get down to what you are evaluating right. So, now, you can see that, there are a couple of things I can do with my integral, I can take out the terms which do not depend on epsilon outside the integral, in that in doing so, I can take out the entire summation. So, I can write down negative summation of m going from 0 to infinity, 1 upon m factorial.

And I can also take out, the m th derivative of psi, with respect to epsilon because, keep in mind that the m th derivative of psi that is a function of epsilon, at E equals to mu is independent of E ok, this derivative is independent of E and what I am left over with is just the now, the integral, 0 to infinity, F prime E into, let us move all this description on the right hand side because, I need some space here, F prime E into E minus mu the power m d E. I think I am not missing any terms here fine. So, now, let us try to solve this by some substitution.

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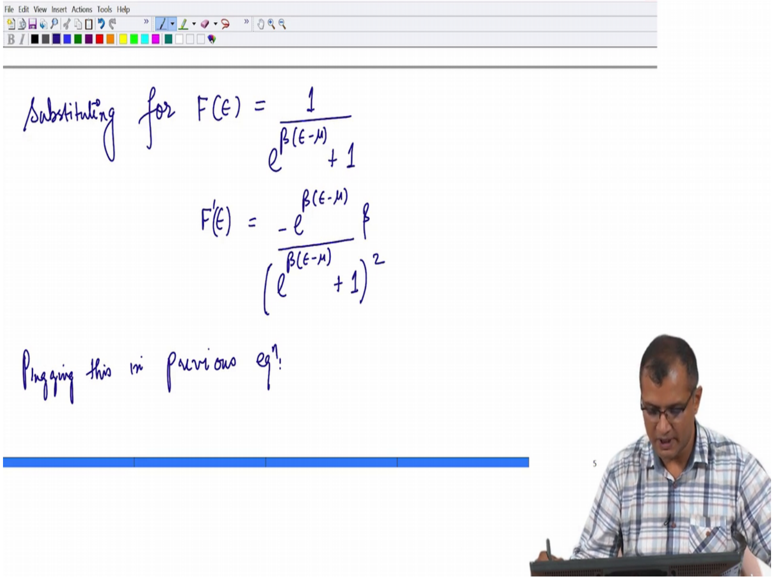

Substituting for $F(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$

$$F'(\epsilon) = \frac{-e^{\beta(\epsilon-\mu)}\beta}{\left(e^{\beta(\epsilon-\mu)}+1\right)^2}$$

Plugging this in previous eq^n:

Well I can easily make 2 substitutions here; the first is I can substitute for the Fermi function. So, substituting for the Fermi function, which is the basically my, the Fermi Dirac statistics, 1 over e raise to beta my energy minus the energy scale the chemical potential plus 1 or F D statistics.

If I take this and perform a first derivative. So, the first derivative with respect to epsilon would simply be which simply give me e to the power beta, E minus mu, into beta, divided by e to the power beta E minus mu, plus 1, the entire thing is squared and the entire thing has to be taken with a negative sign because, that is what comes when you square the denominator fine. So, if you plug this in the previous expression.

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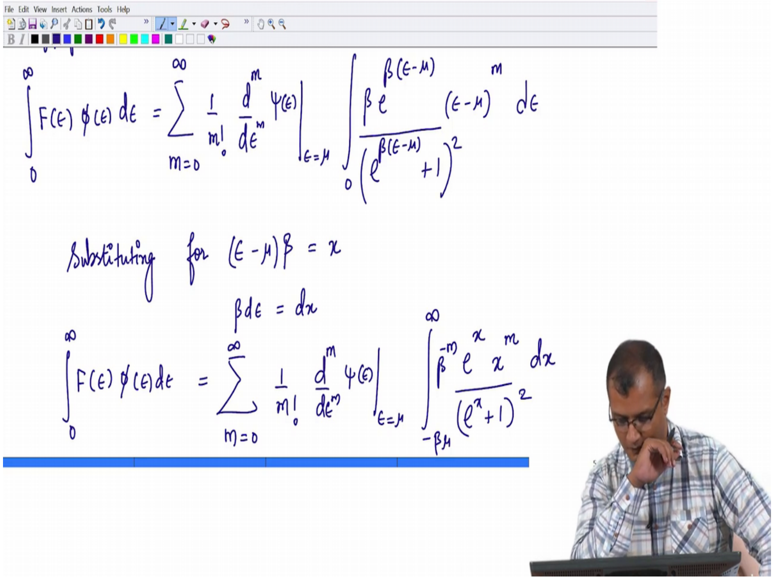

$$\int_0^\infty F(\epsilon)\,\phi(\epsilon)\,d\epsilon = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{d^m}{d\epsilon^m}\psi(\epsilon)\Big|_{\epsilon=\mu}\int_0^\infty \frac{\beta e^{\beta(\epsilon-\mu)}}{\left(e^{\beta(\epsilon-\mu)}+1\right)^2}(\epsilon-\mu)^m\,d\epsilon$$

Substituting for $(\epsilon-\mu)\beta = x$

$\beta d\epsilon = dx$

$$\int_0^\infty F(\epsilon)\,\phi(\epsilon)\,d\epsilon = \sum_{m=0}^{\infty}\frac{1}{m!}\frac{d^m}{d\epsilon^m}\psi(\epsilon)\Big|_{\epsilon=\mu}\int_{-\beta\mu}^{\infty} \frac{\beta^{-m}e^{x}x^{m}}{\left(e^{x}+1\right)^2}\,dx$$

You will get the integral 0 to infinity, F E into the function phi E d E, as there is already a negative sign, that becomes positive now, with the summation m going from 0 to infinity, 1 upon m factorial, dm by d epsilon factorial of a psi, at E equals to mu, into an integral 0 to infinity and what we have here is, F prime which is nothing but, the beta e raise to beta E minus mu, over e raise to beta E minus mu, plus 1 whole square and have beta accorded E minus mu raise to the m, into d E fine right.

So, now just make a small substitution, to solve this integral, for a E minus mu, into beta as sum x. So, we can write down for beta times, d epsilon as d x. So, that will make my integral very simple in terms of x, I can write down my integral as a and I can write down this integral as.

So, limits will well, the lower limit, when epsilon goes to 0 becomes minus mu beta ok. So, the lower limit will transform to minus mu beta because, of the substitution I have made and when epsilon goes to infinity, the upper limit remains infinity and with the substitution E minus mu into beta, my integral becomes e to the power x, into x to the power m, into beta to the power minus m, into d x over e raise to x plus 1 the whole square fine. So, since beta is a function that is independent of x, I am going to write down this outside ok.

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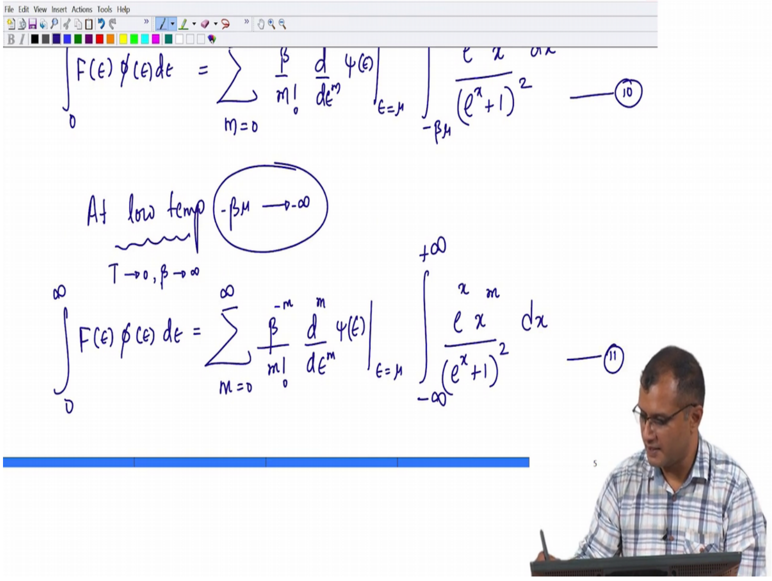

$$\int_{0} F(\epsilon)\,\phi(\epsilon)\,d\epsilon = \sum_{m=0} \frac{\beta}{m!_0}\, \frac{d}{d\epsilon^{m}}\psi(\epsilon)\Big|_{\epsilon=\mu} \int_{-\beta\mu} \frac{e\ x}{(e^{x}+1)^{2}} \qquad ⑩$$

At $\underbrace{\text{low temp}}_{T\to 0,\ \beta\to\infty}$ $\boxed{-\beta\mu \to -\infty}$

$$\int_{0}^{\infty} F(\epsilon)\,\phi(\epsilon)\,d\epsilon = \sum_{m=0}^{\infty} \frac{\beta^{-m}}{m!_0}\, \frac{d^{m}}{d\epsilon^{m}}\psi(\epsilon)\Big|_{\epsilon=\mu} \int_{-\infty}^{+\infty} \frac{e^{x}\, x^{m}}{(e^{x}+1)^{2}}\, dx \qquad ⑪$$

And remove it from here, it is only a function of m ok. So, now we can solve this integral, let us put a label on it, we are at 9. So, let us put the label 10. So, I am going to solve this integral at very low temperatures now, but low temperatures means what? I need a scale, at the beginning of the discussion I said low temperature, by that time if I had introduced to a temperature scale, it would have been meaningless because, I have not discussed the problem in general but, now I have a temperature scale here, which is mu.

So, I can say that very low temperatures means my, beta times mu should go to infinity ok. So, mu over k T should go to infinity, that is the meaning of low temperature. So, I have an energy scale, which is mu, now makes sense and that means, my minus beta mu will be minus infinity, that is a meaning of low temperature right.

Because, low temperature means, T going to 0 or beta going to infinity, you can also say absolutely low temperature but, here I have given a scale argument, any case they both imply the same, this is more beautiful in the sense that, you do not have to take the absolute limit, that T goes to 0 or beta goes to 0.

You can say that at finite T but, as long as mu times beta goes to infinity, I will take this as a low temperature limit fine, that is a more intuitive way of saying low temperature, that you compare 2 scales.

So, now, in this low temperature limit, my integral becomes, the summation m going from 0 to infinity, beta to the power, minus m over factorial m, d raise to m over d E to the power m, please do not be bogged down by this nasty expression, this is a very straightforward calculation.

Which we are simplifying in you know very straightforward steps, there are no very, you know absurd mathematical formulation is a straightforward thing and every step will be explained in detail. In fact, the integral that I am going to run into very soon, will also be explained in the mathematical prerequisites, which is discussed as a separate lecture.

So, that I do not take a long digression into the problem. So, then this integral at the low temperature will have now, new limits going from minus infinity to plus infinity, e to the power x x raise to m, over e to the power x plus 1, whole square d x. Now, it seems that we have to just solve this small integral to compute our answer yes but, it is not a straightforward integral ok. So, let me call this as equation 10, 11.

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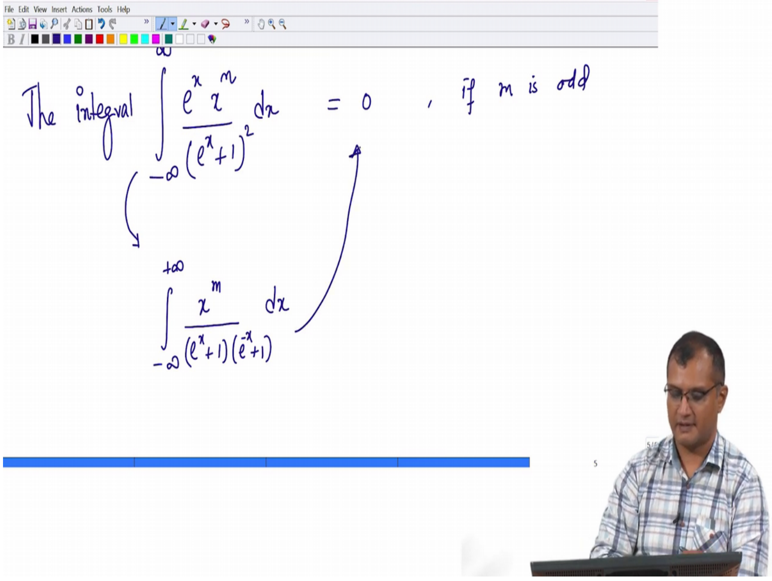

The integral $\int_{-\infty}^{\infty} \frac{e^x x^m}{(e^x+1)^2} dx = 0$, if m is odd

$$\int_{-\infty}^{+\infty} \frac{x^m}{(e^x+1)(e^{-x}+1)} dx$$

So, this integral, minus infinity to plus infinity, can be solved for certain types of m, first of all this integral is 0, for values of m which are odd, look at the interval of the integral, it is from minus infinity to plus infinity. So, if the integrand here is odd then, the integral is 0. Now, the integrand here, which is this function, can be seen as a you know, I can write it as, minus infinity to plus infinity, I can rewrite e raise to x upon, e raise to x plus

1 the whole square as simply multiplying both the numerator and denominator with e raise to minus x, this simply becomes, just rewriting it ok.

Now, you can see that why these evenness or oddness of this integral, depends entirely on the value of m ok. So, I can say that this is 0, if m is odd or you can just say that the since, these are the same integrals, this is the answer to this is nothing but ok. So, it just have to compute for values of m, which are even now.

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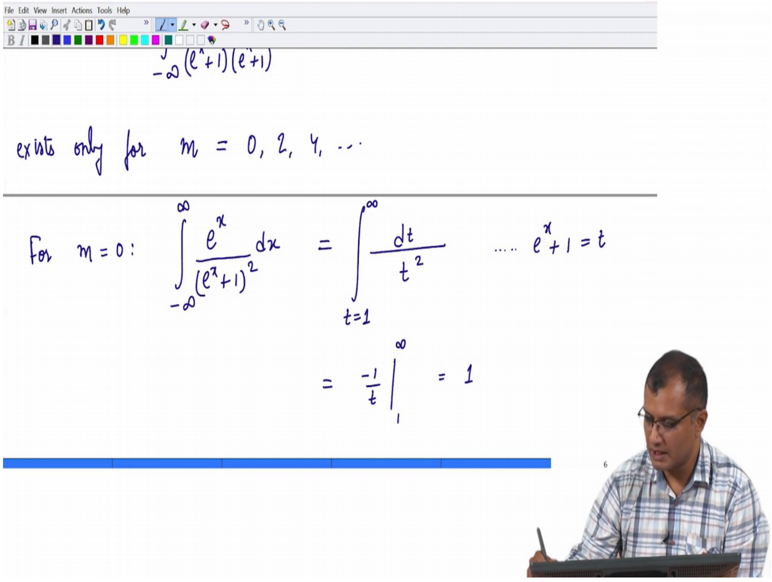

$$_{-\infty}\,(e^{x}+1)(e^{-x}+1)$$

exists only for $m = 0, 2, 4, \ldots$

For $m = 0$: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{x}}{(e^{x}+1)^{2}}\,dx = \int_{t=1}^{\infty} \frac{dt}{t^{2}}$ ….. $e^{x}+1 = t$

$$= \left.\frac{-1}{t}\right|_{1}^{\infty} = 1$$

So, the integral exists, only for m equals to 0, 2, 4 etcetera. So, let us take 2 values and stop there and the reason being higher orders of m will be basically proportional to higher powers of k T for example, m equal to 4 will a term that is proportional to k T to the power 4. And if we are since, we are doing everything at low temperatures, higher orders of temperature would be you know would be vanishing in nature. So, we will stop at the first 2 orders. So, let us take for m equal to 0 and 2.

So, for m equal to 0, my integral will simply become minus infinity to plus infinity, x square sorry, this simply becomes, e raise to x over e raise to x plus 1, the whole square d x ok. And this can be solved by simply substitution of you know, e raise to x plus 1 as some variable t ok. So, you take it square it up and the numerator would just be just d t ok.

Now, what about the limits? Limits have to be you have to be careful, I am substituting e raise to x plus 1 as t, which means the x going to minus infinity limit, will transform to t equal to 1 and the x going to infinity will transform to t equal to infinity.

So, this integral would now be, you know can be straight forwardly computed, it is a minus of 1 upon t and between the limits 1 to infinity, this would simply be 1. So, now you can go back here and say that for m equal to 0, I have 1.

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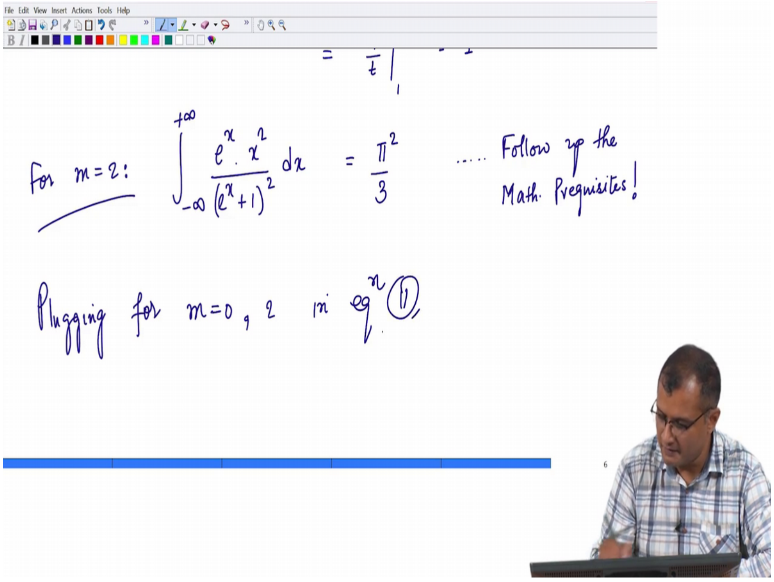

For $\underline{m=2:}$ $$\int_{-\infty}^{+\infty} \frac{e^{x} \cdot x^{2}}{(e^{x}+1)^{2}}\,dx = \frac{\pi^{2}}{3}$$ ..... Follow up the Math. Prequisites!

Plugging for $m = 0, 2$ in eq^{n} ⑪,

Let us compute for one more value, for m equal to 2. So, for m equal to 2, this integral will give me, minus infinity to plus infinity, e raise to x into x square, upon e raise to x plus 1 d x and this integral can be shown to be pi square by 90. I leave that as an exercise and we will do this in the separate tutorial or you can take it up as a personal homework problem but, I am going to say that you follow up the math prerequisites, we will discuss this integral there.

So, let me just confirm the value of this integral, that it is indeed pi square no, it is pi square by 3 I am sorry, it is pi square by 3. So, either you can do it yourself or you can refer to the math prerequisites and we will show it how to get this integral there.

So, now, with the 2 values of m 0 and 2, we plug these values in our final expression, which is equation number 11. So, I will say that, plugging for m equals to 0 and 2 in equation 11.

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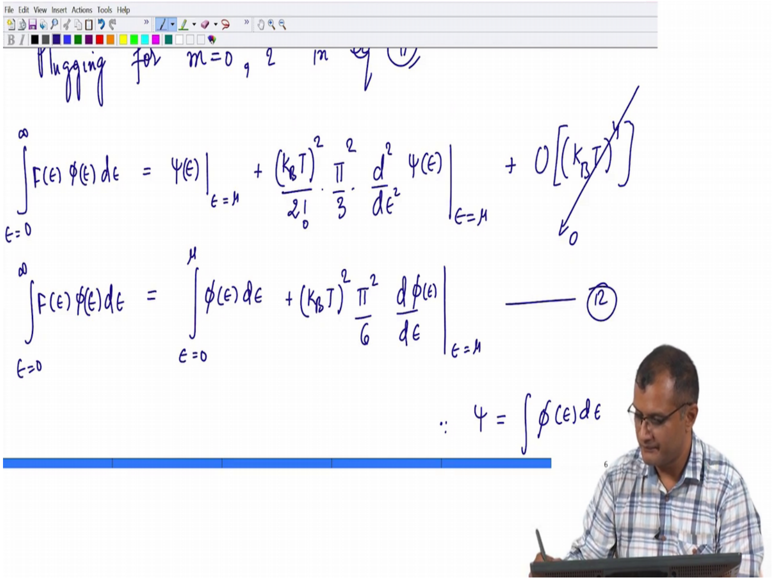

Plugging for $m = 0, 2$ in eq ⑪

$$\int_{\epsilon=0}^{\infty} F(\epsilon)\,\phi(\epsilon)\,d\epsilon = \Psi(\epsilon)\Big|_{\epsilon=\mu} + \frac{(k_BT)^2}{2!}\cdot\frac{\pi^2}{3}\cdot\frac{d^2}{d\epsilon^2}\Psi(\epsilon)\Big|_{\epsilon=\mu} + \cancel{O\left[(k_BT)^4\right]} \to 0$$

$$\int_{\epsilon=0}^{\infty} F(\epsilon)\,\phi(\epsilon)\,d\epsilon = \int_{\epsilon=0}^{\mu}\phi(\epsilon)\,d\epsilon + (k_BT)^2\,\frac{\pi^2}{6}\,\frac{d\phi(\epsilon)}{d\epsilon}\Big|_{\epsilon=\mu} \quad \text{———} \quad ⑫$$

$$\because \Psi = \int \phi(\epsilon)\,d\epsilon$$

We get the final answer as integral 0 to infinity, as a the first term would be just m equal to 0. So, that would just give me psi 8 equal to 0 ok, plus m equal to 2 would give me, k T square by factorial 2, into our pi square by 3, into the second derivative with respect to E of the functions psi, at E equals to mu ok.

So, I think that is it and since I have not gone to m equal to 4, I will say that I have dropped terms of the order, k B T to the power 4 ok. So, that would have been the pre factor of m equal to 4, we are not going to the fourth order because, anyway this is a theory at low temperatures.

So, we will stop at the quadratic term in temperature. Now, what is this the function psi of epsilon this is nothing but, the way we have defined psi, upstairs. Psi is nothing but, the integral of function phi; if you look here the function psi here is nothing but, the integral of our function phi, which is on the left hand side.

So, I am just going to write down this, it is an integral over phi but, this is integral and then it is our function psi computated E equals to mu. So, what I am going to write down is that this is an integral going from, E 0 to mu ok.

That is the value of function psi at mu all right. So, the second term would be nothing but, k B T square and I can write the pre factor here as pi square by 6 and this is a d square psi by d E square. So, psi itself is a. So, I can write this as just, d phi by d E because, our function psi is just integral. So, you can take this as when you take the derivative of a double derivative of psi, you will get, it is equivalent to taking a single derivative of phi ok.

So, then you can say that, if this is for the integral. So, if you can call this as the integral 12 and now you can take some cases for example, you can solve for some thermodynamic properties.

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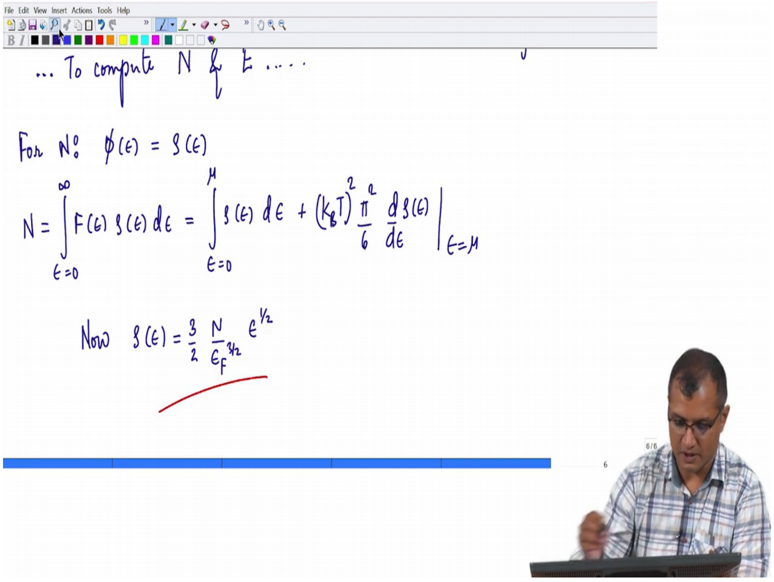

... To compute N & E .....

For N: $\phi(\epsilon) = \rho(\epsilon)$

$$N = \int_{\epsilon=0}^{\infty} F(\epsilon)\,\rho(\epsilon)\,d\epsilon = \int_{\epsilon=0}^{\mu} \rho(\epsilon)\,d\epsilon + (k_B T)^2 \frac{\pi^2}{6} \frac{d\rho(\epsilon)}{d\epsilon}\Big|_{\epsilon=\mu}$$

Now $\rho(\epsilon) = \frac{3}{2} \frac{N}{\epsilon_F^{3/2}} \epsilon^{1/2}$

So, now we are in a position to compute, N and. In fact, I can say N and E ok. So, let us see what we get, if we try to compute N. So, for N, I have to take my function phi of E, as a just the density of states because, if you see the left hand side, N would be nothing but, integral, E going from 0 to infinity, F of E into a density of states d E.

So, putting phi as rho E, this will give me that N, which means in the right hand side, I have to put phi as rho E. So, if you do that, what you get integral E going from 0 to mu, rho E d E, plus k B T, the whole square into pi square by 6, into d by d E of rho E, at E equals to mu.

Now, we know what is the formula for the density of states so, that formula for density of states we are just going to copy it from upstairs, the density of states here, has I need formula ok. So, let us I just copy paste this formula. So, since this is the density of states ok. So, what I am going to do is just plug this value in the previous equation.

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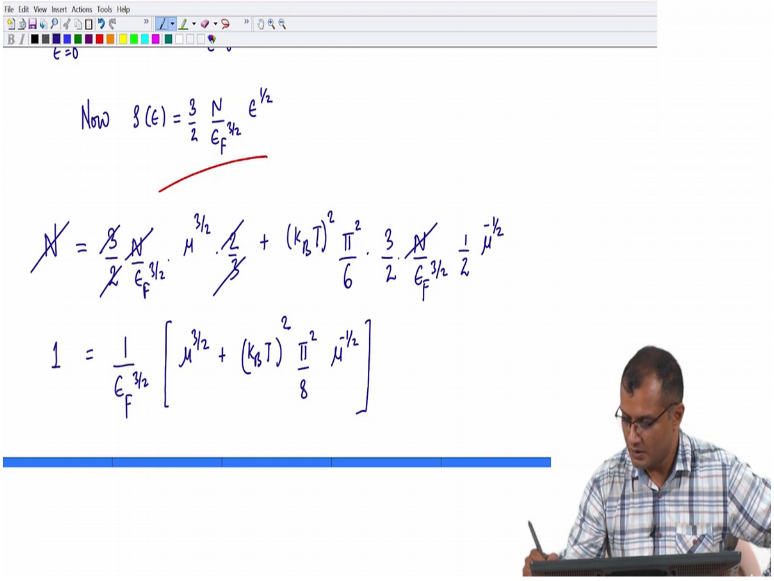

$\epsilon = 0$

Now $g(\epsilon) = \frac{3}{2} \frac{N}{\epsilon_F^{3/2}} \epsilon^{1/2}$

$$\cancel{N} = \frac{\cancel{3}}{\cancel{2}} \frac{\cancel{N}}{\epsilon_F^{3/2}} \cdot \mu^{3/2} \cdot \frac{\cancel{2}}{\cancel{3}} + (k_B T)^2 \frac{\pi^2}{6} \cdot \frac{3}{2} \cdot \frac{\cancel{N}}{\epsilon_F^{3/2}} \frac{1}{2} \mu^{-1/2}$$

$$1 = \frac{1}{\epsilon_F^{3/2}} \left[\mu^{3/2} + (k_B T)^2 \frac{\pi^2}{8} \mu^{-1/2} \right]$$

And this will give me the left hand side is N and if you look at the right hand side I can take a couple of constant pre factors outside, into integral of square root of E that will give me nothing but, E raise to 3 by 2 but, in the limits mu and 0, this will simply become mu, into 2 by 3 fine, plus I will have a k T square, into pi square by 6 and if I take the derivative, I will again get all the same pre factors and the derivative of square root of E is nothing but, 1 by 2 into E raise to minus 1 by 2 but, it equals to mu, it this will be just a mu raise to minus 1 by 2.

So, now you are easily seeing a lot of common 3 factors that, you can probably cancel. So, what you can do straight away here is the following, you can knock off these Ns, from on the sides and what you can also do is now a basically take. So, your right hand side becomes 1, you can take 3 by 2, E F to the power 3 by 2 outside.

So, what you will be left with is basically 2 by 3. In fact, you can do one more thing, you can knock off this also and simply take 1 upon the Fermi energy to the power 3 by 2 common and what will be left is basically mu to the power 3 by 2, plus k B T square, into pi square by 8, into mu raise to minus 2, 1 by 2 ok.

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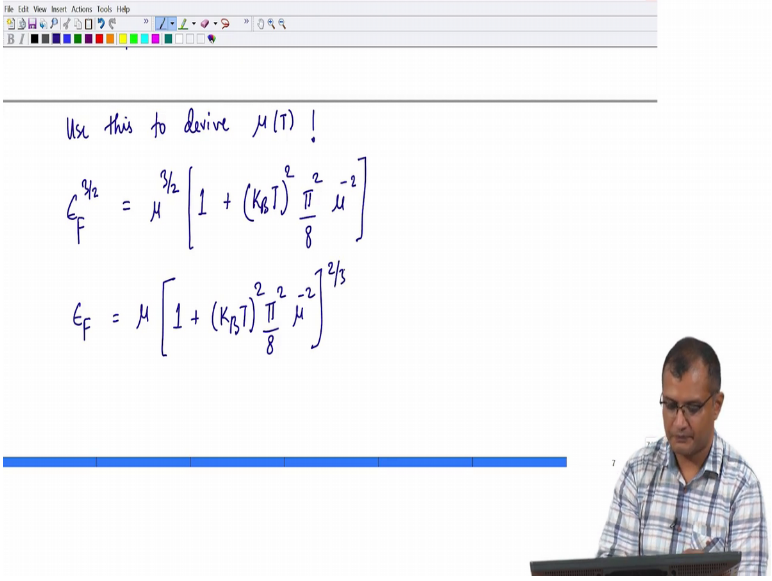

Use this to derive $\mu(T)$!

$$\epsilon_F^{3/2} = \mu^{3/2}\left[1 + (K_B T)^2 \frac{\pi^2}{8} \mu^{-2}\right]$$

$$\epsilon_F = \mu\left[1 + (K_B T)^2 \frac{\pi^2}{8} \mu^{-2}\right]^{2/3}$$

So, now you can use this, to derive an expression for mu as a function of T. So, we have this equation here, which can be used to compute how the chemical potential varies with temperature. So, I have said for one value of temperature that the chemical potential becomes the Fermi energy but, what happens to the chemical potential as you vary the temperature, this is something that you would like to know.

So, that can be computed very easily, you can multiply both sides with E to the power, Fermi energy to the power 3 half. So, what you get is basically, just the equation in the expression the brackets and from that also if you take out the mu raise to 3 half outside. What you get is a just 1 plus k B T the whole square into pi square by 8, into mu raise to minus 2 ok. So, now what you can do here is the following, you can invert this expression for mu.

So, you can say that this is a but, before that you can raise the entire equation to power two third. So, that would become make it E F, is equal to mu time mu into 1 plus k B T, the whole square into pi square by 8, mu to the power minus 2, raise to two third, you raise the entire equation to the power two third. And simply transpose to the opposite side.

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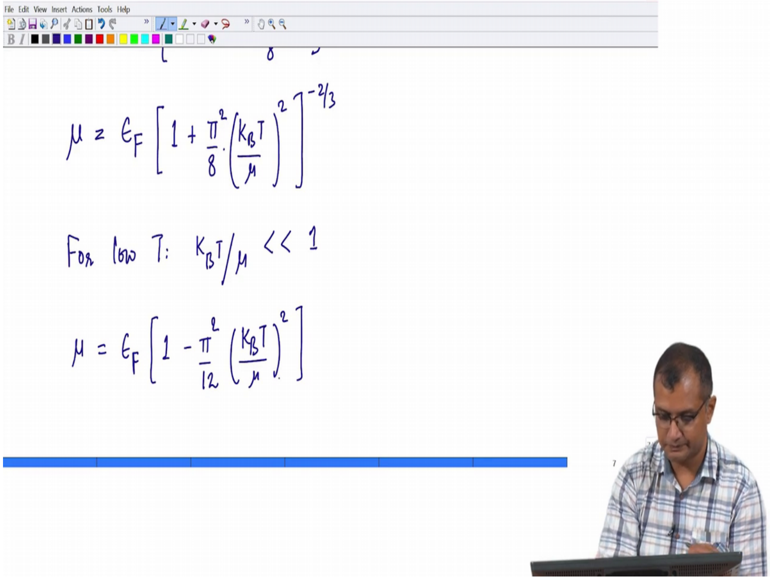

$$\mu = \epsilon_F \left[1 + \frac{\pi^2}{8} \left(\frac{k_B T}{\mu} \right)^2 \right]^{-2/3}$$

For low T: $k_B T / \mu << 1$

$$\mu = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\mu} \right)^2 \right]$$

So, mu becomes E F into 1 plus k B T the whole square, into pi square by 8, into mu raise to minus 2, raise to minus 2 by 3. So, you can write down this as, you can write this entire thing as, pi square by 8 into k B T over mu, the whole square raise to minus 2 by 3.

Now, since we are in the low temperature approximation, for low temperatures meaning my temperatures are a much smaller than the scaled mu. So, I have a scale here. So, such that this ratio becomes much smaller than 1, I can expand this term in powers and to lowest order I can write it as a e to the power, the chemical potential as E F into, 1 minus 2 pi square ok. So, I can write it as a, just pi square by 12, to make things simpler, into this dimensionless number k T over mu the whole square. Now, you can look.

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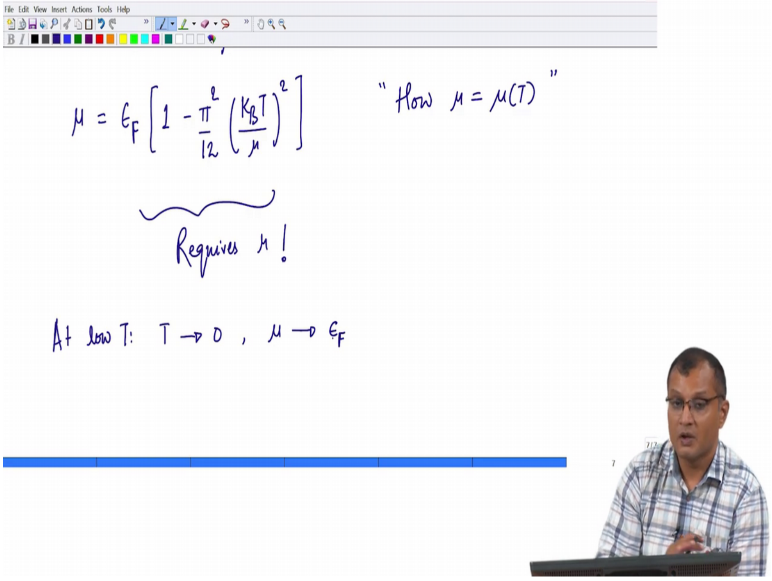

$$\mu = \epsilon_F \left[1 - \underbrace{\frac{\pi^2}{12} \left(\frac{K_B T}{\mu} \right)^2}_{\text{Requires } \mu \text{ !}} \right]$$

"How $\mu = \mu(T)$"

At low T: $T \rightarrow 0$, $\mu \rightarrow \epsilon_F$

Now, this is how basically, this tells us how mu varies with temperature ok. So, I will say that this equation tells me, how mu varies with temperature. So, for that too tell me for that too basically be valid, you can see that my equation requires the left hand side, in the right hand side to be computed ok.

So, this requires basically. So, how do I circumvent this problem, I am going to find out mu but, that requires me to compute mu for the right hand side. So, we can at least in the sense that, when mu is very close to the Fermi energy, then to lowest order we can tell how mu will vary.

So, at very low temperatures, again the idea is that my temperature is very close to 0, tending to 0, I can say that my mu is nothing but the, Fermi energy only on the right hand side ok.

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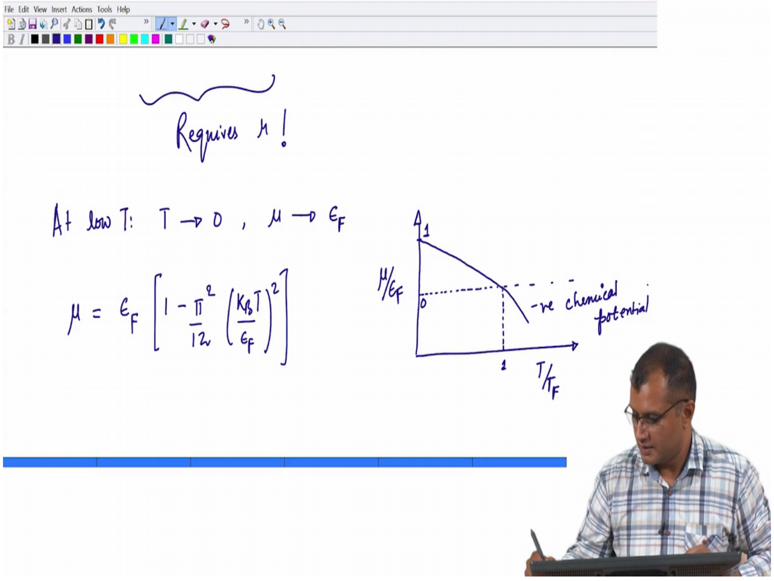

Requires μ !

At low T: $T \to 0$, $\mu \to \epsilon_F$

$$\mu = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{K_B T}{\epsilon_F} \right)^2 \right]$$

So, then I can write down my chemical potential as a, mu that goes as E F minus pi square by 12, into k B T over E F, this is like saying that if I was it at 0 temperature then, how would the right hand side behave, the right hand side would behave like that, as T goes to 0, the right hand side would behave like this. It is not exactly 0, because if exactly 0 is it when, mu becomes T F. So, this is how typically one expects the chemical potential to behave with respect to temperature. So, if I plot the chemical potential over the value at T equal to 0, as a function of T over T F.

Because, I can take my k B T over E F as, T over T F then, I can start from 1 because, that is what it equation would predict at T equal to 0 and eventually somewhere along the lines where, this is of the order of 1, you would be hitting the 0 of the chemical potential and from that point onwards, your chemical potential will turn negative. So, this is the area of negative chemical potential.

But, you already know that this is a temperature which is very high, only of temperatures comparable to Fermi temperatures your chemical potential turns negative. So, you need to really climb up in temperatures that much for the chemical potential to turn negative, another quantity that we can quickly compute, is the total energy.

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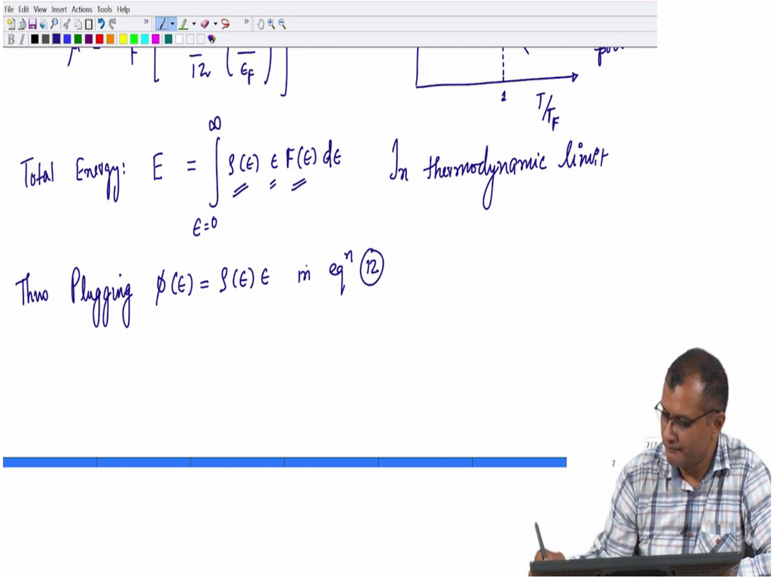

T/T_F

Total Energy: $E = \int_{\epsilon=0}^{\infty} \rho(\epsilon)\, \epsilon\, F(\epsilon)\, d\epsilon$ In thermodynamic limit

Thus Plugging $\phi(\epsilon) = \rho(\epsilon)\,\epsilon$ in eq^n ⑫

And that is the quantity that will tell us, how heat capacity behaves at low temperature, due to these Fermi electrons ok. So, the missing component of the overall heat capacity that is due to Fermi electrons will now be addressed. So, we are not the final leg of our discussion. So, the energy as we know is given by an integral, 0 to infinity, of E times the density of states, into the Fermi function, this is what we know in the continuum limit ok, in the thermodynamic limit.

That we already know, our sum has become an integral, the density of states is rho E and this is our Fermi function and that is the value of energy ok. So, the total energy is nothing but, this integral. Now, we already know an integral of this sort. So, here I have a formula. So, if you look at equation 12, in the left hand side, if I replace phi of E as E times the density of states then, it is nothing but, the expression for total energy, in the right hand side I simply plug phi as rho times E.

So, what I am going to do is basically plug phi as rho into E ok. So, I am going to. So, just plug phi as rho E into E in equation I believe 12. So, what I get is the left hand side is the energy.

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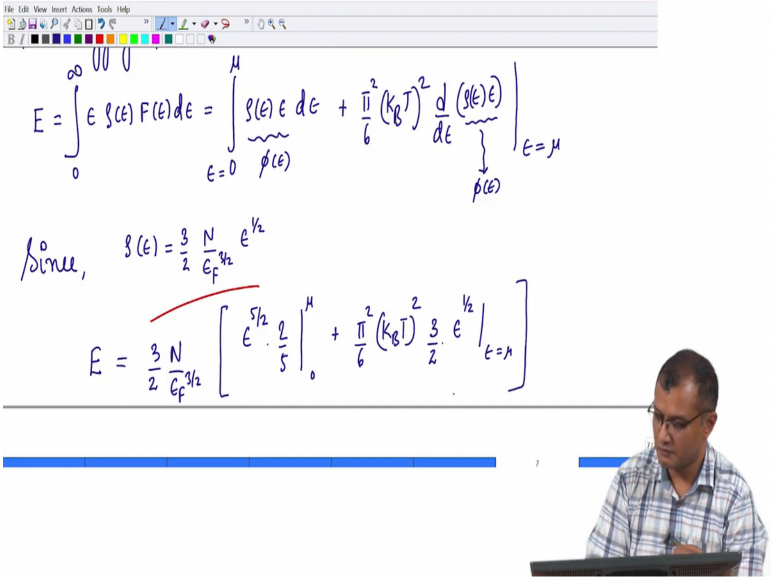

$$E = \int_0^\infty \epsilon\, \rho(\epsilon) F(\epsilon) d\epsilon = \int_{\epsilon=0}^{\mu} \underbrace{\rho(\epsilon)\epsilon}_{\phi(\epsilon)} d\epsilon + \frac{\pi^2}{6}(k_B T)^2 \frac{d}{d\epsilon}\underbrace{(\rho(\epsilon)\epsilon)}_{\phi(\epsilon)}\Big|_{\epsilon=\mu}$$

Since, $\rho(\epsilon) = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}}\epsilon^{1/2}$

$$E = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}}\left[\epsilon^{5/2}\cdot\frac{2}{5}\Big|_0^{\mu} + \frac{\pi^2}{6}(k_B T)^2\frac{3}{2}\cdot\epsilon^{1/2}\Big|_{\epsilon=\mu}\right]$$

So, this would be 0 to infinity because, we are at finite temperature all energies are possible now, energy times the density of states into the Fermi function and at the right hand side was nothing but, integral 0 to mu, rho E times E d E. So, this was our function phi of E ok, plus pi square by 6, if I am not wrong, pi square by 6 into k T square into d phi by d E, into d phi by d E now, my phi is again as I said, rho E into E, at E equals to mu.

So, this is nothing but, my phi of E but, I already know that my rho of E the density of states, which was written here, this expression ok. So, you can, I know the density of states is this. So, I can compute what is my the integral. So, it is very simple now because, I can write down this integral as keep all the constants outside. So, my entire energy E, what nothing would be nothing but, I can take a 3 by 2, into N over E F to the power 3 half and I believe the same will come out from the second term also.

So, I am going to keep it outside, a big bracket and this would be an integral of E raise to 3 by 2 because, there is already an E outside. So, E times rho would be E raise to 3 by 2. So, that will be nothing but, E to the power of 5 by 2, into 2 by 5 and the limits would be 0 to mu, plus pi square by 6, into k B T the whole square and again I have taken all the constants outside. So, I will be taking derivative of varies to 3 by 2. So, that would be 3 by, 2 into e raise to half, at equals to mu fine.

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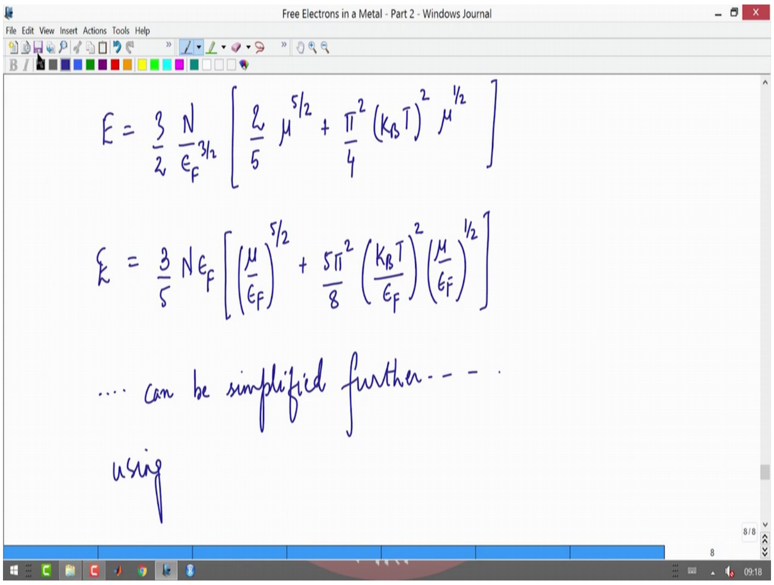

$$E = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}}\left[\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}(k_BT)^2\mu^{1/2}\right]$$

$$E = \frac{3}{5}N\epsilon_F\left[\left(\frac{\mu}{\epsilon_F}\right)^{5/2} + \frac{5\pi^2}{8}\left(\frac{k_BT}{\epsilon_F}\right)^2\left(\frac{\mu}{\epsilon_F}\right)^{1/2}\right]$$

.... can be simplified further.... .

using

So, now you can see that, this becomes a nice expression. So, I can write it as 2 by 5, mu to the power of 5 by 2, plus pi square by 4 k B T the whole square, into mu to the power half right. So, what you are now looking at, is an expression of the internal energy of the Fermi gas, as a function of temperature. So, one final hurdle, before we derive the expression for heat capacity, is to simplify this expression. So, what I am going to do now, is just rearrange the terms a little bit here.

So, what I will do is I will pull out a factor of 2 by 5 also, from the bracket and that will make the pre factor outside as 3 by 5, into N to the power. So, please cut it tell me.

Student: (Refer Time: 71:25).

All right. So, what we can do here with this expression is to rearrange the terms a little bit here and I can pull out a factor of 2 by 5, that will make it 3 by 5 inside, I am sorry outside and this will be E in the denominator, E to the power of 3 by 2, Fermi energy to the power 3 by 2 and what I will have inside is mu to the power 5 by 2, plus 5 pi square by 8 ok.

So, this is multiplied to k T square, into mu raise to half. Now, what I can do now is just take mu, you know I not dimensionalized this mu, by simply saying that I am going to keep a factor of Fermi energy outside and what I will do is a take a Fermi energy inside to the power 3 half.

So, that becomes mu over E F to the power 5 by 2 and I will do the second term, as a I just want to start from the beginning, I am not getting the right words, not getting the right words.

Student: (Refer Time: 73:35) right.

So, what I can do now is basically simplify this expression, by pulling out a few factors. So, I am going to pull out a factor of 2 by 5, that makes my pre factor outside as 3 by 5, into I am going to write down this as N times the Fermi energy and I will write the first term inside the bracket as mu over the Fermi energy, to the power 5 halfs and the second term inside the bracket I am going to write it as a, 5 pi square by 8 ok.

Because, I pulled out a factor of 2 by 5 ok, into k T over the Fermi energy to the power 2 and the last term I am going to write as mu over a Fermi energy, to the power 1 half ok. So, I can see the terms inside the parentheses have become dimensionless, each energy scale is divided by another energy scale, what has happened is that, outside the parentheses now, the energy scale has come, that is free energy that is the Fermi energy, which is which sets the scale for my total energy ok.

So, then I can now utilize the I can say that this is my expression for energy, this can be simplified further, using my already derived expression for the chemical potential.

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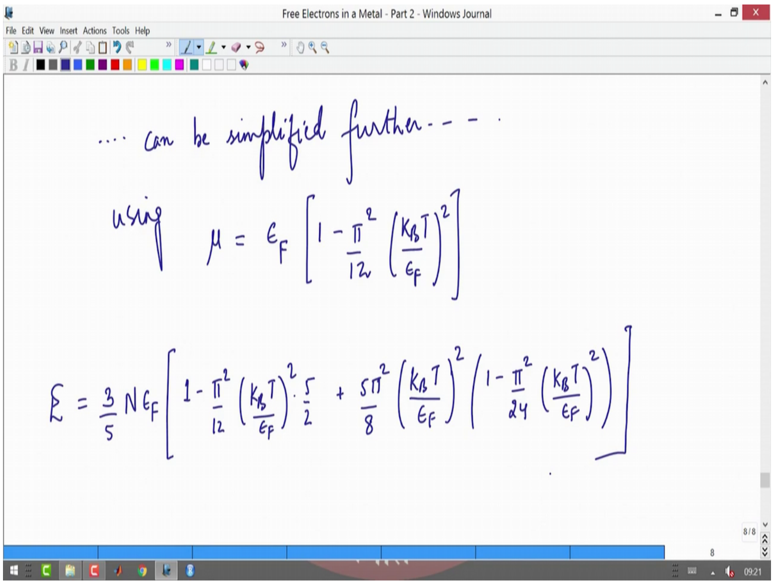

.... can be simplified further.... .

using $\mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2\right]$

$$\mathcal{E} = \frac{3}{5}N\epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2\cdot\frac{5}{2} + \frac{5\pi^2}{8}\left(\frac{K_BT}{\epsilon_F}\right)^2\left(1 - \frac{\pi^2}{24}\left(\frac{K_BT}{\epsilon_F}\right)^2\right)\right]$$

So, I am going to use this definition of the chemical potential ok. So, this nice definition of the chemical potential, comes to our use. So, you can see that, I can write mu over E F as 1 minus pi square by 12 into k T upon F, E F raise to 2 ok. So, I can write down my energy, as E equals to 3 by 5, into N times the Fermi energy, this was the pre factor outside, mu over E F to the power phi half is just 1 minus pi square by 12, into k T over Fermi energy to the power 2, raise to 5 half.

Now, in the limit of small temperatures, when k T or E F is very small, I can expand this power to just a linear orders and say that this is nothing but, 5 by 2, into k T square over E F square and the second term is of 5 pi square by 8, if you look here, into k T square over E F square and then it is multiplied to mu over E F the power 2. Which again I can just to linear order, write as 1 minus pi square by 24, into k T over E F the whole square ok. So, what I have done here, we have made some simple approximations.

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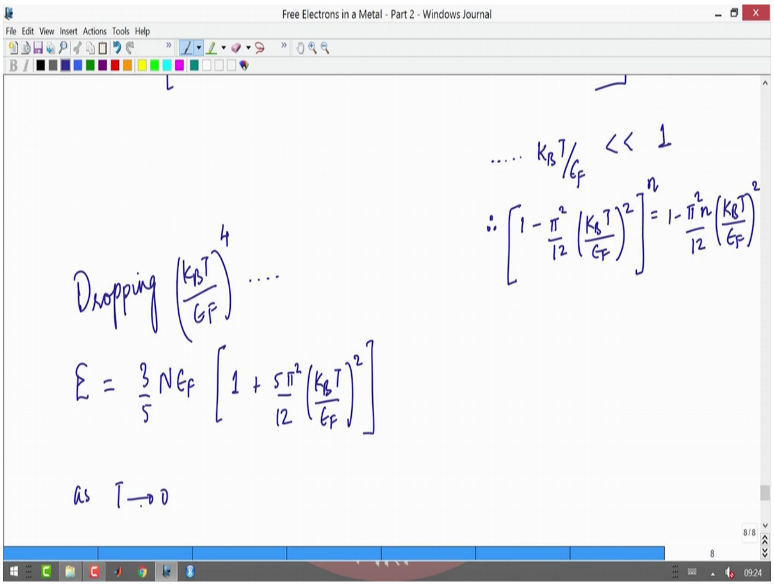

$$\ldots\ldots \; K_BT/\epsilon_F \ll 1$$

$$\therefore \left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2\right]^n = 1 - \frac{\pi^2 n}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2$$

Dropping $\left(\frac{K_BT}{\epsilon_F}\right)^4 \ldots\ldots$

$$E = \frac{3}{5} N\epsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2\right]$$

as $T \to 0$

So, whatever we have done here is basically, we have used the fact that k T over E F is a number that is much much smaller than 1 and 2 linear order, the fractional power of 1 minus pi square by 12 into k T by E F square simply, reduces to the approximations that we have taken. So, I am going to write down just 1 of them that, 1 minus pi square by 12 into k T over E F, the whole square raise to the power n is simply, 1 minus pi square into n over 12, into k T by E F to the power 2, that is the approximation I had used ok.

So, let us see what finally I have now, easily you can see since, were in the approximation k T over F much smaller than 1, I am going to drop a terms that are higher than quadratic in nature. So, dropping definitely the fourth order term ok, I am going to drop this term, in comparison to the quadratic term, my expression reduces further and I get the expression for the energy as 3 by 5, N times E F, into just the 2 terms which are quadratic in temperature and you can see it is basically 5 by 8 minus 5 by 24, which I have to do. So, that would be just 1 plus 5, I am going to write it as, 5 pi square by 8, into k B T over E F, a whole square into 1 minus 1 by 3 ok.

And this is nothing but, I can write it as a 2 by 3 and so, this is nothing but, 5 by 12, just a simplifications, that I have used here. So, now you can see that the leading term, as a T goes to 0, that is an absolute 0 temperature.

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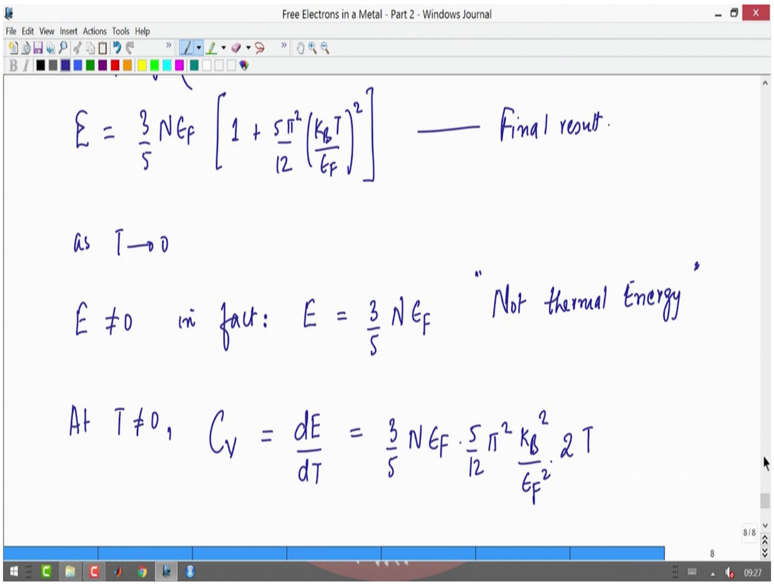

$$E = \frac{3}{5} N \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$ — Final result.

as $T \to 0$

$E \neq 0$ in fact: $E = \frac{3}{5} N \epsilon_F$ "Not thermal Energy"

At $T \neq 0$, $C_V = \frac{dE}{dT} = \frac{3}{5} N \epsilon_F \cdot \frac{5}{12} \pi^2 \frac{k_B^2}{\epsilon_F^2} \cdot 2T$

The energy of my system is not 0. In fact, it is given as, the energy of my Fermi electron gas at absolute 0 is not 0, it is given as 3 over 5, N times E F ok. But, this is not a thermal energy, this energy is not thermal in the sense that, at T equal to 0, if I touch a metal, which has these electrons having a total internal energy of 3 by 5 N times E F and the E F is of the order of 1 electron volt, its a very high amount of energy, I still you know feel as cold as you know, as cold as hell because, there is absolutely no room for electrons to conduct, at T equal to 0, the electrons have occupied all the energy levels up to the Fermi level.

And hence in spite of the fact that this energy is very large, these electrons cannot conduct heat to my palm or my hand, which is touching them and so, I feel nothing but, very cold as T tends to 0, this energy is not thermal in nature, it cannot give rise to any conduction and this energy purely is the our consequence of the Pauli’s exclusion principle, that drives electrons to sit at different energy levels in a quantum system ok.

So, at temperatures that are nonzero, I can write down my energy with this expression, let us call this as the final result and I know that the heat capacity, which is a C v a heat capacity at constant volume, is given as the derivative temperature derivative of the total energy.

So, if I use the above expression, that I have derived for energy then, the first term the Fermi energy component provides no component and that is it clear why it provides no component to the heat capacity because, it is not thermal in origin. The second component provides contribution and that is 3 by 5 N E F, constant pre factor into 5 by 12 pi square, k B square over E F square and 2 times T. And you can already knock off a lot of terms here.

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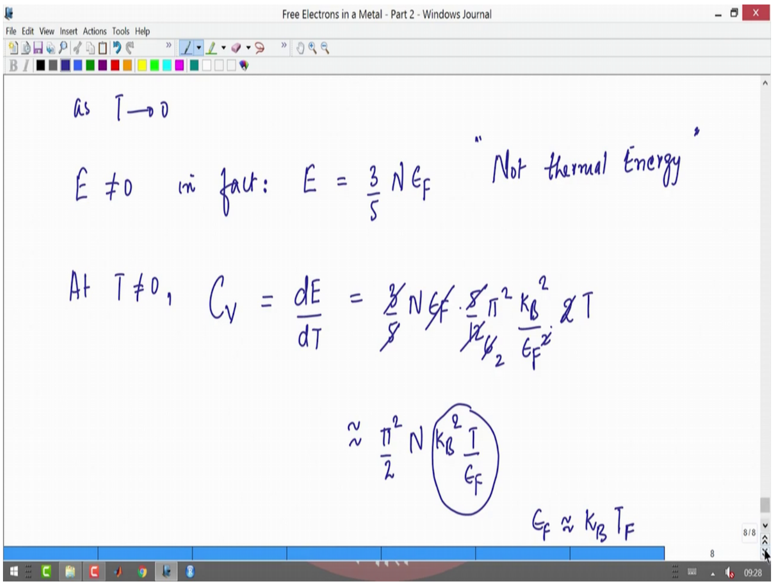

as $T \to 0$

$E \neq 0$ in fact: $E = \frac{3}{5} N \epsilon_F$ "Not thermal Energy"

$$\text{At } T \neq 0, \quad C_V = \frac{dE}{dT} = \frac{\cancel{3}}{\cancel{5}} N \cancel{\epsilon_F} \cdot \frac{\cancel{5}\pi^2}{\cancel{12}\,\cancel{6}\,2} \frac{k_B^2}{\epsilon_F^{\cancel{2}}} \cancel{2} T$$

$$\approx \frac{\pi^2}{2} N \boxed{k_B^2 \frac{T}{\epsilon_F}}$$

$\epsilon_F \approx k_B T_F$

So, you can knock off this and you can knock off 1 E F and this becomes 6 and that becomes just 2. So, what you have in the end is just, pi square by 2, into N times k B square, T over E F ok. It should have the units of N times k B, heat capacity is a natural units because, k T is energy scale. Now, I can do something with this, I can say that my

Fermi energy is you know, in order to express in terms of a temperature scale, I am going to write down this as k B T F.

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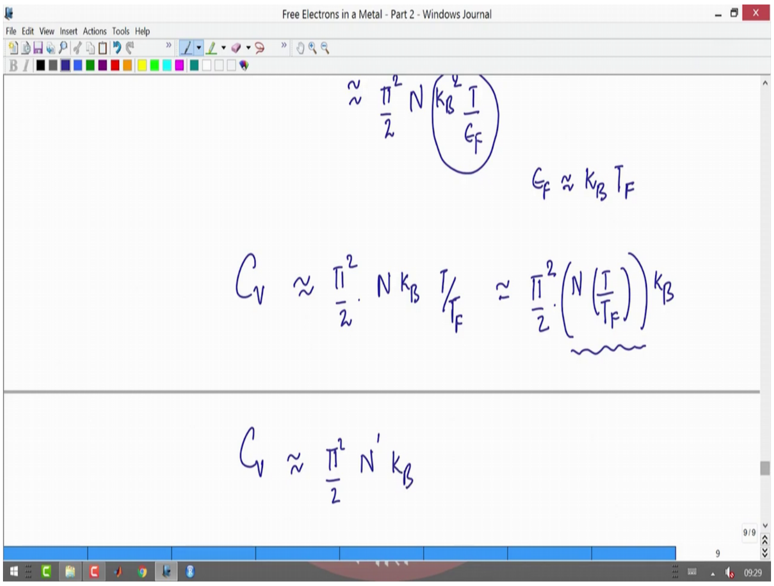

$$\approx \frac{\pi^2}{2} N \left(\frac{k_B^2 T}{\epsilon_F}\right)$$

$$\epsilon_F \approx k_B T_F$$

$$C_v \approx \frac{\pi^2}{2} \cdot N k_B \, T/T_F \approx \frac{\pi^2}{2} \cdot \Big(\underbrace{N \left(\frac{T}{T_F}\right)}\Big) k_B$$

$$C_v \approx \frac{\pi^2}{2} N' k_B$$

So, my heat capacity is now, due to thermal energy is pi square by 2, into N times k B into T over T F and I am going to rearrange this basically in some way. I am going to write it as pi square by 2, into N into T by T F, the entire thing times T entire thing, times k B ok. So, this factor here, is an extensive factor. So, I can call my C v now as, pi square by 2, into some N prime, into k B.

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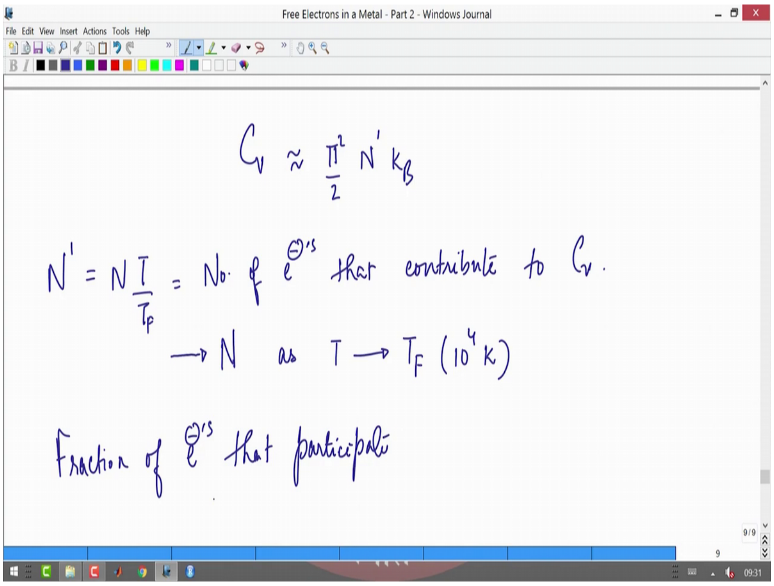

$$C_v \approx \frac{\pi^2}{2} N' k_B$$

$N' = N \frac{T}{T_F}$ = No. of e^{-}'s that contribute to C_v.

$\longrightarrow N$ as $T \longrightarrow T_F \; (10^4 K)$

Fraction of e^{-}'s that participate

And I am going to call this N prime here, as the number of electrons, that contribute to heat capacity ok. Because, heat capacity is an extensive quantity and this number smoothly tends to the total number of particles, as the temperature becomes much higher and becomes the Fermi temperature, if you see here the N prime here, is N into T by T F ok.

So, I am going to write this here just to remind all of you, that this N prime is N into, T by T F. So, the number of electrons that contribute to the heat capacity and that tends to N as T goes to T F, which means at very very high temperatures of the order of I know that Fermi temperature is of the order of 10 to the power 4 Kelvin for most metals, at such high temperatures, it is expected at all the electrons in the system will be contributing to the heat capacity because, all of them will be thermal, you do not have a quantum picture at that you know, at that temperature, you are in your strongly in the classical limit.

So, this then also tells us, what is the fraction of electrons close to 0 Kelvin, that participate in the heat capacity well, you can now say that the fraction of electrons that, participate in conduction and hence heat capacity in conduction and hence heat capacity.

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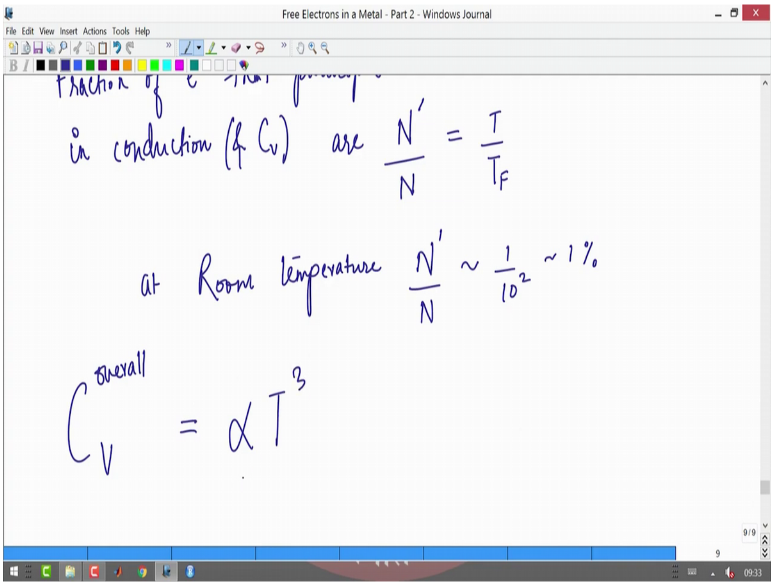

fraction of e that

in conduction (& C_v) are $\frac{N'}{N} = \frac{T}{T_F}$

at Room temperature $\frac{N'}{N} \sim \frac{1}{10^2} \sim 1\%$

$$C_V^{\text{overall}} = \alpha T^3$$

If they do not conduct, they do not contribute to heat capacity and heat capacity are basically. You know, you can you have already written that these are the number of guys participating in conduction and I am asking you, what is the fraction of electrons that participate in conduction or heat capacity, a fraction is basically the guys who are participating over the total number of guys and this is nothing but, T over T F.

And at room temperature you can think of this number, you can now see that this is a very very small you know fraction. So, this fraction of electrons that, participate in heat capacity ok, at room temperature this fraction is usually, you know 1 upon of the type of the order 1 upon 100 because, your room temperature is 300 Kelvin. So, it is like of the order 100 whereas, the Fermi temperature is of the order of 10000. So, I am going to say that this is a fraction of the order of 1 percent ok.

So, this is like 1 percent, 0.01 is like 1 percent, 100 times this would be 100 you know 1. So, this is the you know, this now brings us to the final picture that we were chasing from the beginning, that I am now in a position to write down the overall heat capacity of a solid at low temperatures. So, I know that there is a term which comes from the lattice vibrations and that is proportional to T cube.

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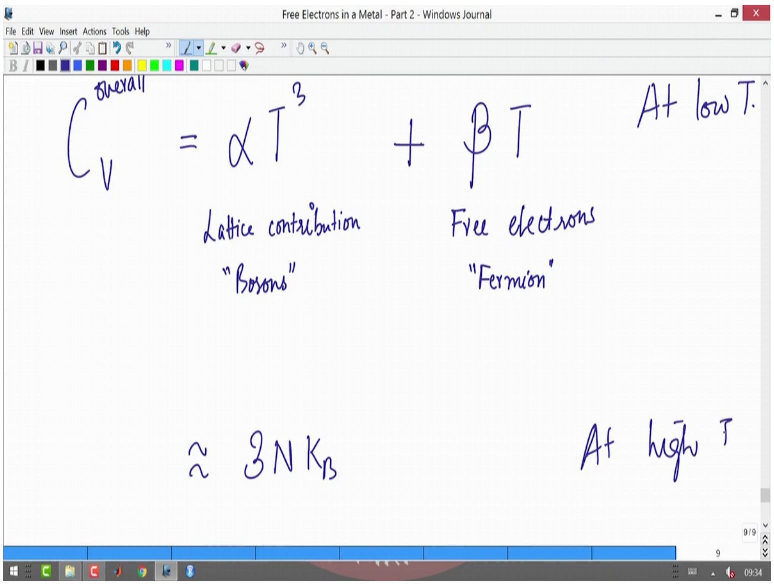

$$C_V^{\text{overall}} = \alpha T^3 + \beta T$$

At low T.

Lattice contribution "Bosons"

Free electrons "Fermion"

$$\approx 3 N K_B$$

At high T

So, this was the Debye contribution. So, I will say, the lattice contribution, vibration of ions, that is meant by lattice contribution and this is basically I would say the contribution coming from bosons, the vibrations are quantized in a terms of phonons, that are bosons and we have today seen a contribution coming from, the free electrons and that is linear in temperature ok.

So, this is due to the free electrons and these are fermions. So, both bosons and fermions together constitute the heat capacity at low temperatures and you can see that very very low temperatures the heat capacity would be dominated by only free electrons because, the T cubed term would get much much smaller.

So, it becomes linear at very low temperatures then, become dominated by T cube, eventuate high temperatures none of these contribute, it becomes the Dulong-Petit 3 N k at high temperature. So, this is given as a thrice N K, at high temperatures well, I am going to assume that high temperature would mean this T becomes a much much larger than the Fermi temperature or even before that. So, this ends the discussion of the 3 electrons in a metal and we will discuss the correspondence from correspondence between the classical statistical mechanism.