Non Equilibrium Statistical Mechanics
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Lecture 35- Critical
Phenomena (Part 7)

So we have looked at the icing universality class of systems for which we wrote down a Landua energy functional and from which we got the equilibrium configuration. Now I would like to do that in slightly more general context and show what happens when you introduce fluctuations on the one hand and the idea of inhomogeneity is on the other and there will be relaxation to the equilibrium state.

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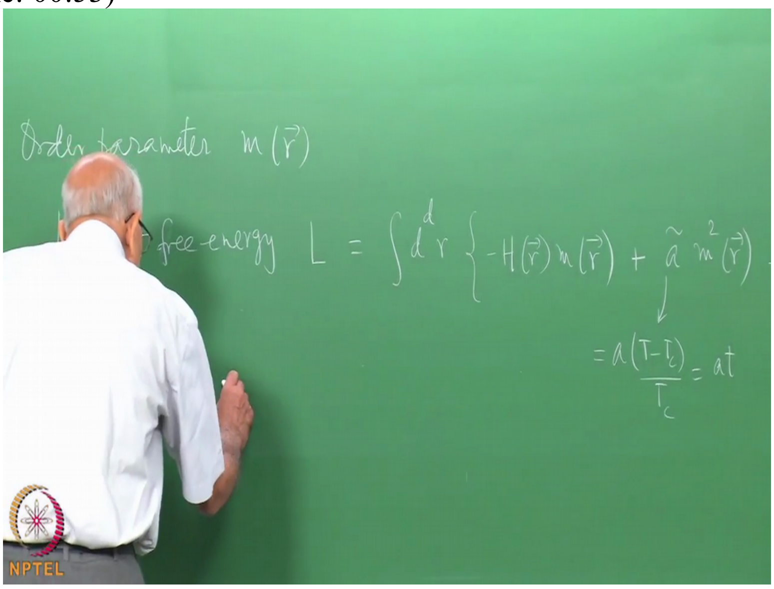

So to recall to you we begin right away with the order parameter m of r the magnetization or whatever the order parameter be and if (I) if you recall the Landau free energy is not exactly the Hemoltz or the Gibb's free energy.

We saw that we constructed a functional in such a way that you got the correct equilibrium value above and below the critical temperature. This free energy L let me just call it L this is equal to an integral the in D dimensions of R and then the terms inside, first of all the absence of an external field there is no linear term, but if you put in an external field there is offcourse a term proportional to that. So let's put in a field H or r so this is minus H of r, M of r there is such a term and the next term is a quadratic term but if you recall it had a coefficient which was proportional to T minus T c.

So that it would give you the right critical behavior so that term let me call it A tilda m squared of r and this A tilda is a positive coefficient A times T minus T c over T c, just A times T, little t was this difference between this T minus T reduced temperature T minus T c over T c and in the magnetic case there was no cubic term but only the fourth order term and the temperature dependence of that term the coefficient was irrelevant.

It just had to be positive so that you would have stability about the equilibrium, stable equilibrium. So this was B times m to the power 4 of r and then to include inhomogeneities cause just by the fact that you have an H of R but in general. You have a term which include gives you the gradient energy when you have inhomogeneities in the magnetization and this was of the form one half that half is just for convenience, some coefficient times gradient of m of r Mod squared ok. So that was the Landua free energy.

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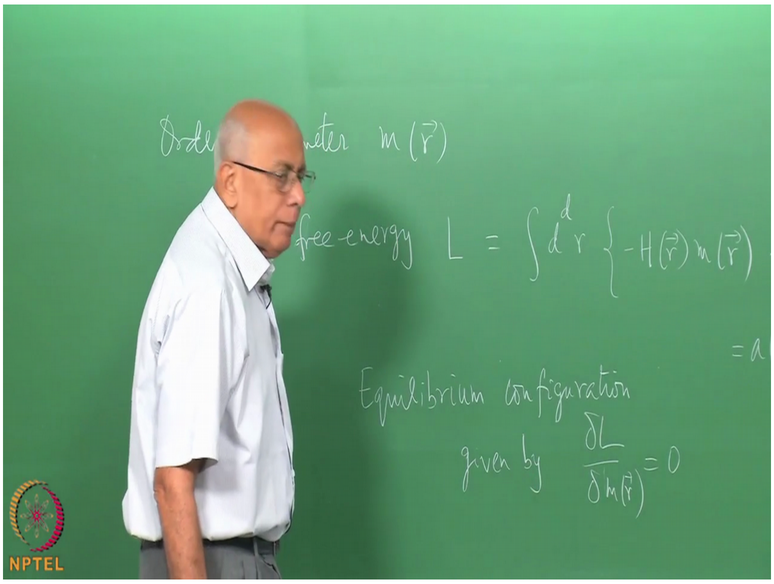

The equilibrium solution is found by minimizing these free energies, so equilibrium configuration given by delta L over delta m of r equal to zero, and we also have to ensure that it is a minimum rather than the maximum but the structure of this makes it clear that it will be a minimum.

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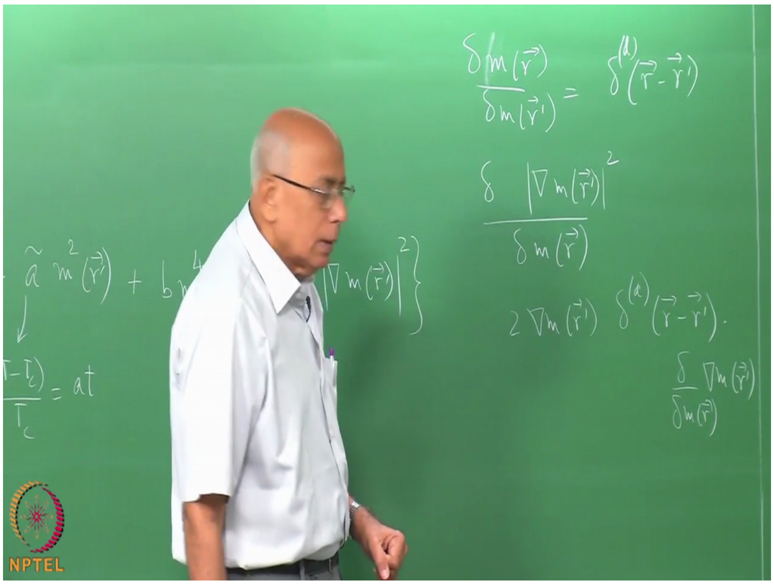

So what is that equal to when we use the rule for functional differentiation make this R prime everywhere and use the fact that the functional derivative of m of r with respect to delta m of r time, this is equal to the delta function D dimensional of r minus r prime, that is the basic rule.

Then offcourse the only non-trivial term is this and I argued that the function derivative of gradient m of r mod squared r prime say divided by delta m of r that is what you going to have to differentiate out here. So if you make all these guys R primes, yeah I need to do the integral as well but let me, look at what is going to happen here, this is going to be twice so it is going to give you a twice gradient of m of r prime times derivative D or whatever it is of r minus r prime times the functional derivative of the gradient itself.

But the functional derivative because it is got to be a vector right, so this is times dot-dot it to delta over delta m of r gradient m of r prime ok and this you do integration by parts yeah.

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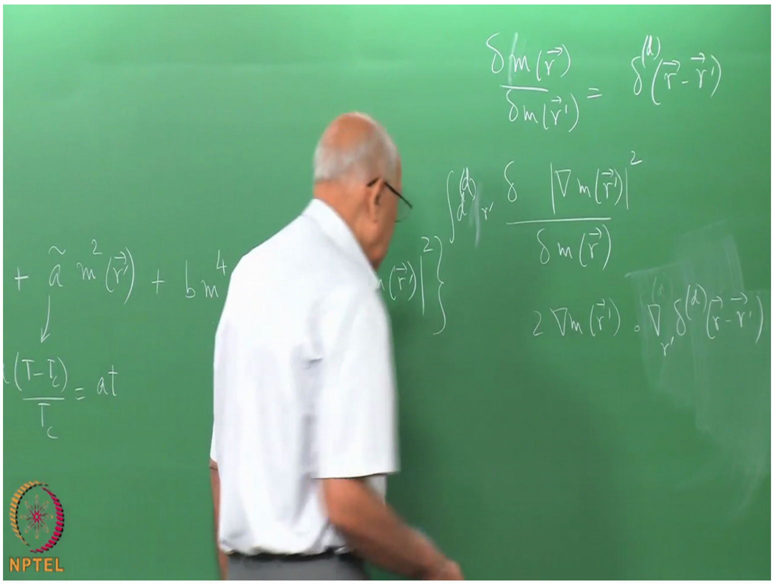

Only one? Oh yes so you are right, gradient of the delta function, thank you, yeah. I used this, gradient with respect to r prime, yeah so this is r prime. But what I really have is in integral over d d r prime so really you should insert this d r prime on both sides and when you do that and you integrate by parts this delta function is going to five and this gradient is going to operate on this factor here. So you got a delta or del and then you have del squared.

This already half so just gives you minus del square ok. So let's cut that short and write this whole rule as delta over, so once you put that in we can write the solution down. So let's just do that.

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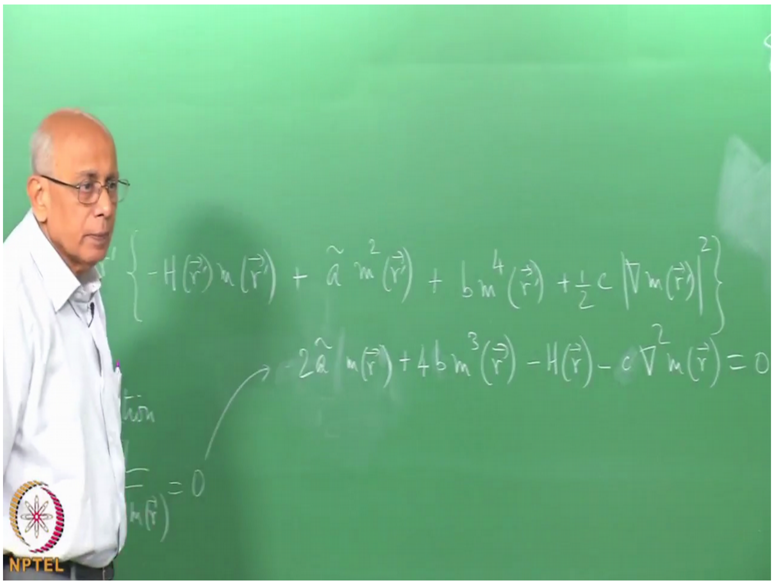

So this configuration is given by minus I want to write this properly. So 2 A tilda m of r prime plus 4 b m cubed of r sorry the r everything is r, r prime is gone minus h of r minus c del square m of r equals 0, that is the equilibrium configuration.

So it is a solution to a partial differential equation in space like the ordinary equation. But there is a non-linearity here, there is an inhomogeneous term which is not the series, but if you add just this term would be nice be linear but unfortunately there is this non-linearity ok which you can't get away from. So the equation the thing from the start it is obviously a non-equilibrium situation, a non-linear situation ok alright.

Now let suppose we have solved this in principle and now I ask the following question, if there is a local fluctuation say due to thermal noise from this equilibrium configuration, how does this system comeback to it. So this is now in the spirit of our old in-friend linear response theory and we like to find out what is way that relaxation is going to proceed? Ok and now this no rigorous way of doing this except in a specific model but you guess the following exactly as we did in the very Langevin equation is, you guess the following.

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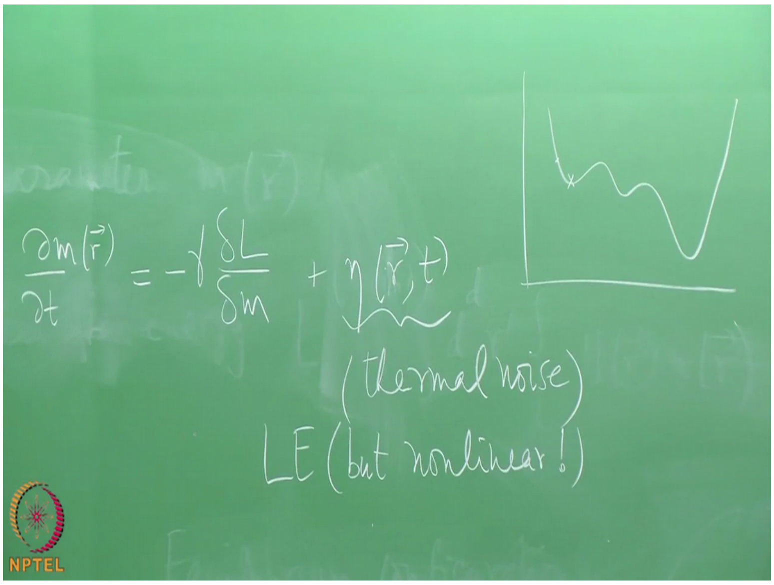

It's a well a good assumption good be to say ok for a given configuration delta L over delta M I am not going to write all the arguments, this quantity is zero in equilibrium and away from equilibrium for a non-equilibrium configuration this measures the deviation from the equilibrium. So a good way to find out what the relaxation is like a good guess, is to say delta over delta T, a configuration m of r relaxes to equilibrium by minus this times a constant times this deviation from the equilibrium. So that is the relaxation equation, a typical relaxation equation.

Exactly in the same spirit as we saw relaxation occurs in the Langevin equation or in the Boltzmann equation etc. But there is no guarantee that you give a configuration it doesn't relaxed to local minimum. So in general once you give me an equilibrium configuration of this solved by this there are local maxima and minima and there is (())(11:42) a global minimum which is a thermodynamic equilibrium state. But if you give me a model of this kind with some initial condition in configuration there is no guarantee that it doesn't tend to a local minimum and stick there.

That doesn't happen in actual practice because if you plot this m as a function of configuration variable it might have this kind of behavior and you, if you start here you might relax some play to this point whereas you really have a thermodynamic equilibrium state over there. So it is clear them as we some fluctuation which take you out of these local wells and put you into the global minimum. Therefore to this you must add a noise term which mimics the effect of fluctuations. But this noise must itself be inhomogeneous because the function of r because you have got an order parameter which is a function of r now.

So it is a field plus eta of r and t. So this is thermal noise and this is exactly the structure of a non-linear Langevin equation ok. So its solution is formidable because you have a stochastic differential equation for a field and it is gotten non-linearities. It is space dependent so it is a partial differential equation, space and time derivatives are on top of it you have non-linearities. So its formidable and the only way you can make any headway with it is by functional integration methods.

But we need to specify what sort of noise? So we make the simplest assumption that it is uncorrelated Gaussian noise.

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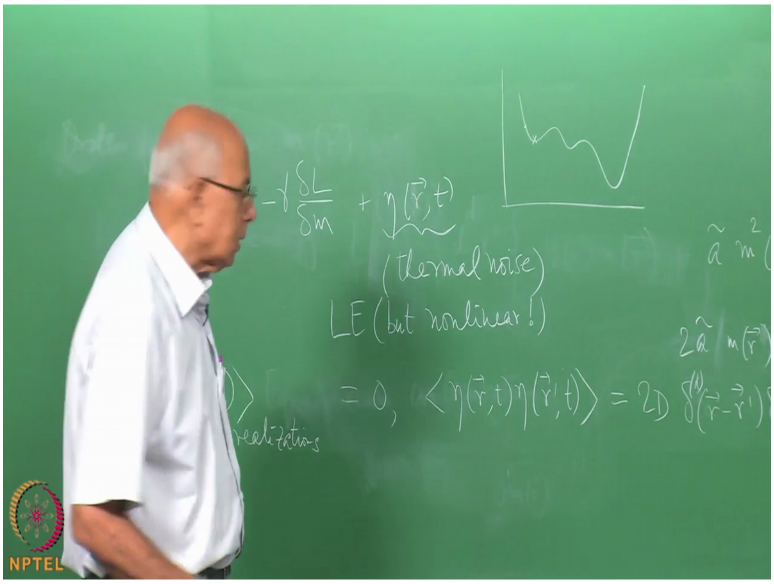

So will start with by making this assumption that eta of r and t over all realizations of this noise, over all realization will say that its got assumed that it is got zero average and moreover here is a crucial assumption eta of r t eta of r time t prime over all realizations this is proportional to delta functions. So we assume that the noise is uncorrelated at different time points and at different times. So this is equal to some constant 2 D delta of r minus r prime delta of T minus T in D dimensions offcourse ok.

This is not so obvious and I am slurring over certain technicalities here for instance if you have what is called a non-conserved parameter then this is no longer true and consistency demands that we should Del squared over here acting on the delta function. But in the magnetization case that doesn't apply, so this is ok as it stands. Then we need to specify the probability distribution of this eta, so what would one do?

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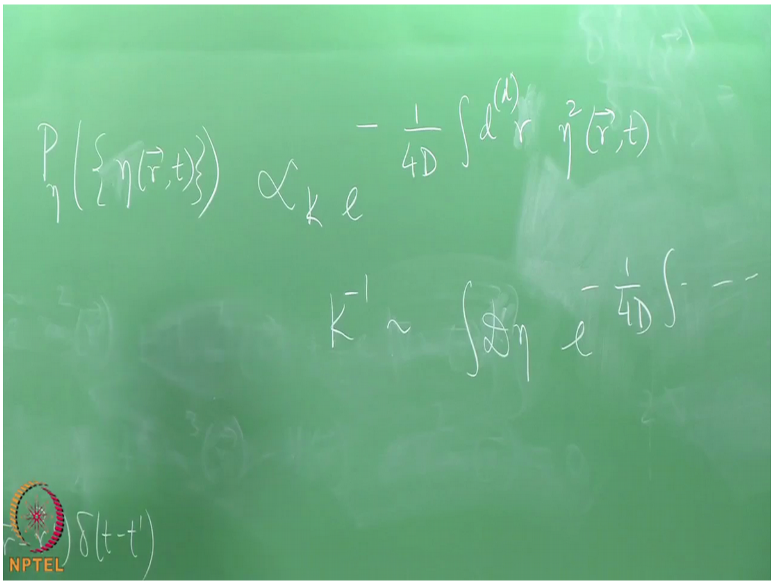

You would assume that the distribution of eta so let me call it P sub-eta the probable yeah

Student: (())(15:43)

This is not quote -unquote dissipate about a parameter, it is magnetization unlike for example, concentration, that relaxes to an equilibrium configuration but there is no conservation of total magnetic movement or any such thing ok. So there are two classes of problems and if time permits I will mention the other class but in this case this is a consistent (tenure) ok. So you have P of eta of some configuration m of (r) of eta of r comma t this probability is proportional to e to the power minus some variants in because you got a 2 D here you need to have a 4 D out here integral D d of r eta squared of r and t.

So that is the probability distribution of it is a Gaussian probability distribution generalized to a field. This constant of proportionality whatever is out here is the normalization, it will be a functional integral over all eta's, so this there (should) be some constant and K inverse goes like integral well D eta e to the minus whatever 1over 4 D integral etc ok. So I am not going to get into we will not try to normalize that, we just need the assumption that it is a Gaussian ok.

Which means you are in principle know all the joint probability distribution as well. Once you make this assumption that this delta correlated it is like noise, this is the sort of special space dependent extension of white noise that we did in the Langevin equation ok and we had now talking about the probability distribution of whole configurations of eta at every point r for a given for each given T.

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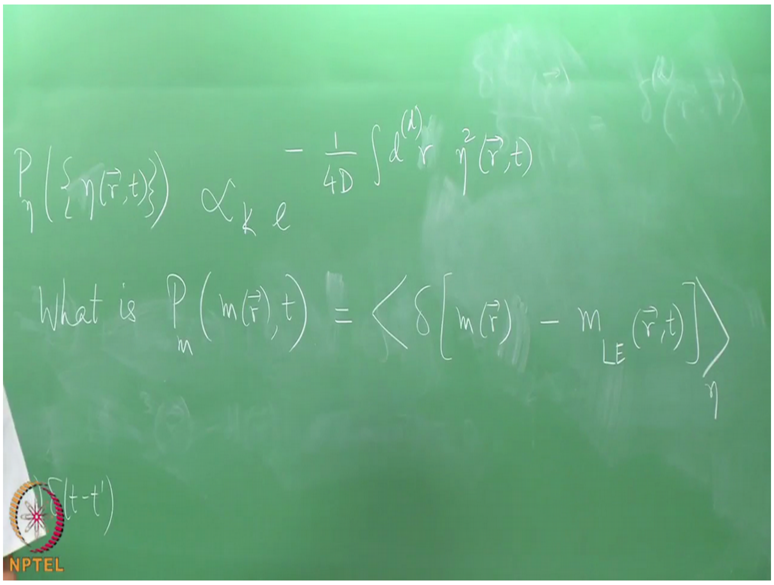

Then not surprisingly one can actually write down a corresponding Fokker-Planck equation because we need to know what is the probability distribution of m, what is the probability distribution of the configuration m of r at any time T, given this. So it is the old question exactly that in the same as what we solved in the case of the ordinary Langevin equation. Given a Langevin equation this statistics of the eta the fact that it is Gaussian etc, how did you get the Fokker-Planck equation from it?

Now I just made the statement that we have a in the original in the other case that we have a Fokker-Planck equation we look at its equilibrium and so on. But one can make this a little more rigorous. One can make this one can do that fairly easily as follows and we want to do this in this functional case. So lets go about it in the following way. This quantity must obviously be equal to by definition the expectation value of a delta function generalize delta function of m of r T minus m of r for a given configuration minus the M that you get from the Langevin equation.

So let me write it as m L E by solving the Langevin equation for each value of the noise each realization of the noise and then averaging over all noise in realizations. So if I call that m L E this is the function of r and t over eta. In other words the average is of this delta function (over) weighted with P sub-eta. That is by definition the probability distribution of this configuration in m ok.

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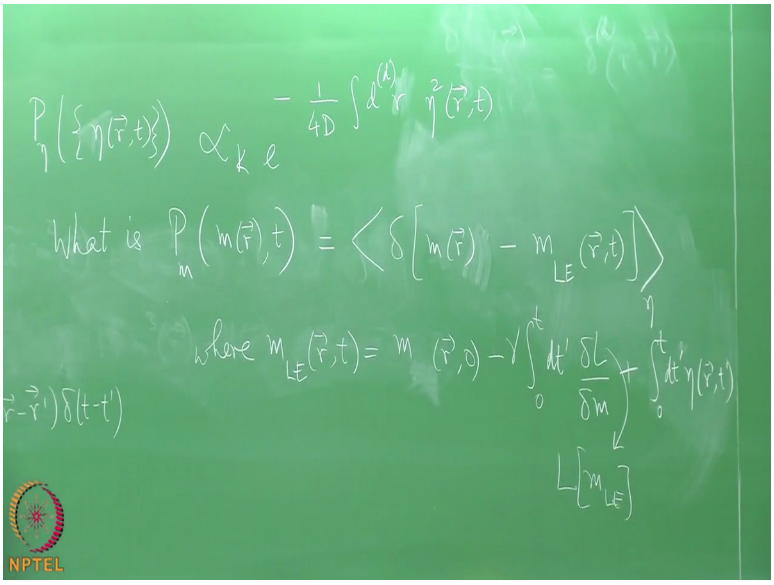

Where n L E of r and t equal to m L E of r and 0, you give me the initial configuration and then you have to solve this equation. So it is just minus gamma times integral 0 to T dt prime delta L

Student: (())(21:13)

This is what I give, there is no need for L E it is whatever initial configuration give minus delta L over delta n, I won't write the argument here plus integral 0 to T dt prime eta of r t prime ok. That is the formal solution offcourse this quantity here involves that derivative with respect to L so that is the formally double non-linear object but in, so it is not a solution, it is just a representation of this m L E and you have to put that so everywhere here in bracket the argument is m L E itself consistent way offcourse, so I should really mark the argument this is L of m L E ok.

And then I have to take this delta function multiplied by P of eta and integrate to overall realizations sum over all realizations right. So that is the way one won't do this and you could start by saying we don't write the solution down I find the derivative of this with respect to time. So I am going to head towards a Fokker-Planck equation and for that I need the derivative with respect to time that will give me a derivative of this quantity. So it is a theta function to start with and then this going to be derivative of this.

So all the T dependence is sitting here and for this quantity I go back and use the Langevin equation here. So that is way you derive the (Langevin) Fokker-Planck equation for a given Langevin equation. Even in the ordinary case the finite degrees of freedom case. So the sum and substance is that you end up with the following Fokker-Planck equation.

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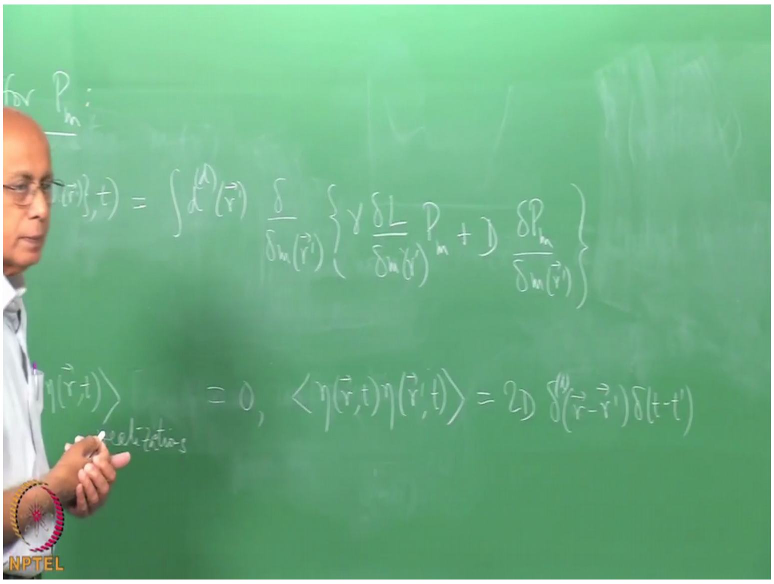

So here delta over delta t P m of configuration at anytime t, this quantity is equal to not surprisingly you are going to get gamma times integral d D of r prime times gamma times that is the drift term.

There was a minus sign in the Langevin equation and remember in the drift term you get another minus sign so that gives you a plus out here, plus and the way we normalize this with a 2 D here you have to do one over this guy. So this becomes plus D times delta P n over delta n ok. So there is a second derivative term because the noise in this case in not multiplicative it is caught pure delta functions, no r dependency here by assumption. So it is D times that which is what we expect plus this guy here. This is the drift term.

The only difference is it is not linear, this thing is completely crazy it is not linear equation at all. But we can write down what is the equilibrium value, what is going to be the equilibrium distribution. That is found by putting this equal to 0, which is equivalent to putting this equal to 0.

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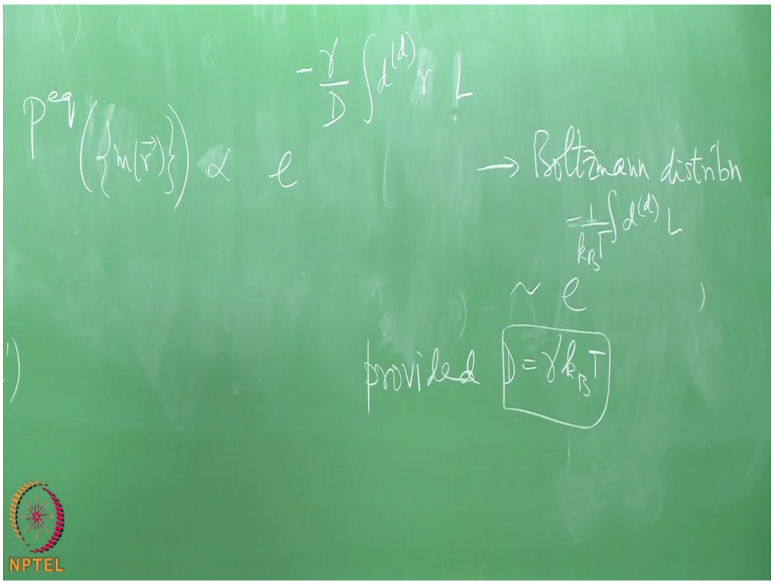

So the solution the equilibrium solution lets write it down, the equilibrium configuration it is not a function of time, this fellow is obviously apart from normalization constant its E till the minus gamma over D because essentially its equating this to zero (right).

So it is E to the minus gamma over D times integral d (d) r prime of r lets say the function of r here I am little confused here because at the notation, no it will be L because I wanted to check of this equation is valid or not right.

Student: (())(26:45)

It is just going to be L, it is just L, right because it is very differentiated I am going to get delta L over delta m here. But we would like it to be the Boltzmann distribution right, remember this the energy density and you are going to have to integrate to get the full Landua energy. So it is of the form E to the minus whatever the energy but we would like it to be the Boltzmann, so this goes to the Boltzmann distribution which goes like E to the minus integral d (d) L minus gone over K Boltzmann T provided D equal to gamma KT.

That is the fluctuation dissipation theorem right. that provides a consistency check ok. I am sorry for using the same symbol D that we used for diffusion in the possession space earlier. This is diffusion in the velocity space the analogue of that because it is the Fokker-Planck equation for the analogue of the velocity.

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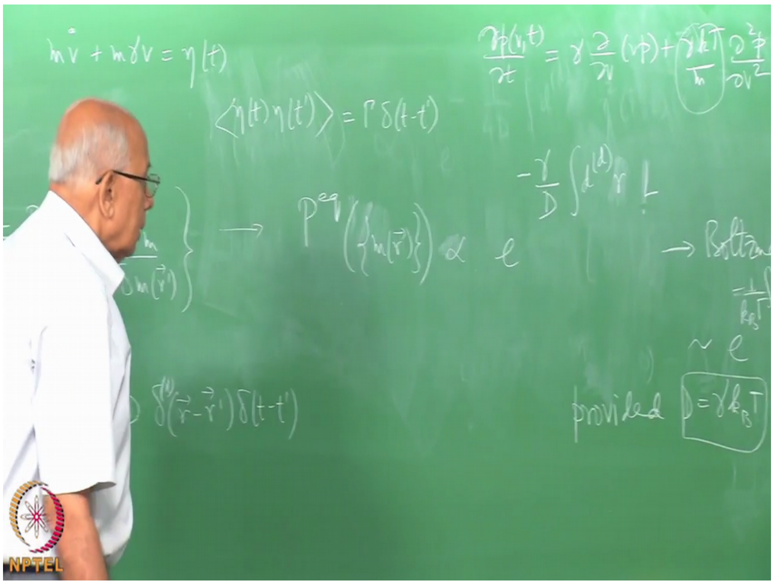

If you recall what I was way back when it was m v dot plus m gamma v equal to this force eta of t and we assumed the strength capital gamma for this guy here.

So we assume that eta of t prime was equal to gamma delta of t minus t prime and then we got gamma's two little m gamma K t but more important remember the Fokker-Planck equation for it we had delta o P of the P t over delta t equal to gamma times delta o over delta v, v P plus gamma K t over m D2 P over D v 2. So this was the diffusion constant in velocity space. Now the m has essentially by put equal to 1 there is no m sitting here. So it is not surprising that you get t. Capital gamma plus the role of D capital D here plus over gamma ok. So that is way the consistency thing works out ok. So we have some idea.

Student: key statement here is that, that exists a L from which that equation is coming.

Exactly, so there is specific L we model this L we took care of the n square term, the m4 term, the gradient energy term etc such that you get the correct equilibrium distribution and we impose this condition here. Now the next question is to ask how does it relax? How does it relax to equilibrium? That is a harder question because you really have to go back and ask look at the Langevin equation itself , but that is a harder question however one can do the following.

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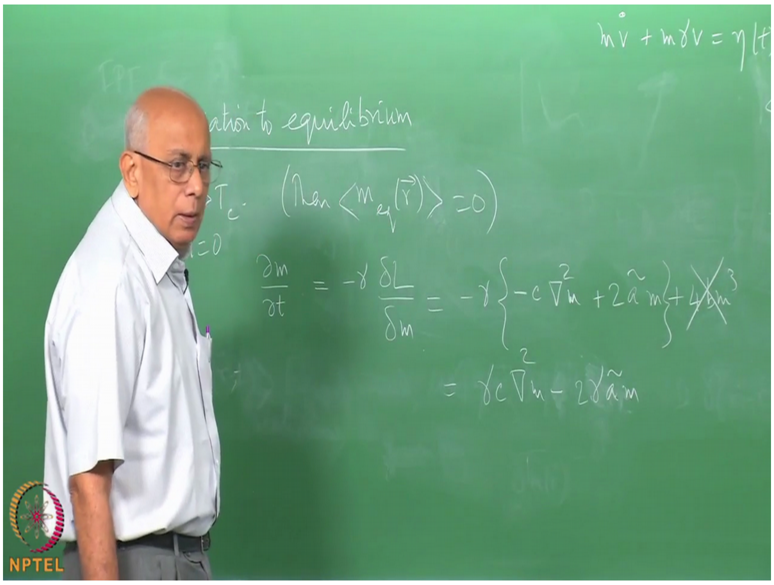

One can linearize exactly, so we won't assume, so relaxation to equilibrium and lets do this in the simple case when T is greater than T c because then m equilibrium of r this guy is equal to zero I use this for the average itself so right. I should really put brackets here because I have use this m in the Langevin equation without putting brackets but ok. So we write an equation for this M if I go back to the Langevin equation I have delta M over delta t this quantity, little bit away from the equilibrium is equal to minus gamma delta L over delta m which is equal to minus gamma and now you have to tell me what all those terms were.

There was minus a minus C del squared M, so lets do it in the absence of an external field, plus 2 A tilda M plus there was a 4 b M cube I am goanna throw this out, right, we taking the average values not going to integrate with (())(33:00) especially. Everywhere there is average so the eta term has gone away. So this is equal to gamma C del squared M plus 2 A tilda M I am unhappy with the sign minus-minus delta m gamma ok.

Because there is a del squared term here and this m is spaced dependent the obvious thing to do is to resolve it into Fourier modes take Fourier transform. Lets, let me put that tilda for the Fourier transform ok.

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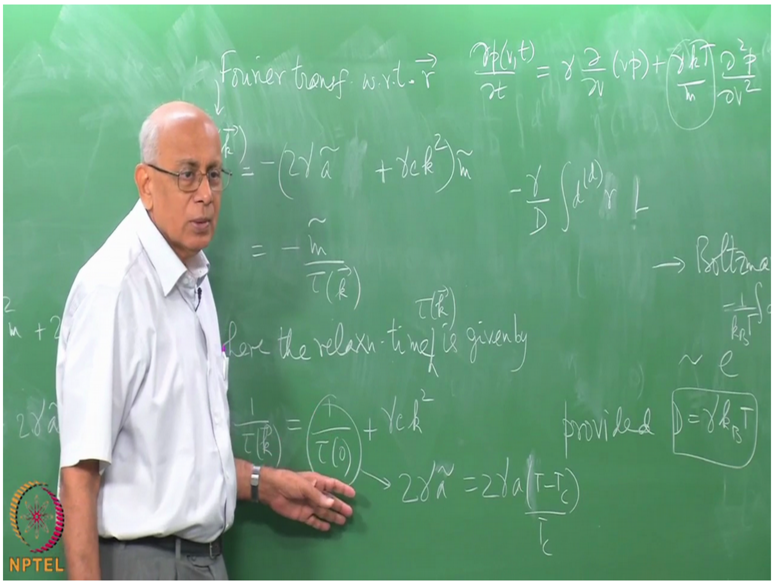

So it says delta m tilda over delta T for a given K ok, so this Fourier with respect to the space variable R, special Fourier transform. This fellow is minus 2 gamma A tilda m tilda and then this is going to give me K square, the del square with a minus sign.

So there is going to be minus gamma C K square and I pull out the N tilda ok right. I do a Fourier transform here I am going to get a minus K squared because it is I K the whole squared and this is the m tilda. So this quantity is equal to leaving out non-linear term so for small deviations from the equilibrium the linearize equation gives me this. Incidentally this Langevin equation this relaxation equation the full non-linear relaxation equation is called the time dependent Ginzburg-Landua equation in (super) when apply to super conductivity.

So just this alone with the full non-linearity is the time dependent Ginzburg-Landua equation and then you add to it the noise term it becomes a stochastic differential equation and it helps you to analyze the way fluctuations lead to state of equilibrium. In principle within this model it tell you everything about the time dependent magnetic configurations. If you include the external field then it tells you in principle everything ok.

But whether the original assumption this is valid or not is a different question. It is clearly reasonable and plausible sufficiently closed to equilibrium because you are saying the rate at which it relaxes to equilibrium is proportional to the deviation from the equilibrium. It is in absolutely a linear response (theory) statement, just like fixed law of a diffusion heat conduction etc it is a linear response statement ok.

Student: (())(37:04)

Linear in the field, it is linear in the field not linear in the variable. In the order parameter no, but it is linear in the field, that is what linear response does right ok. So in particular it will give you some of the statement about the susceptibility because remember the susceptibility is the derivative of the order parameter with respect to the field at zero field. So this thing is equal to minus m tilda over Tau of K where the relaxation time is given by relaxation time Tau of K is given by 1 over Tau of K equal to 1 over Tau of 0, that is this term plus gamma C K square right.

Tau of 0 by definition this quantity here is 2 gamma A tilde. So the infinite wave length or K equal to 0, the uniform background configuration, relaxes at this rate with Tau of 0. But this is equal to 2 gamma A times T little, I shown to you it is T, T minus T c over T c and as T goes to T c this goes to zero so 1 over Tau 0 goes to 0 and therefore Tau 0 goes to infinity, that is called critical slowing down. So this is where critical slowing down comes.

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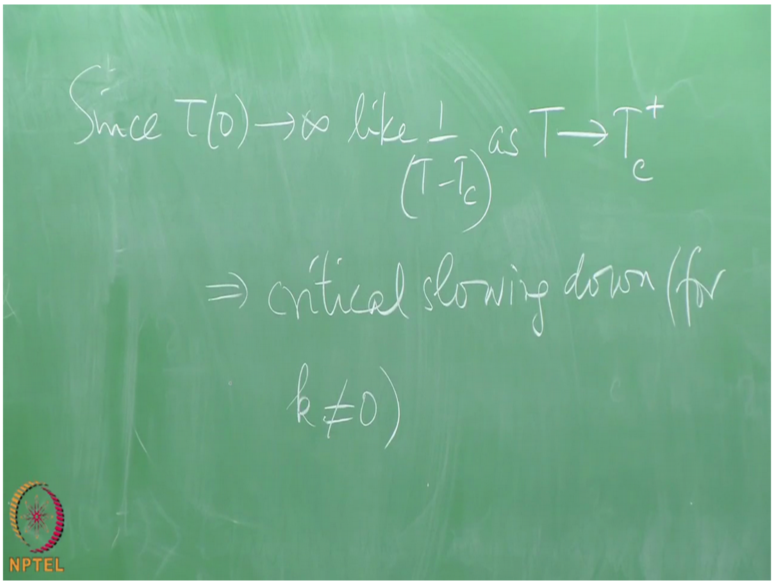

Remember we are in T c greater than T greater than T c we started with that assumption. So it diverges goes to infinity implies critical for K knot equal to 0, so finite wavelength fluctuations even at the critical point will be ok, they have to relax with the finite time, they go like K squared one over Tau K goes like K squared ok. But as long as K this thing is zero if K is zero then you have a divergence of the relaxation time.

But the finite fellows don't do that ok, which is reasonable, it is only the infinite long wavelength over all background mode that makes that gets slowed down ok. The shorter wavelength one's will have finite relaxation time because those are controlled by this quantity. As this becomes larger this is finite and therefore you have finite relaxation time ok. So this helps us formulate it is a start of something called dynamic scaling, the dynamic scaling hypocrisies. What we have therefore is the following and from here

Student: ((())41:23)

Yeah the starting exponents will get related now will see when you have so let me call it dynamic scaling.

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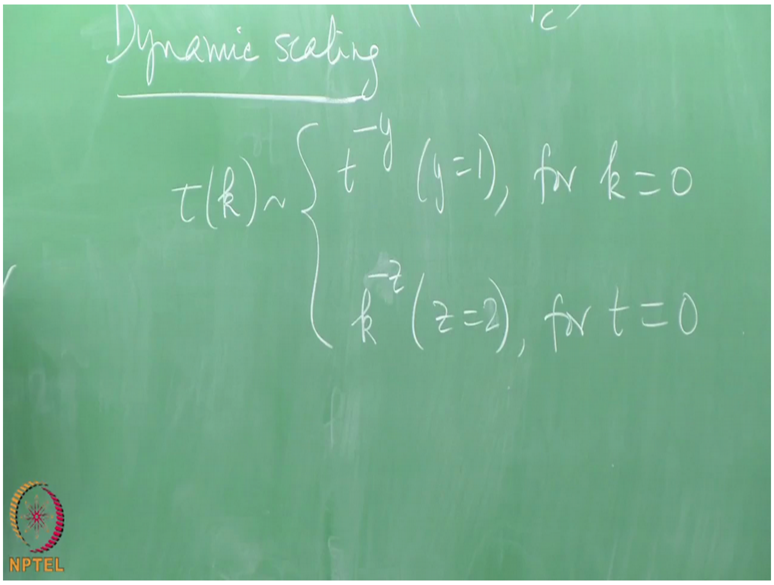

Look at what Tau K did, Tau of K it went like T no let me use T for T minus T c over T c, there is no time appearing explicitly anyway in this formula so it should not confuse, this goes like T to the minus Y, Y equal to 1 for K equal to 0, right, and it went like K to the power minus Z it is a standard symbols, Z equal to 1, and a 2, for T equal to 0. At the critical point, the other modes the one over Tau K was exactly proportional to K square and therefore Tau K is proportional to K to the minus 2 ok.

So have introduced two new exponents Y and Z here, now both these can be subsumed in one relationship by again making a hypothesis that at the critical point this guy here is some power multiply and that closed to the critical point this guy is some power-law in little T multiplied by a scaling function exactly in the spirit of (())(43:16) scaling right and then experiment to will have to tell you that is correct or not ok.

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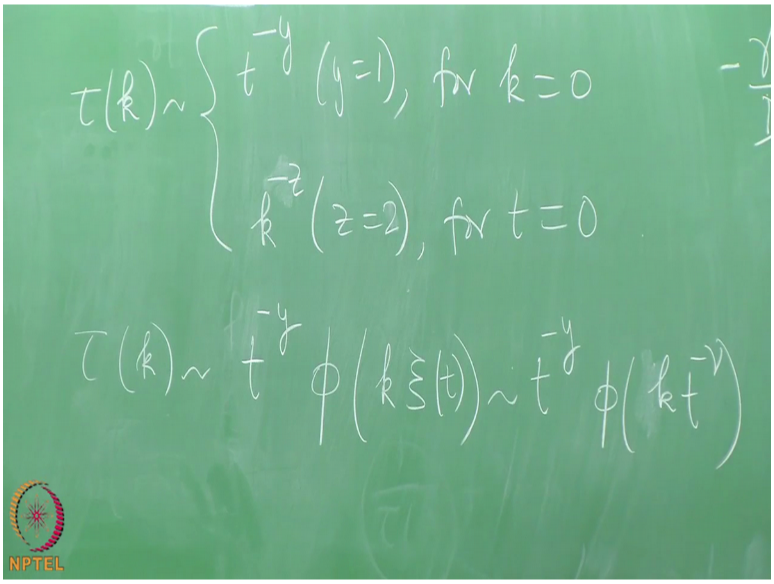

So one hypothesizes that Tau of K goes like T to the power minus Y is this in general even when this kind of simple mean field theory is not valid multiplied by some function of K times the correlation length, the old order parameter correlation length to the which is the function of little t , and now we need to get these two from it for that you require since, so let me write this fellow as the critical region T to the minus Y Phi of K times T to the minus Mu apart from some constants, because this diverges like this exponent new if you recall which was one half in mean field theory right.

So how is this going to be reconcile with that? If you put K equal to 0, you should have this divergence here which is this term here already provided Phi of zero is finite.

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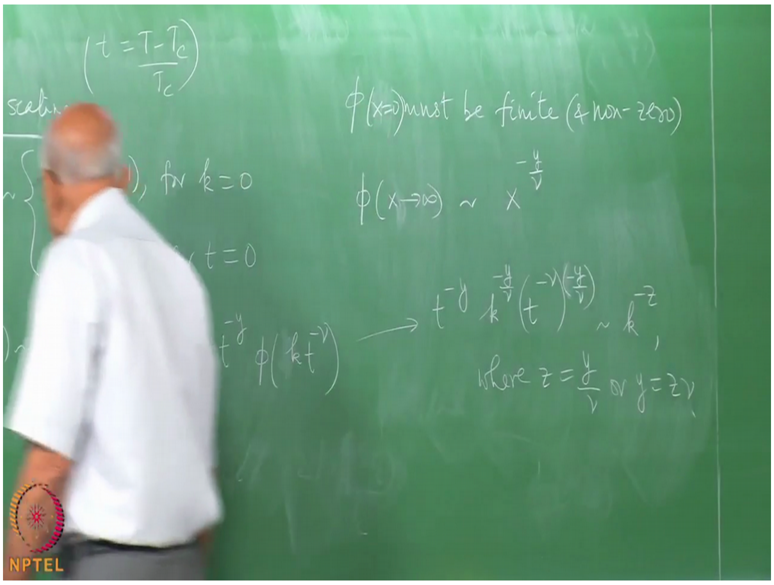

So we require of this function Phi, Phi of 0, must be finite and non-zero. Some finite constant. But as T goes to 0 this guy goes to zero you want this behavior that means this thing must be cancelled out, with a T to the power Y. So we want Phi of whatever its argument Phi of X lets say Phi of X, Phi at X equal to 0 must be finite and Phi at X tends to infinity must go like X to the power minus Y over Nu right.

Because then this term will go like this will then go to T to the power minus Y K to the power minus Y over Nu T to the power minus Nu to the power minus Y over Nu, which will give me T to the Y which cancels this and gives me a one over K to the Y over Nu. So this goes like K to the power minus Z where Z equal to Y over Nu. Because remember we wanted K to the minus Z. so it is a simple trick it is the same trick being played all over N that you have two different limits then forces the scaling function to have this behavior, that is the only way which you can be consistent. So this will immediately imply or Y equal to Z Nu.

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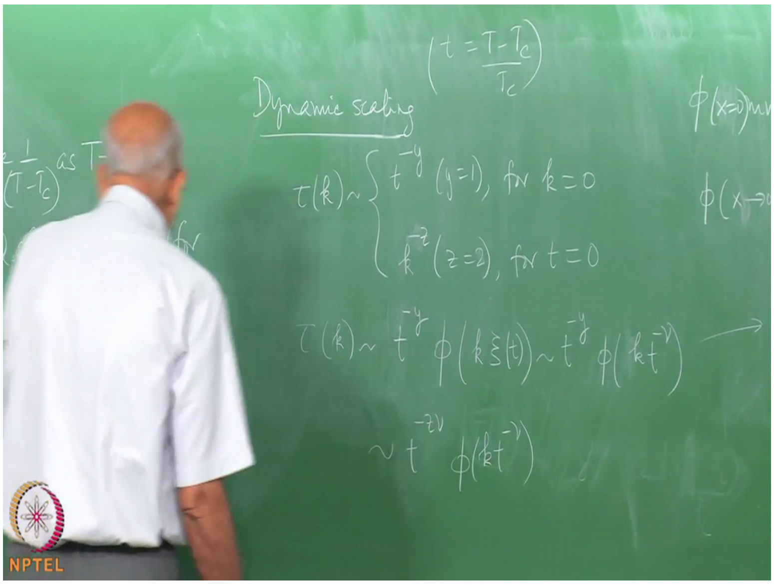

So this implies that this Tau K goes like T to the power minus Z Nu Phi of K to the minus. So we introduced a dynamic scaling exponent Z, there is one more exponent here. Now we can relate this to the susceptibility.

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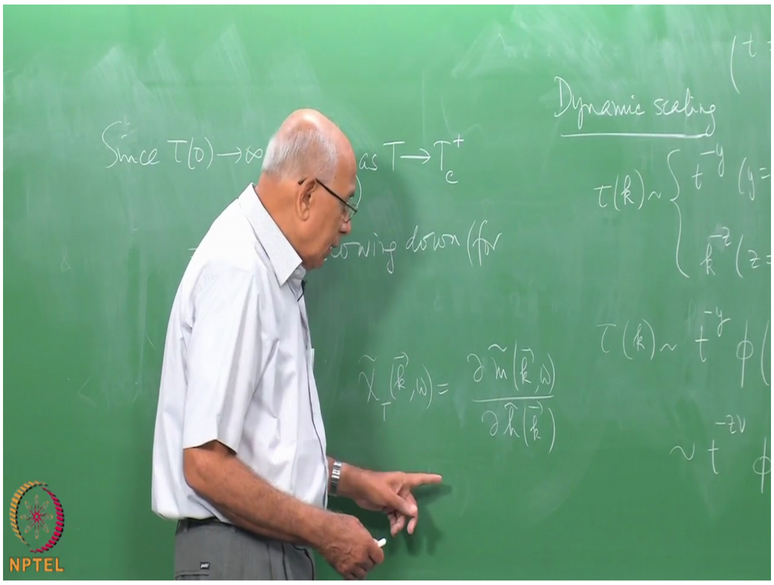

Because what you need is the formula the susceptibility Chi T, now we do a Fourier transform of K omega this follow here is the derivative of M tilda in Fourier transformed in space and time and average taken with respect to delta H tilda ok.

So we can now write and we know that there is gamma here in the exponent gamma divergence near T equal to zero for the statics susceptibility right. so once again we can make a statement about the dynamic susceptibility make a generalize scaling hypothesis here. So essentially you will have to assume that this follows here is some power of t little t multiplied by a function of K Xi and omega Tau knot that Tau knot is this (thing), the K equal to zero relaxation tank right and that turns out to give you relationship between this dynamic exponent and the other exponents ok and so on.

Now this whole business can actually be generalized to much more general class of problems including the way this is the beginning of the dynamic scaling theory which is now apply to a very large number of problem, both in and out of the closed equilibrium and far away from the equilibrium, such as the growth of surfaces (())(49:18) decomposition etc-etc ok. So the wide variety the trick is again the scaling I haven't talked this is the starting point of the re-normalization growth approach to critical phenomena.

Which is where this comes into full play the power of this scaling arguments ok and you can rigorously show what the relations are between various scaling exponents, what the upper and lower critical dimensionalities are, what the nature of the critical point is in every case etc.

Student: (())(49:57)its we got all this static exponents by looking at just trying to we got relations when we try to write everything in terms of the object which came from the correlation Langevin, so now what we are doing is we have got from this, since we have got something which is weighty correlation in length they took correlation time (Exactly) I need only one more (Exactly).

So that is the whole point. Once you have a relation like that we got rid of this intermediate thing Y, and we already know Nu and the time behavior is given this thing here will tell you what dynamics.

Student: is it true for, it should be true for all.

With suitable modifications, yeah with suitable definitions of a say etc yeah. The power of the this whole thing is not apparent here because unless we do the re-normalization group which is another way of saying that we use scale invariants near the critical point the system becomes scale independent. All fluctuations on all lengths scale and time scales become equally important and the trick was as oppose to the original ways of tackling equilibrium statistical mechanical problems where you tried to solve or trace over a fine partition function to trace over all degrees of freedom simultaneously.

This divides and conquers. So it breaks things up depending on the K value or the wavelength of the fluctuations integrates over variables which either varies slowly or very-very long period in time long special extent and then or the other way about and then ask for a case impose the condition that system will look scale invariant on all scales. So that forces you to have certain relationships between various exponents among other things right.

It also gives you the calculation method of computing systematically computing critical exponents outside the framework of main field theory ok. I use the symbol Y and Z here all though in the case we have look at Y was 1 and Z was 2, but the whole idea is that you hypothesis that these exponents can have different values other than these values ok and in the D2. There is a closely related to this Langevin equation there is another one for growth of random surfaces, growth for aggregation for instance call the K P Z equation the Kardar–Parisi–Zhang equation which again is like the diffusion equation with a noise term added to it ok.

We saw the original diffusion equation was for a probability distribution but now we saying there is noise added to a diffusion equation itself, in the same spirit as this, this is already partial differential equations and on that we added noise as similar kind of approach there ok. That leads to so called roughening exponents in similar results ok. So think I will stop here with this topic and refer you to some texts for the rest ok. There are couple of (interested) good textbooks on this many-many good textbooks on critical phenomena but I will write out a list of these useful books and give it.