Non Equilibrium Statistical Mechanics
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Lecture No 31
Critical Phenomena (Part 3)

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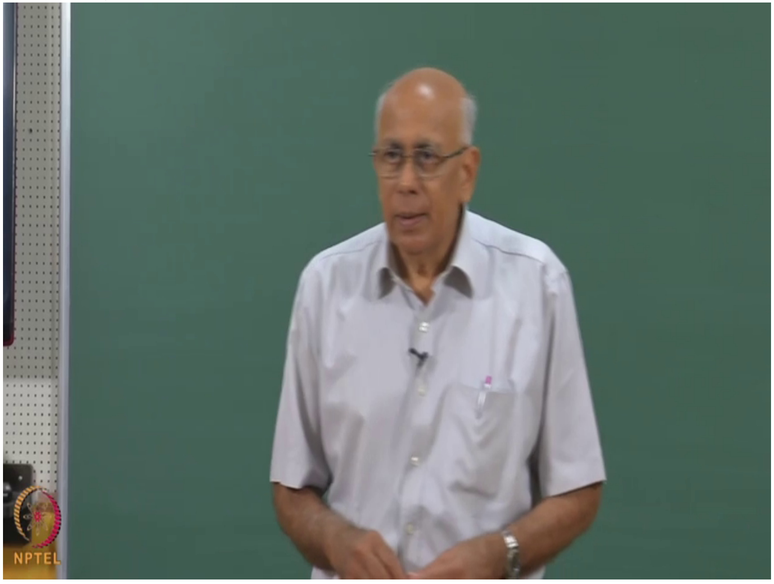

Right, so today we will look at the fluid magnet analogy in a little more detail and I will introduce the so-called mean field critical exponents. So just to recall to you quickly what we have said, we said if you have a collection of spins each of which can point in one of

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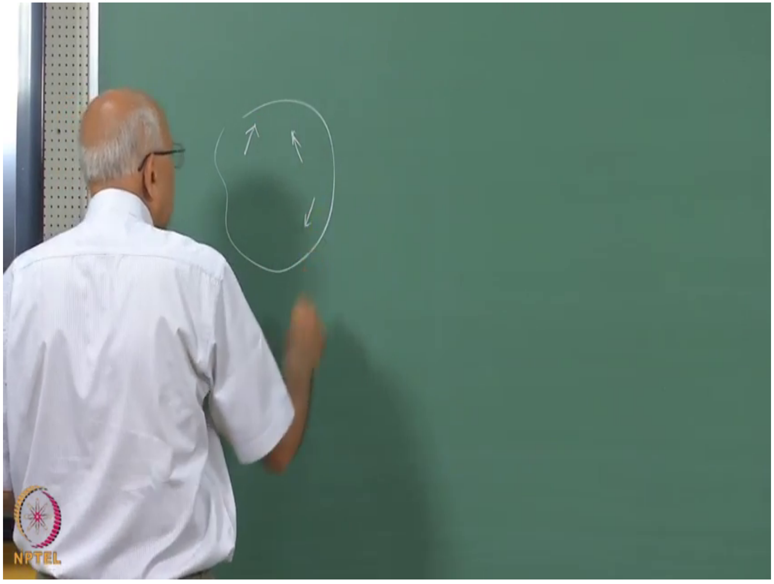

two directions, up or down in the simplest case then we had the simple formula for the magnetization in an applied magnetic field H and the magnetization per particle, I denoted it by m and this quantity was equal to tan hyperbolic mu H over k Boltzmann T when mu is the magnetic moment, magnitude of the magnetic moment

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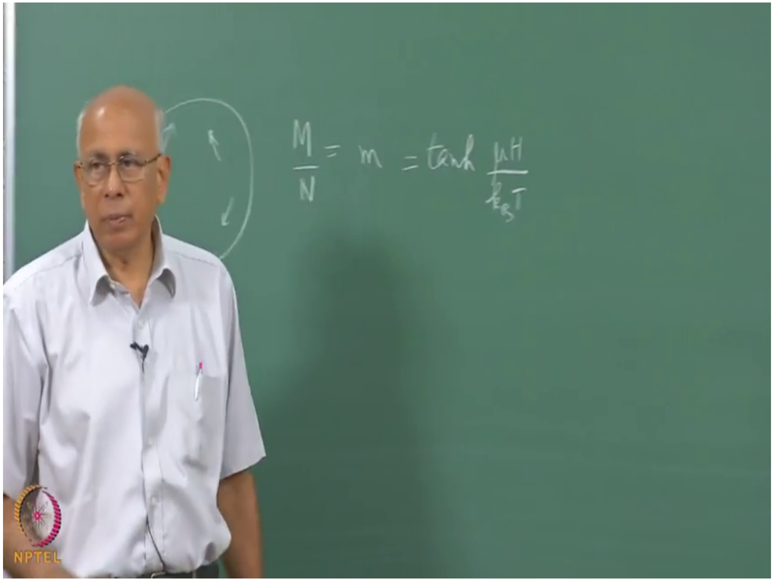

of each particle. Pardon me?

(Professor – student conversation starts)

Student: 0:01:13.3

Professor: I defined it as M over N mu, yeah.

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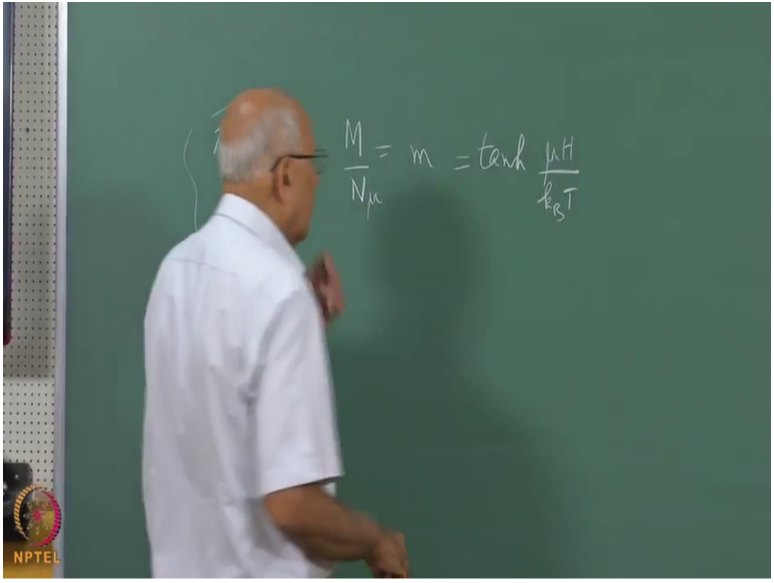

That was equal to this. And now of course, we can plot this m versus H diagram, so here is H, here is m. So here is H, here is m. And it runs from between minus 1 and plus 1 in this fashion.

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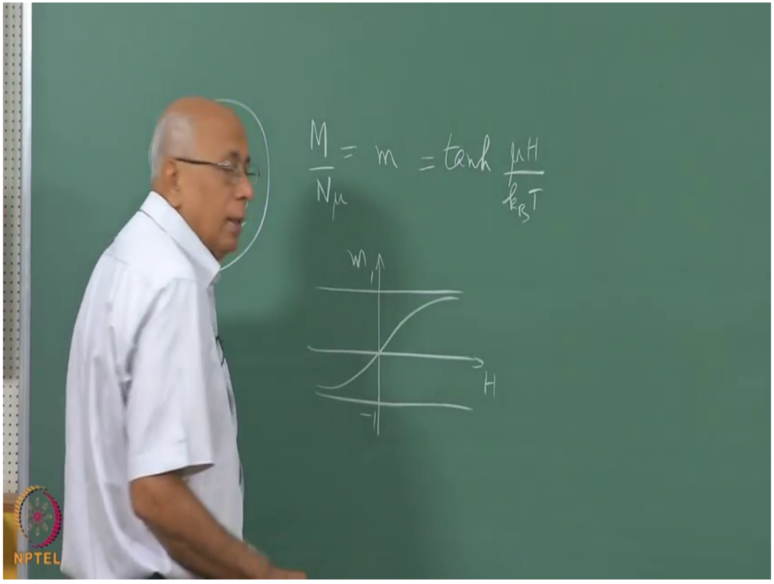

And the isothermal susceptibility, it is the derivative at zero field out here, this quantity kai T was equal to delta m over delta H at constant T at H equal to zero, this quantity was equal to mu over k Boltzmann T. It is proportional to 1 over T which is the Curie Law for paramagnets,

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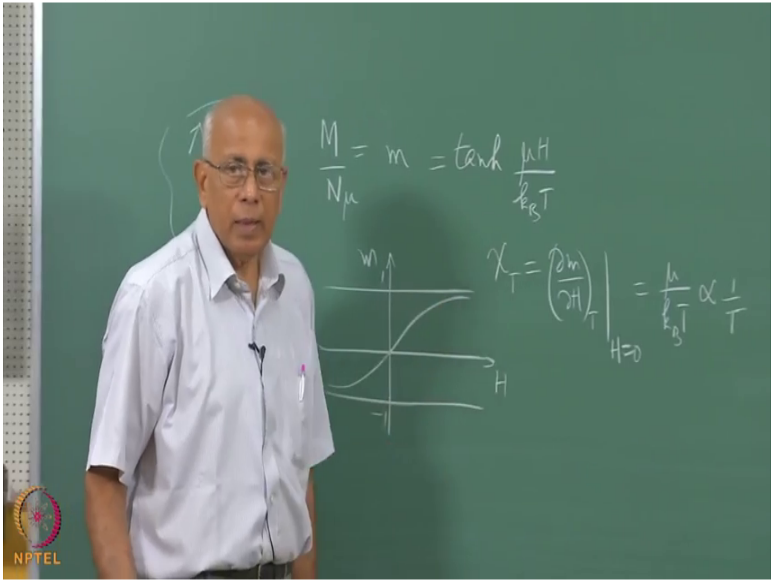

right?

(Professor – student conversation ends)

Now the question is how are you going to include interaction between these various magnetic moments leading to the phenomenon of ferromagnetism itself. But while we do that, and I am going to do that in the theory called Mean Field theory, let me qualitatively draw graphs and show you what the actual results are. So that you can see immediately where we are headed for, Ok. And the graphs are as follows.

Again the so-called fluid magnet analogy which said

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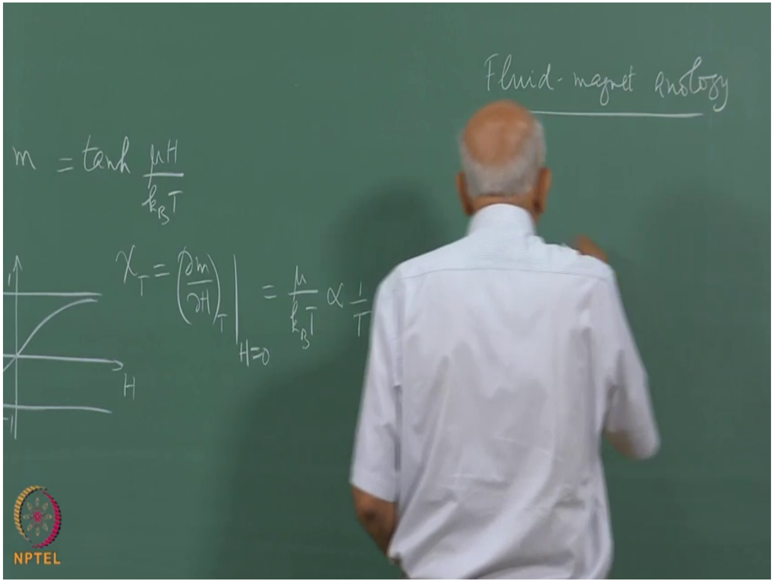

that for a fluid, you had P versus V and these isotherms were things like this below the critical point and then you had a critical isotherm of this

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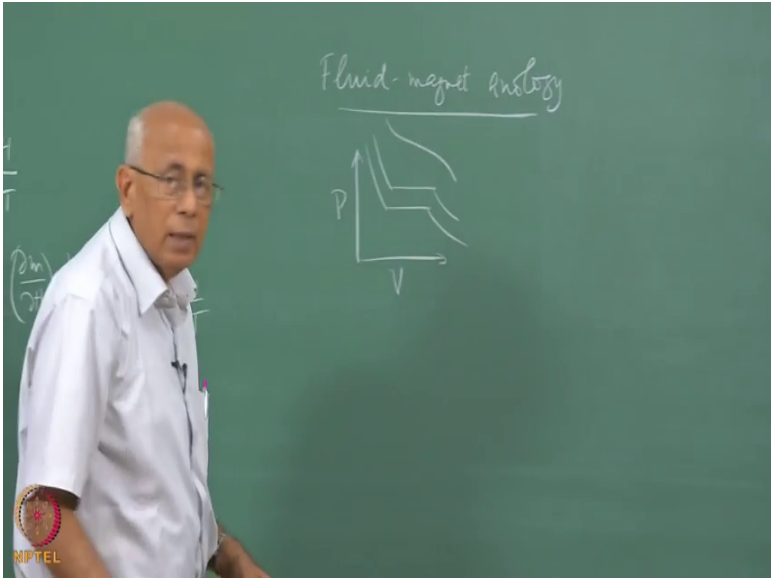

kind. And above the critical point you had things like that. So this point, this inflection point is V c P c and of course, for T c. So this is T equal to T c, isotherm, this one

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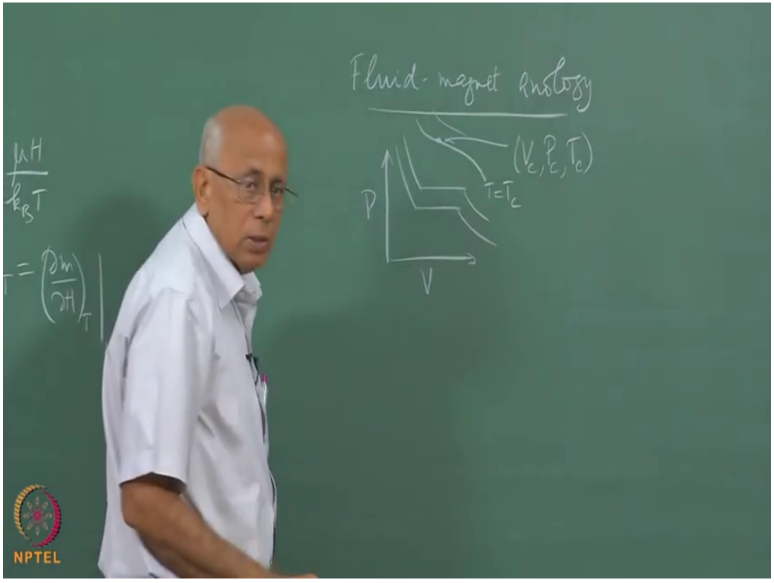

thing. Corresponding to that we have the m versus H diagram here in this fashion.

So this region above T c corresponds to these isotherms here, Ok. Then let me just say what the result is. This critical

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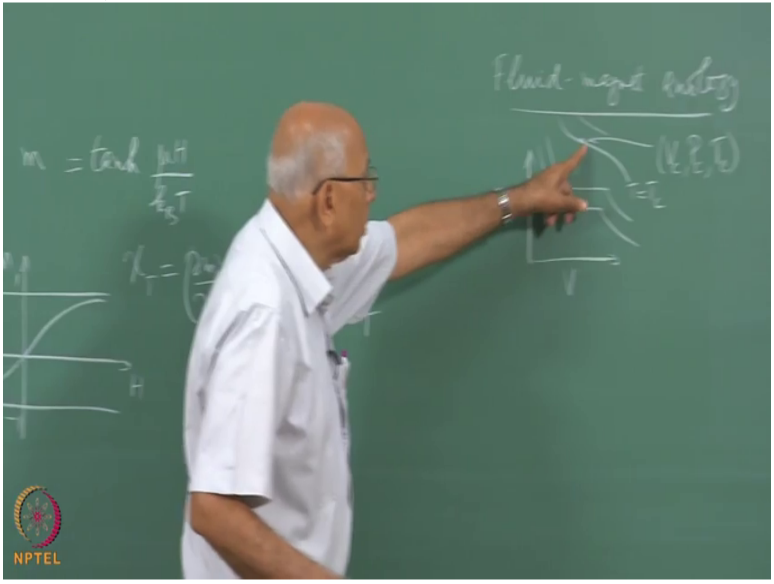

isotherm T equal to T c with an inflection point corresponds to what is called a critical isotherm here, like this and this is for T equal to T c. This is for T greater than T c.

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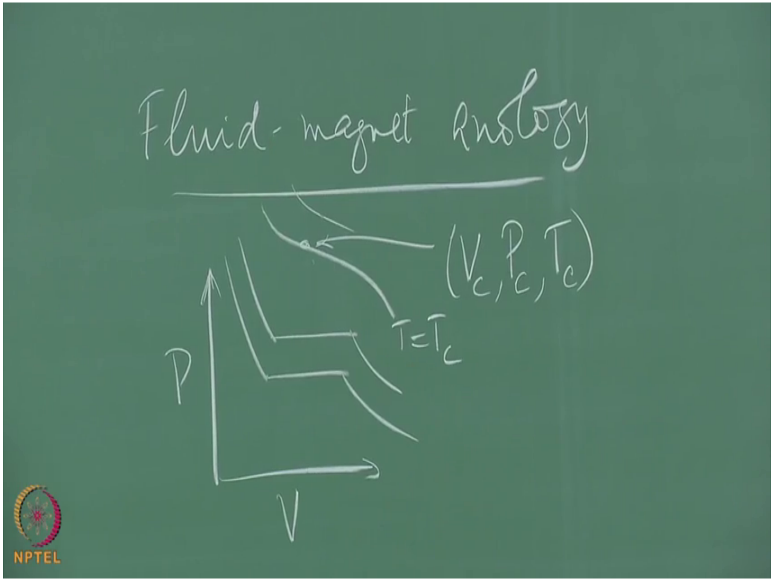

Just as this is for T greater than T c and this is for T less than T c.

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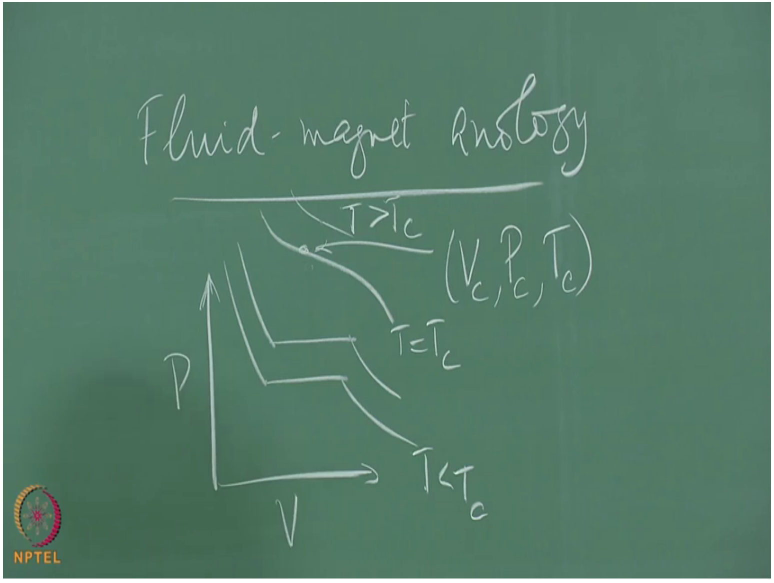

Of course you immediately see a difference which is that while the pressure here at the co-existence region changes with the temperature, that does not happen here. They are all at the same point here and now you can ask what happens if you go below T c for this system here? This model will not show anything because it is equivalent to the ideal, classical ideal gas. In other words there is no phase transition. P V equal to R T there is no condensation because there is no interaction between the molecules assumed. The moment you put in interaction between molecules, then condensation can happen if there is an attractive interaction and there is, we know in real life, an attractive interaction at long range and that is what is responsible for condensation below T c here.

The same thing happens here and we will see this explicitly. What happens actually is you come acro/across, come till here, and then you discontinuously jump out here. So this is an isotherm, this thing here for T less than T c.

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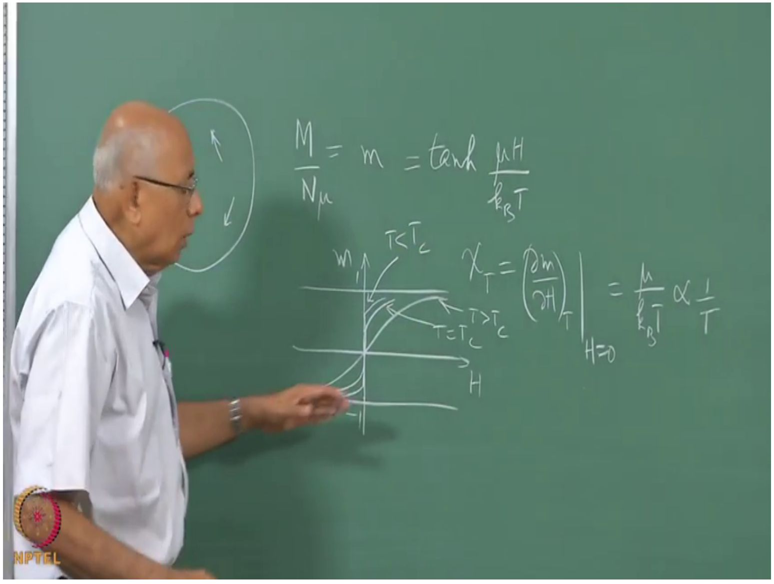

And all the isotherms for T less than T c are of this kind. They have a discontinuity on the H equal to zero line, and they jumped off. So in a sense, all these points, these flat portions are all compressed on to one line, Ok in this case.

So this is the connection between those two fellows. Now you can ask what is the susceptibility. Well I defined it, delta m over delta H at H equal to zero; you could ask is a there susceptibility in the ferromagnetic phase? Now why do I call this the ferromagnetic phase? Because even I switch off the field, there is a remnant magnetization. That is the characteristic of a ferromagnetic phase. So if I switch off the field from the positive direction, then at that temperature, the system will show this magnetization. But if I switched it off from the negative direction, from opposite sense of the field, it would show this value here, Ok.

Just as out here, if I decrease the pressure to this point, you would be here.

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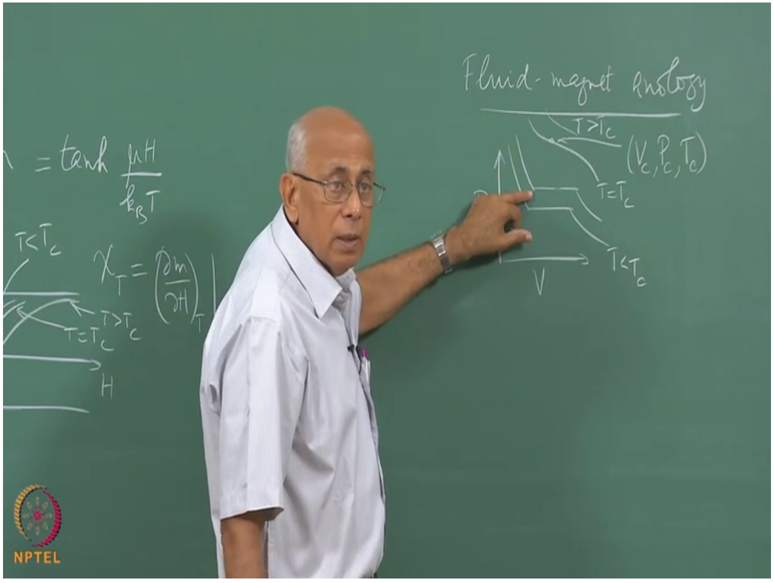

If I decreased it on the gas phase, you would be here in exactly the same way. So it looks like the analog of the liquid phase and the gas phase here, it does not, it has more symmetry. It is a magnet with the magnetization positive versus a magnet with a magnetization negative here. The analog of the fluid homogenous phase out here is the paramagnetic phase when there is no magnetization at all. Ok. Now this become

(Professor – student conversation starts)

Professor: Yeah

Student: Inside the thermal noise 0:06:50.1 suddenly starts doing something when the temperature is less than T c?

Professor: Yeah, so we will have to see what this T c is and what the difficulty is. We are going to see this in some detail, Ok? What happens is, in a nutshell, his question is does thermal noise start doing something at T c or below T c and the answer is yes.

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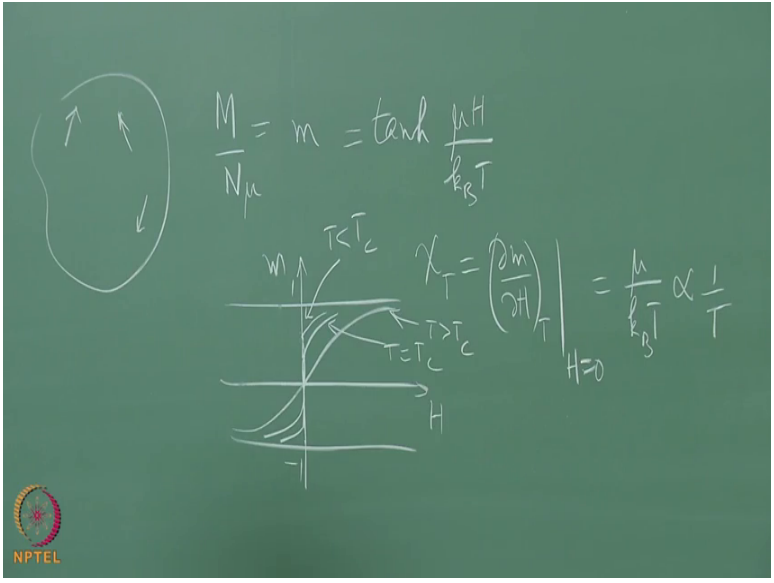

It is the other way about.

(Professor – student conversation ends)

The interaction tends to align things. Beyond a certain temperature the disorder effect wins and the ordering effect does not. That is essentially, in a very crude way, what happens. There is always thermal noise present, thermal fluctuations. The typical strength of a unit of thermal fluctuation is in energy units, is k T.

Now if the characteristic energy associated with ordering is much less than k T then you are in the so-called high temperature regime which is disorder and if it is small compared to, if it large compared to k T then of course you are on the lower temperature regime, Ok. So ultimately as we will see today, the strength of the interaction decides where this critical point, where the critical temperature is. There is always thermal noise of course.

Many other crazy things happen at the critical point and that is why it is so hard to understand in elementary terms. One of the things that happens is, and this is a modern way of understanding, the correlation length of the system diverges at the critical point and we will in fact establish that. We will see how it diverges, today. Ok. So this picture has to be kept in mind, the P V and the m versus H.

Then the next one was, in the fluid case we had the, the fluid I am doing here so, we had the T here and we had P and remember that this was liquid, this was gas and this is the critical point. So this point is T c, this is the critical pressure

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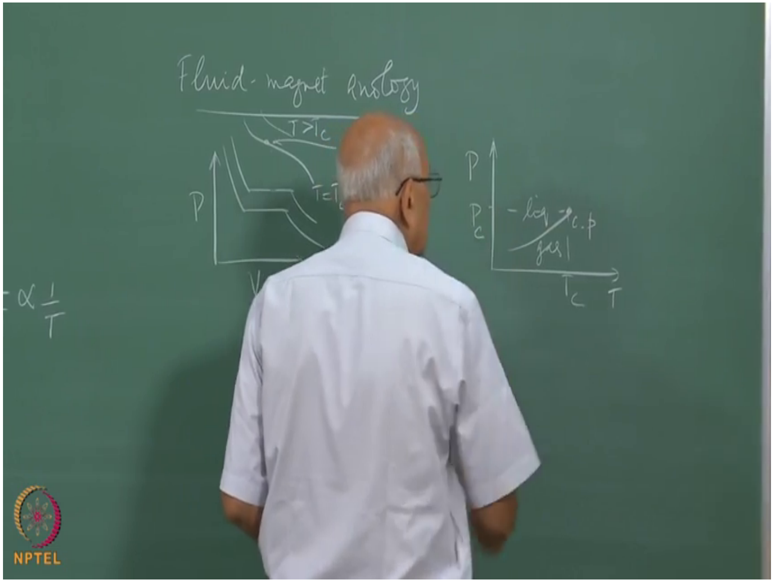

P c, Ok.

The question is what is the analog of that here. I have already said that the two phases, liquid and gas correspond to magnet up and magnet down in this case and the fluid phase above the critical point, the critical region corresponds to the single phase there, the one phase region corresponds to the paramagnetic region here. So what does the analog of that graph look like? Well it is like this and I will draw it here.

Of course there P and T are non-negative so only the first quadrant is involved. But here, you certainly have P will run all the way from zero upwards but m of course can be positive or negative,

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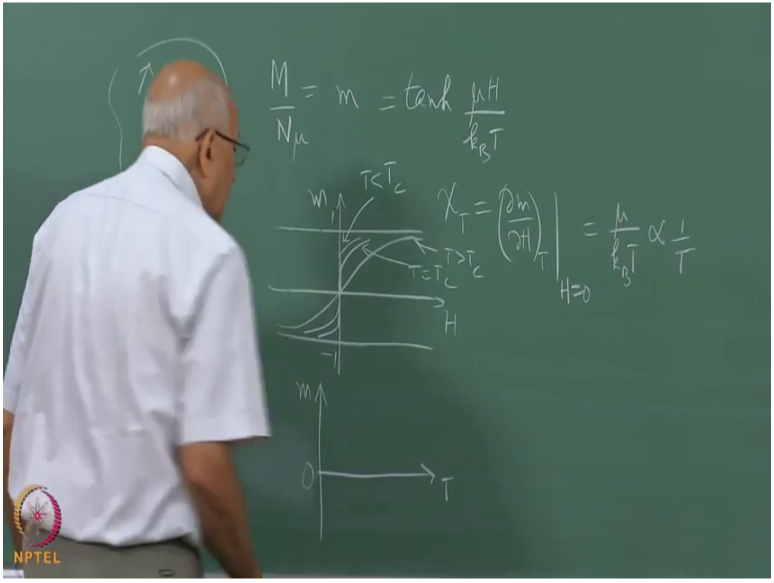

H

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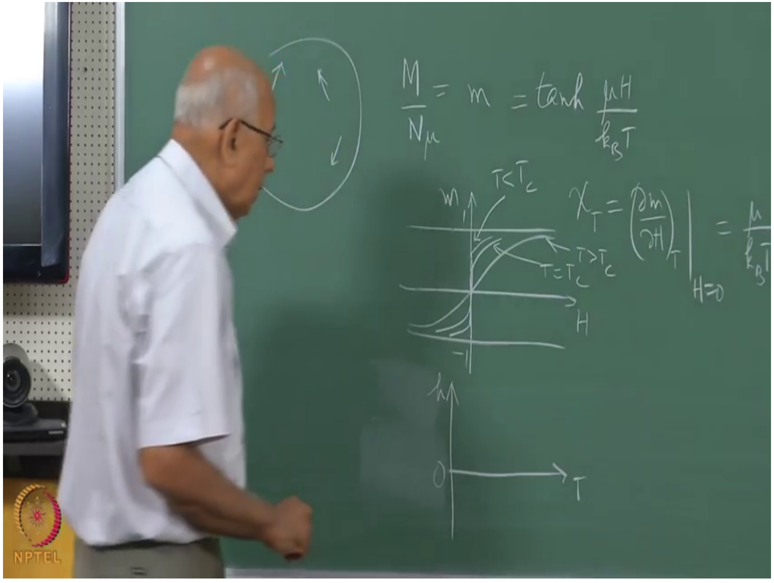

can be positive or negative It is a function of this field. And the question is what is the analog of the P T curve. It has got to be H versus T curve

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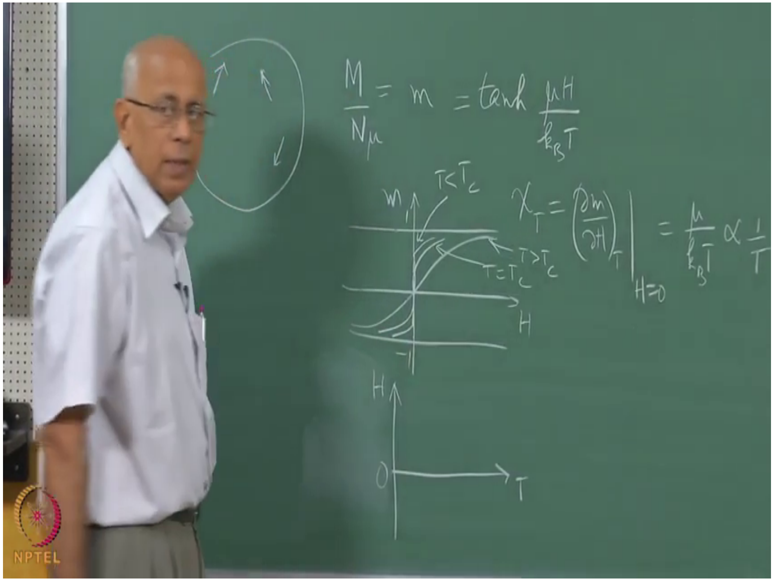

here, Ok. So what do you think is the analog?

Well, let us remember what happened here. We know that the slope, slope of co-existence line d P over d T equal to delta S over delta V by the Maxwell relation, Ok. And then I argued

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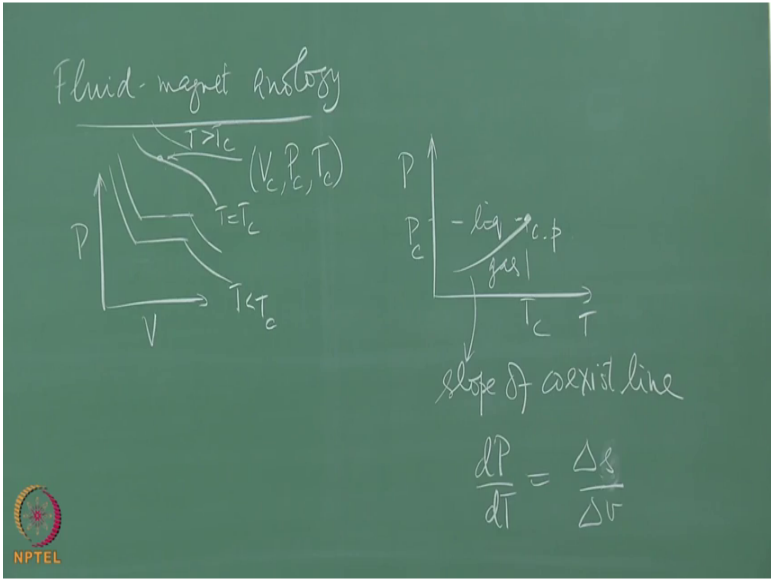

that generally delta V between vapor and liquid is very, very large.

So you have large denominator and the slope is very small in magnitude. It is also always positive because that stands for S entropy of gas minus entropy of liquid divided by volume of gas minus volume of liquid specific volumes, then both numerator and denominator have necessarily to be positive. And the slope is small. Here the same thing happens but now you have the symmetry between up and down. Whatever you got here with a certain switching off the field here, exactly the same thing with a minus sign appears here. This is reflected here in this. This is completely symmetric.

You can see in this graph, that if you change the sign of H, you change the sign of m. So it is completely symmetric in the situation. Therefore this graph has no choice but to be a straight line here. It cannot be in any particular direction. Because if it were up, it would violate this up down symmetry. So it has got to be like that and then at T critical.

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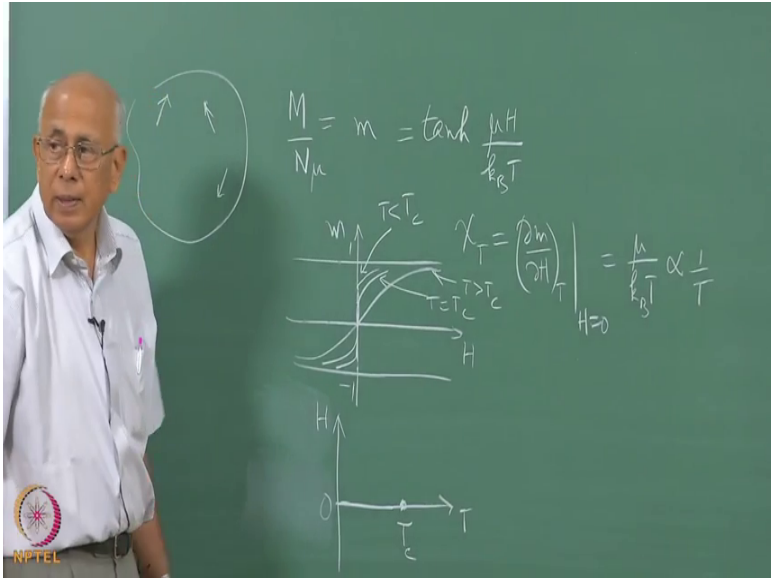

And that is the coexistence line between a phase which is not liquid but ferro up, let me just denote it by this. And this is ferro down.

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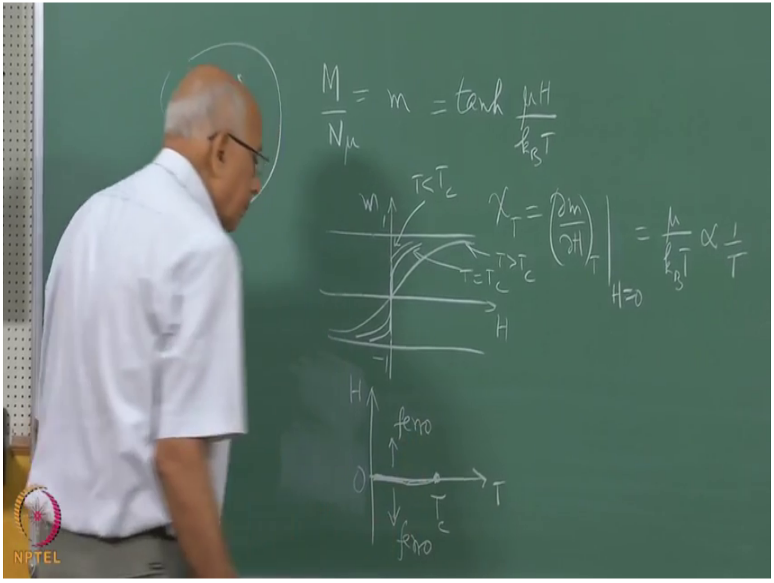

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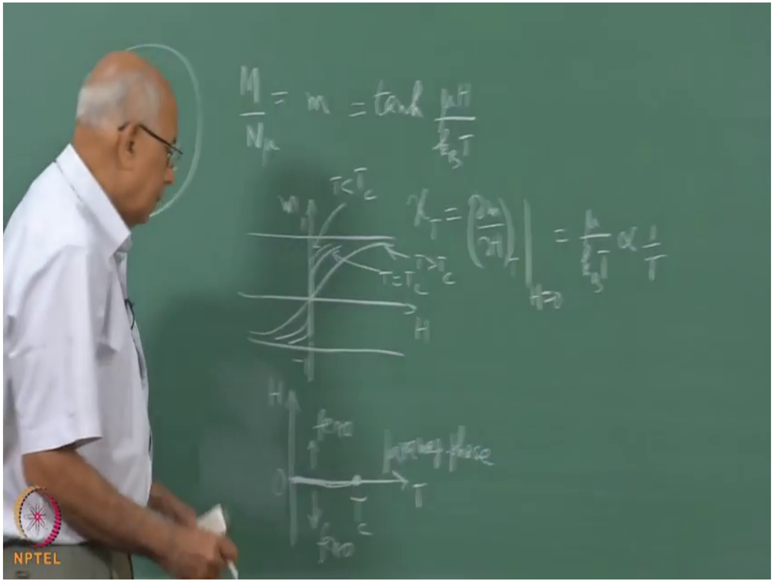

And on this side, just as you have a homogenous fluid phase out here, on that side you have the paramagnetic phase. So you already see by symmetry that the slope here, d H over d T of the co-existence curve must necessarily be zero.

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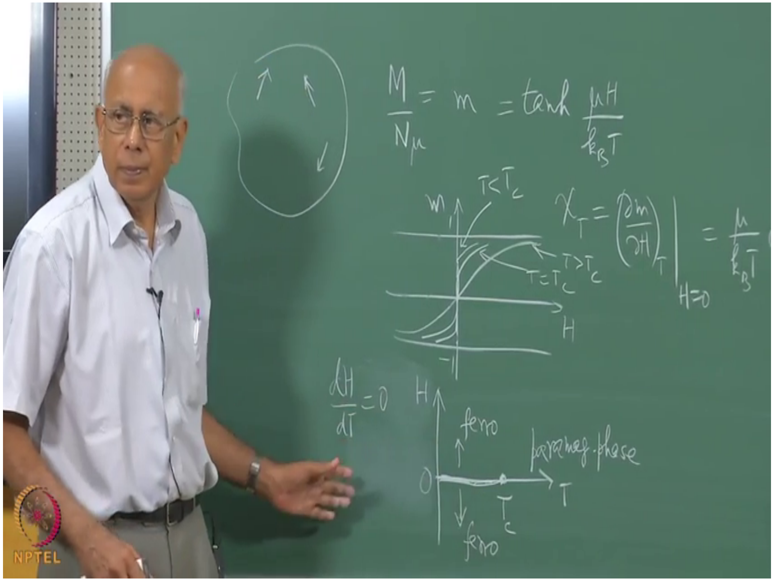

Ok, right? I just argued this by words saying that look, whatever happens up here, when you reverse the sign of the field, the magnetization changes and it has got to be so.

But we can now ask, no, no, no, let us just blindly apply the Clausius Clapeyron equation. Let us just blindly apply the analog here.Then why should it be zero? d H over d T,

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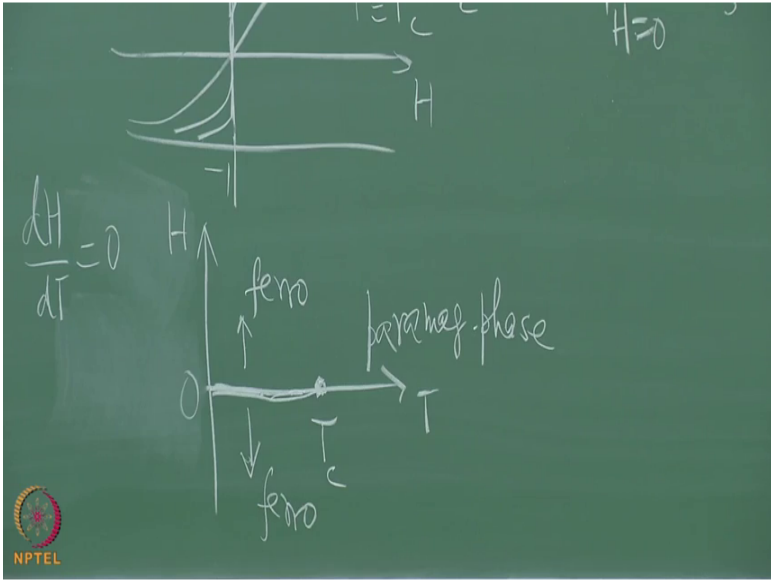

that graph must have a slope which is delta S divided by delta what? What is the analog of the volume?

(Professor – student conversation starts)

Student: Magnetization

Professor: The magnetization, delta m.

(Professor – student conversation ends)

And if this is zero what does it mean?

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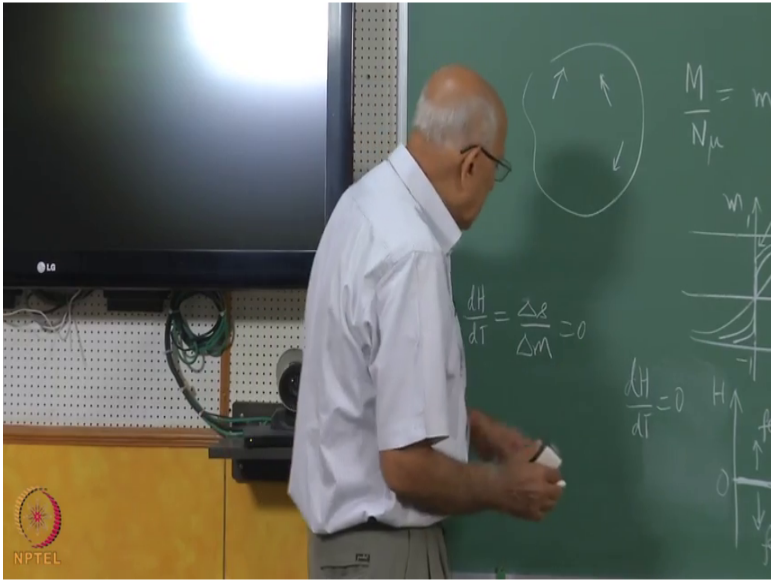

Well, delta m is finite because this is m for zero field and this is m from the

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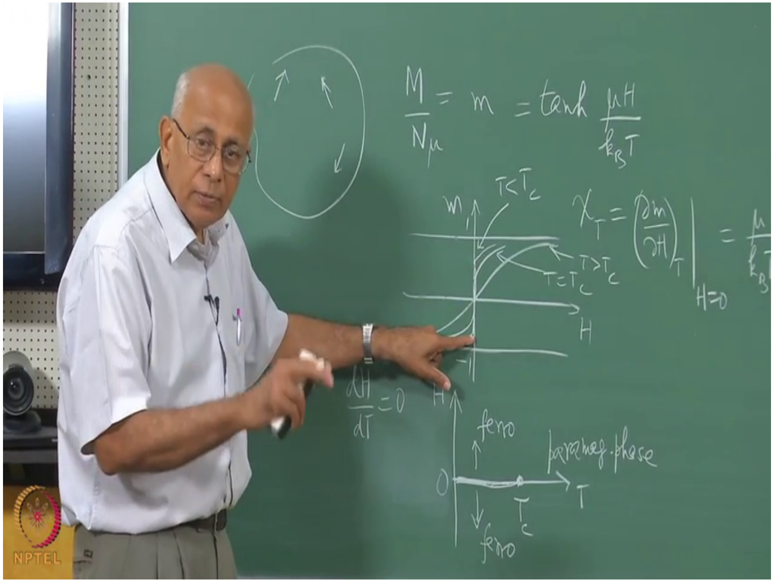

negative side. So delta m, the denominator is finite. It is this difference, Ok. But the numerator is zero because, how after all do you get this graph? How do you get a point here? You have a specimen like this in which everything can point only up or down, then if this is, remember this is an average magnetization over the whole sample, so it is clear that if you were at absolute zero of temperature, you applied the field, infinitesimal field in the up direction, everything would align in the up, you would be here.

Similarly if you applied the negative field you would be here. The reason you have this gap here that has not gone all the way up there is because of thermal noise, thermal fluctuations. So what you have is large islands of up, up, up and then there are small regions of down, down etc

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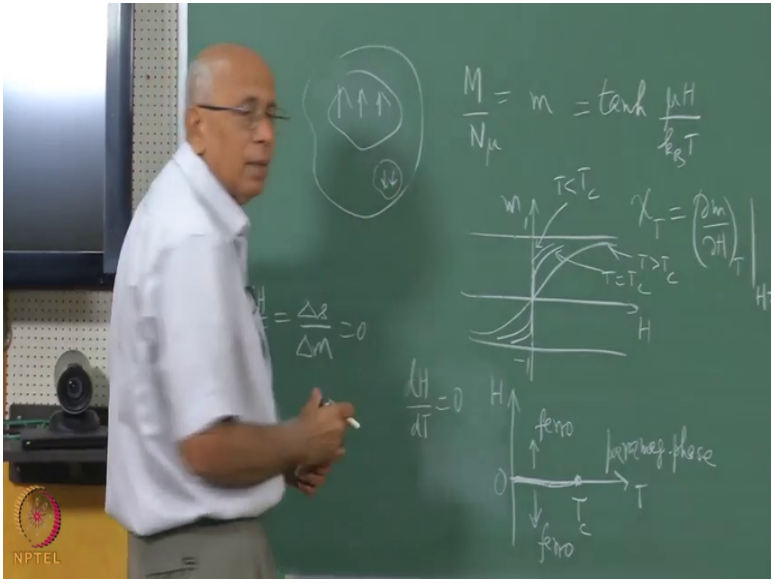

as you bring the field down towards zero. This, on the average there will be more pointing up than down and you have a net up magnetization.

As you lower the temperature further it takes less and less field in order to align. So you are going, this intercept is going to move up, ditto for the other side, move down towards minus 1. Now it is clear by this symmetry that the entropy of the system which is a measure of the disorder in the system is equal on both sides, because whatever domain you had here for up field and down domains in the up field, exactly the opposite happens in the case of a negative field.

So these would become down and these would become up here. So it is not surprising that the entropy is zero, change in entropy is zero. The moment that happens the slope here has to be zero. Because the denominator does not vanish, the numerator vanishes by this symmetry.

Ok. So that is a rigorous argument that this curve, coexistence curve between up and down has to be flat here.

It is a good thing it is flat because you see otherwise look at what will happen. Out there, if I start with a liquid at atmospheric pressure say,

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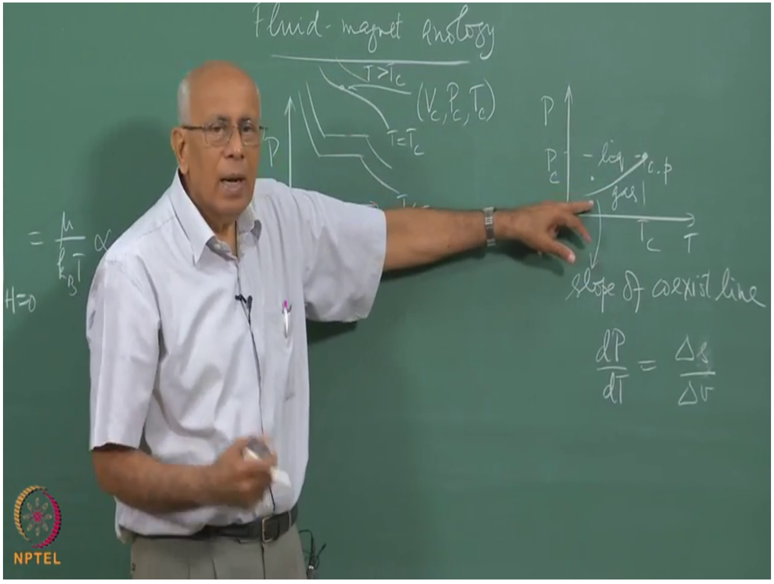

and I heat it. I increase the temperature; it will boil once you cross 100 Celsius. So it will go, cross, so it will certainly boil, because you can go across this line out here. You could also get it into gas form by dropping the pressure down this way or anywhere across which cuts this line here.

On the other hand, if you start here with ferro up and you just increase the temperature, it is going to go up like this. It is not going

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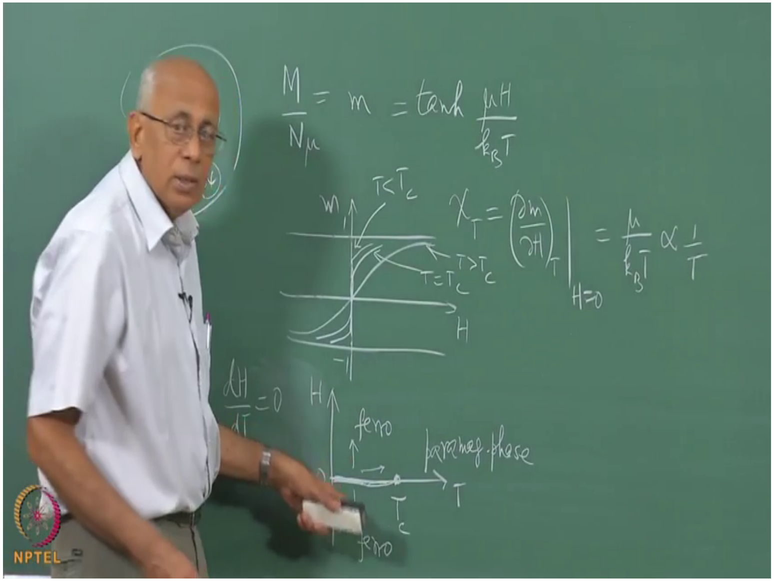

to transform into a magnet down without changing the field. You have to change the field to do that. On the other hand, you can boil water, you can boil water it becomes gas without changing the pressure but you cannot make an up ferro magnet down ferro magnet without changing the field just by increasing the temperature.

So that itself tells you that this graph has to be flat by symmetry.

(Professor – student conversation starts)
Student: That is really for para to ferro which are going to discuss here.
Professor: We are now going to talk about para to ferro that is a different matter out here, what is going to happen here. For that I need the third of these graphs. And the third graph was of course, in this case, in this case, it was the density, sorry T versus rho graph and it was roughly like this. This was the coexistence region. This here was the gas phase, this is the liquid phase, this is the critical point, Ok. And this region was the coexistence region.
(Professor – student conversation ends)

In the van der Waal's isotherms

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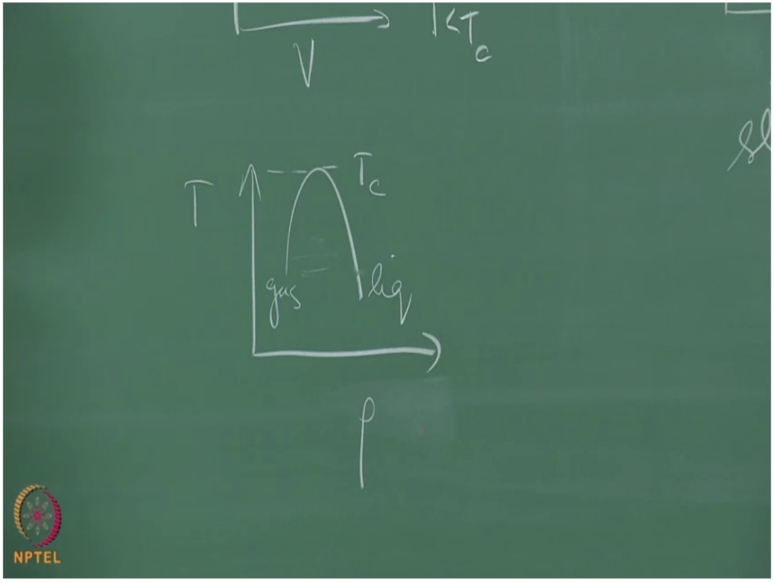

if I quench something suddenly to a point here, it would phase-separate into a part liquid and part gas, saturated vapor above it. And the exact ratio how much happens can be found by so-called lever rule which I am not getting into here, but the fact is the two coexist here. On the other hand there is no coexistence region here at all because let us look at what the analog of that graph is.

So on this axis I draw temperature, I draw magnetization here and since m can be plus or minus, I draw this. Then here is T c. First

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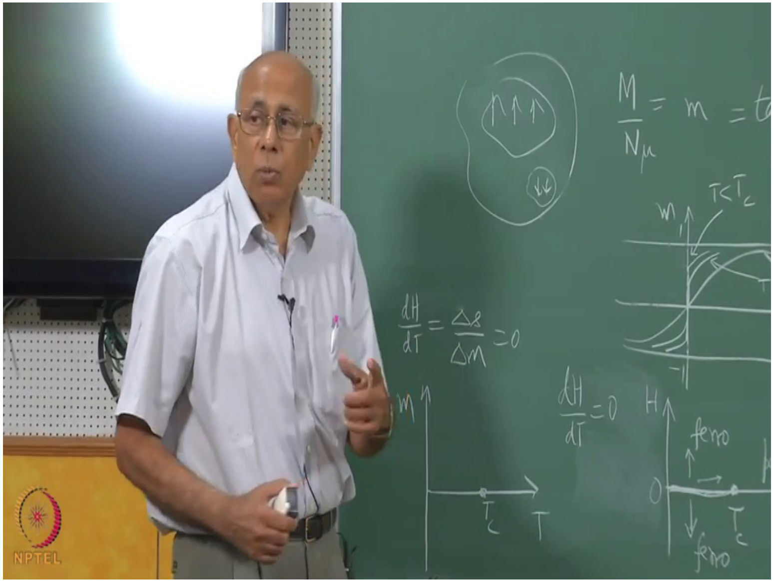

let us look at what happens if I have zero field. Then depending on whether I switch it off from above or from below, above T c I have zero, at T c you are just about taking off from

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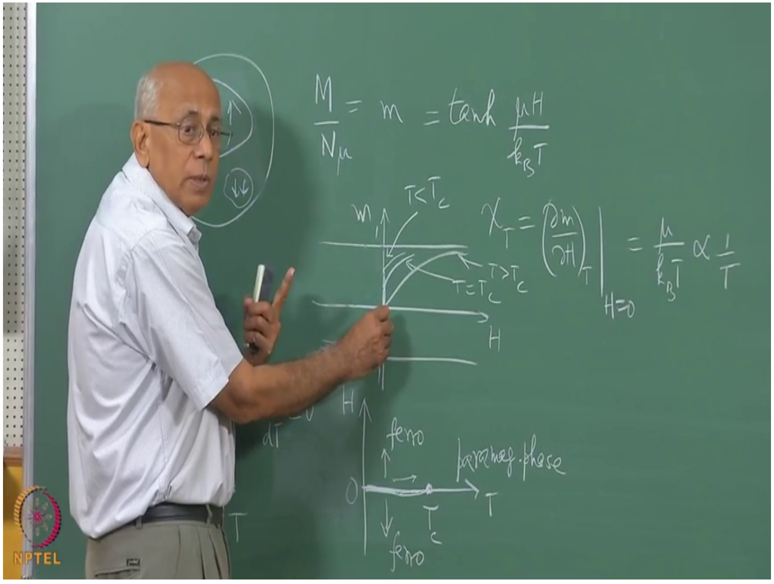

zero and then these intercepts go towards the saturation value. So this means that they are going to go like this, Ok.

And this branch corresponds to the other part of it. So let me call this m naught to show that it is the remnant magnetization in the absence of a field. So m naught is either this or this at different temperatures.

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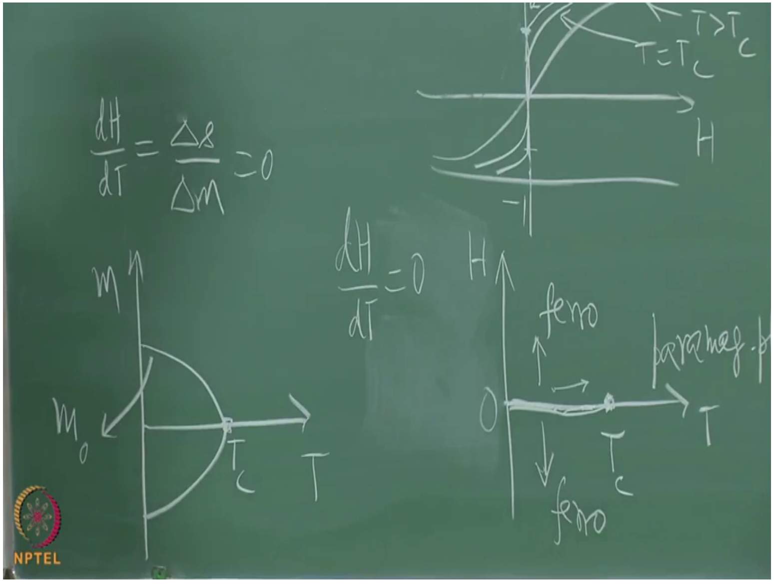

As I go towards absolute zero, this tends to 1 in the units we have chosen and this tends to minus 1 out here, Ok.

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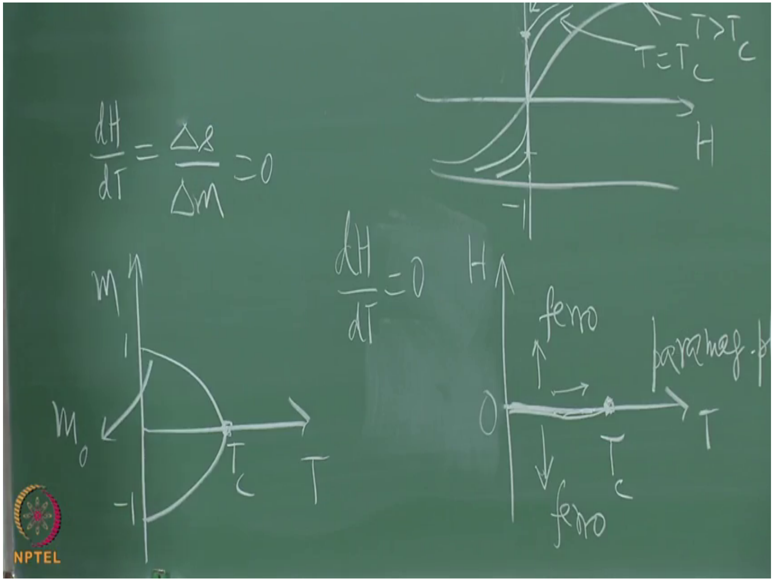

The reason we call it the continuous phase transition as opposed to a discontinuous phase transition is that when you, when you boil liquid here the density changes discontinuously.

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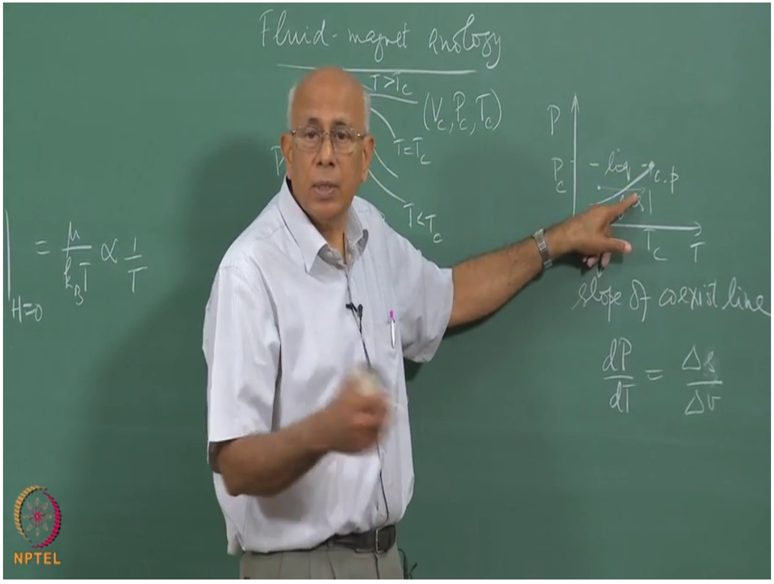

There is latent heat given to the system and there is a discontinuity perhaps in the order parameter. But that discontinuity disappears as you get to the critical point. In exactly the same way, this intercept for m naught starts off

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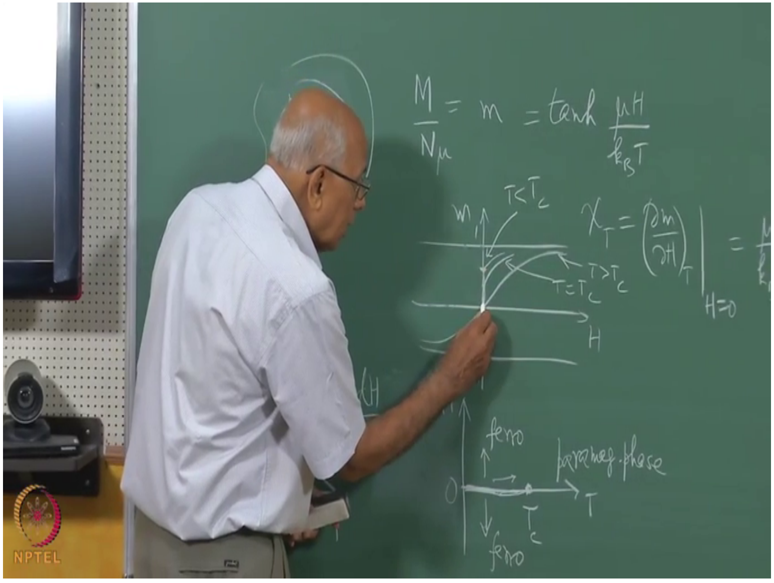

smoothly from the origin and goes up.

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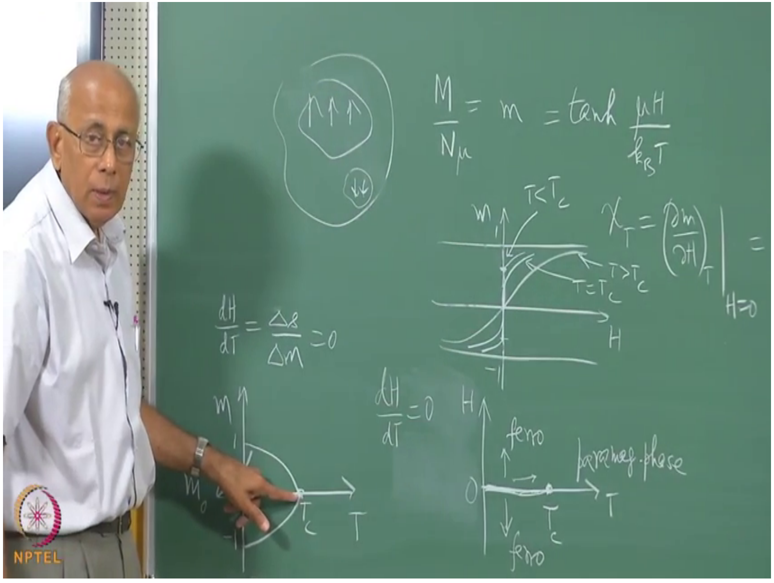

But it is zero outside, in the ferro magnetic phase, in the paramagnetic phase but in the ferro phase it starts from this point, then it jumps suddenly and then it goes up. And that is the other branch, going down here.

Now you could ask, suppose I looked at these intercepts

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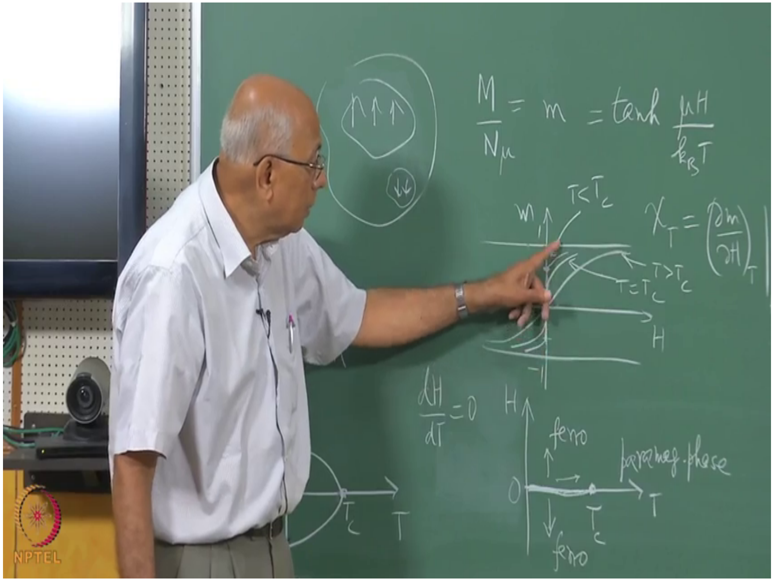

for small positive values of the field. I do not switch off the field. I keep it very small and positive. Then out here of course you would have, for large thing, you would have for this field you would have this much magnetization. As you lower the temperature you are going to get more and more of it and you are not crossing any discontinuity or anything like that at any point.

So this would in fact start like this

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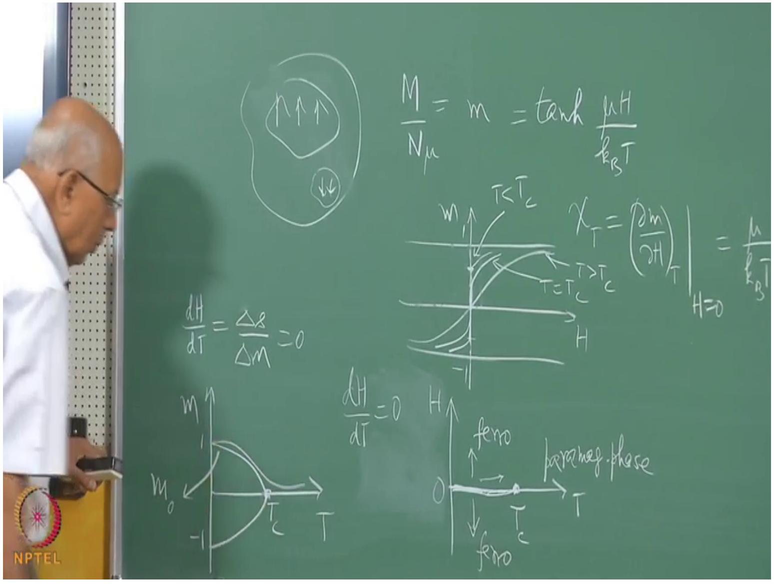

and go up like that. It cannot be more than the saturation value therefore it saturates. On the other hand, in a perfectly symmetrical way on the negative side it would do this out here.

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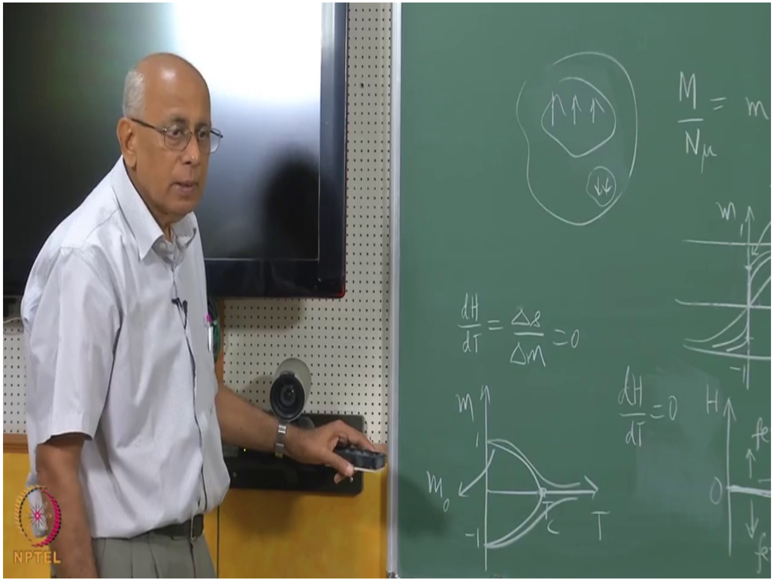

As you come down in the field to zero and then go towards negative fields, the magnetization jumps discontinuously to

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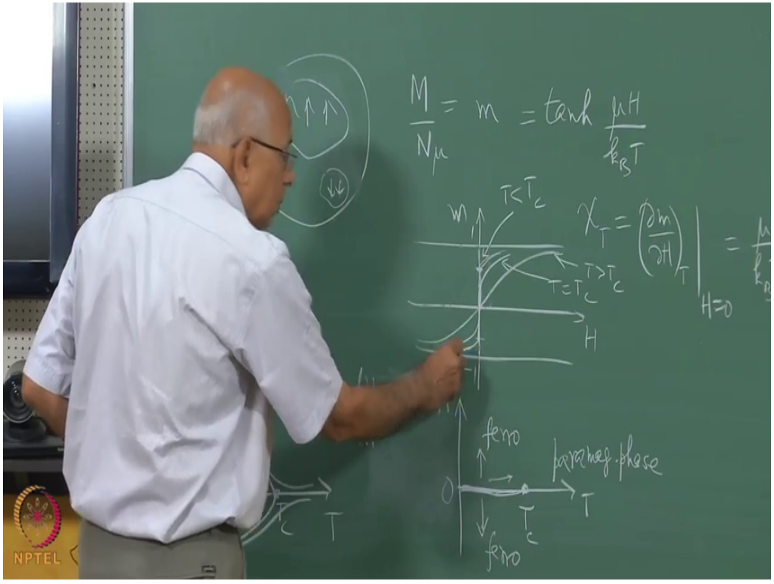

this point and goes here. So you have a jump from here to here. And then it goes for the down, Ok.

So there is no coexistence region here. This region does not, what happens essentially is that the free energy, if you plot the free energy will have an unstable minimum, unstable extremum at that point, Ok. Since we are going to talk about Landau theory let me say that right away. If you plot the free energy, the corresponding free energy F as a function of m, magnetization

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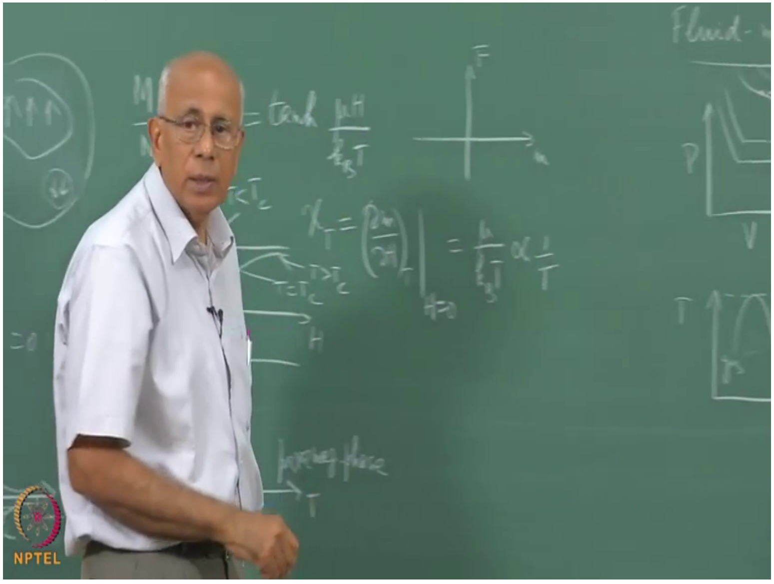

then in the so-called paramagnetic region, this is in the absence of an external field, so F a function of m, H, T and now I set H equal to zero.

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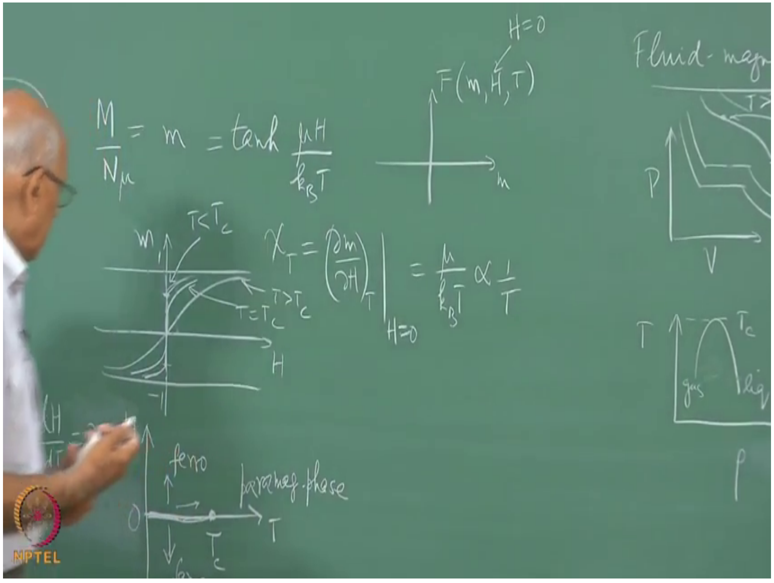

Since at H equal to zero, the only stable solution is zero magnetization for T greater than T c and that is the stable thermal equilibrium state, this free energy must look like this for T greater than T c,

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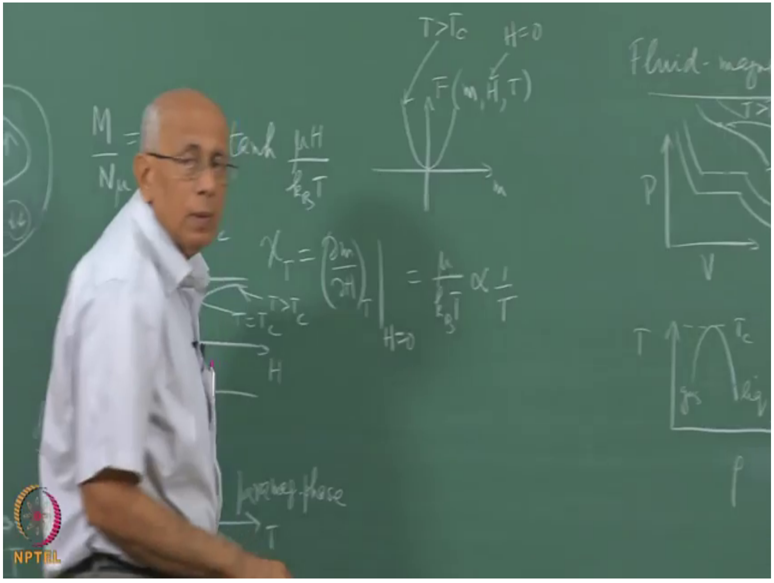

Ok. On the other hand when you come to this point here,

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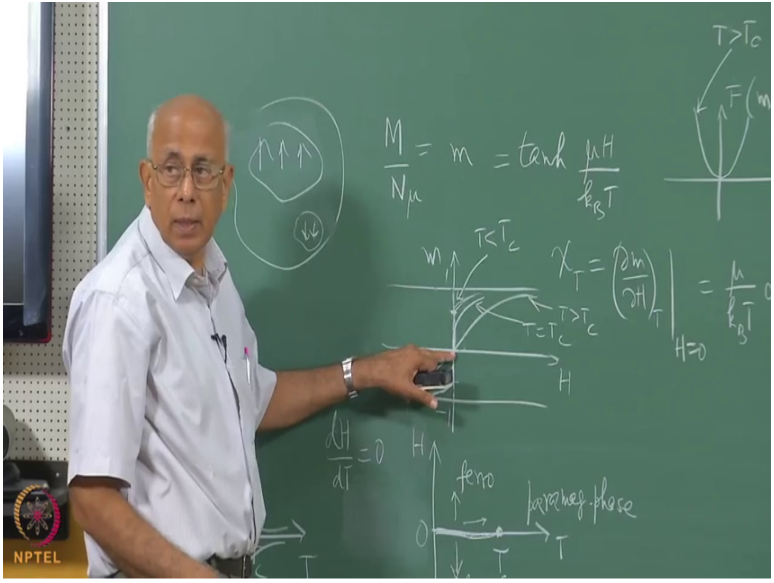

the critical region, this free energy, you still have zero as a solution, so it gets extremely flat in this fashion, in fact the leading term starts with the fourth power rather than the second power, so it is not a parabola

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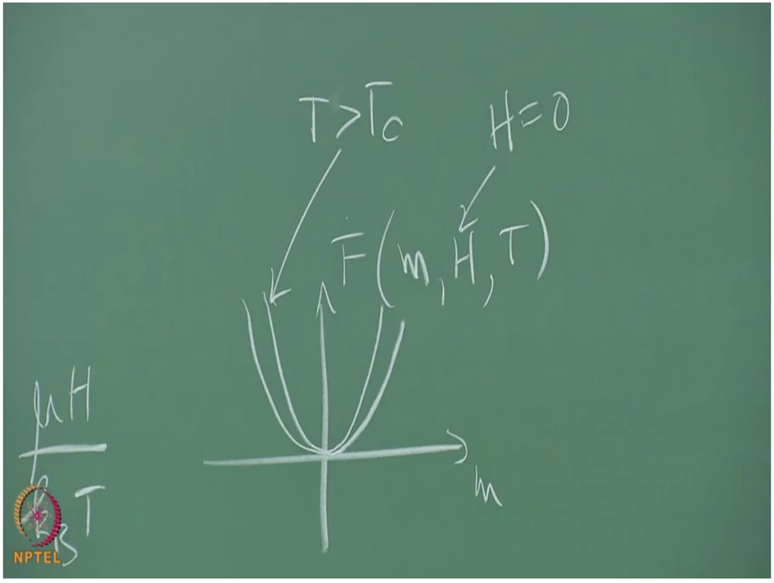

but it has got higher derivatives as zero as well and when you come down below T c it does this.

That is the way this parabola of this single maximum, single minimum

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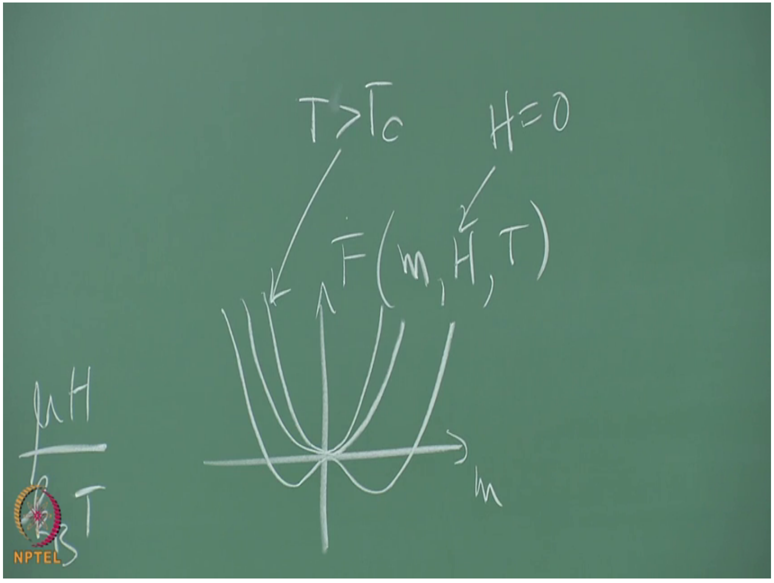

it breaks up, finally it degenerates into a curve which is not a parabola anymore near the origin but a fourth power, it starts and then for T below T c, even below T c this higher order minimum splits into two minima and a maximum in-between them, Ok. This is the

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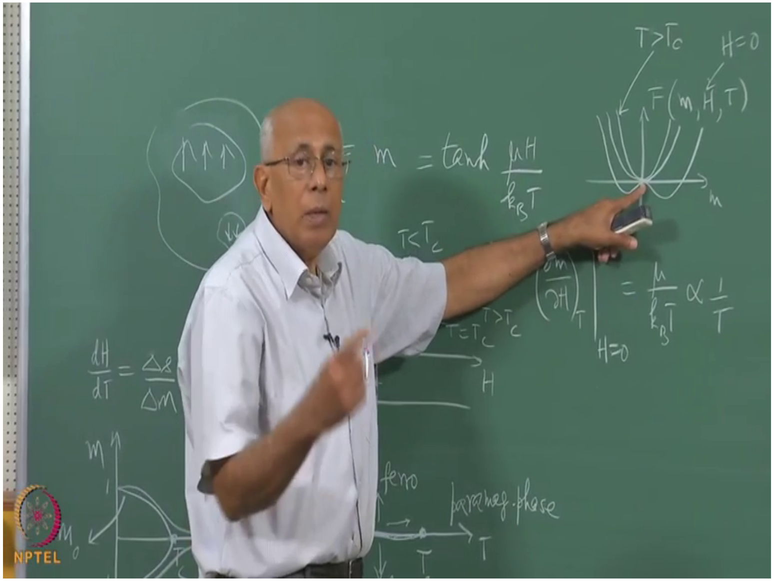

m equal to zero solution, that is unstable so you see this becomes unstable, this solution.

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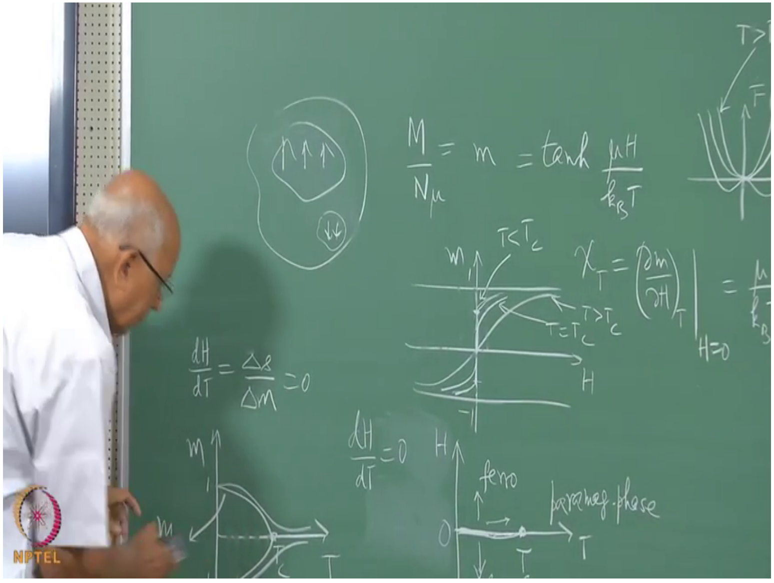

But you have 2 stable minima as solutions. They correspond respectively to this point and this point.

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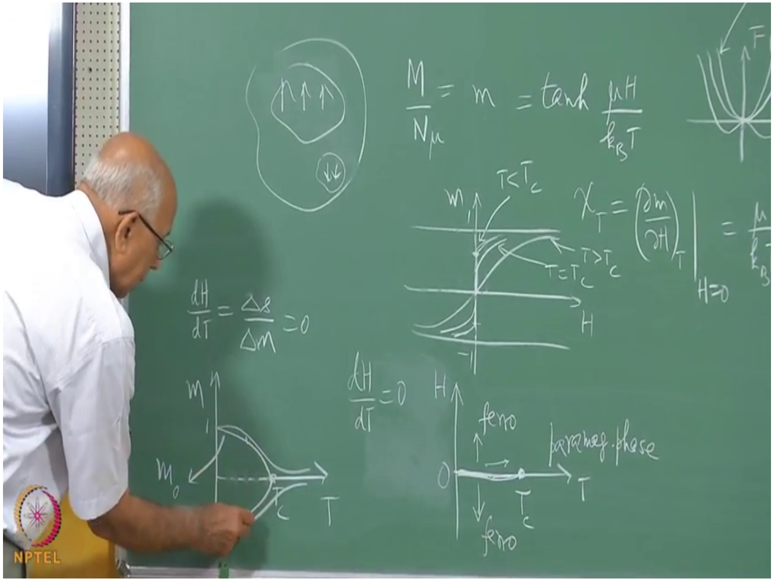

Since this happens smoothly and continuously as you cross T c it is called a continuous phase transition. There is no discrete jump, finite jump in the value of the order parameter in the magnetization.

(Professor – student conversation starts)

Student: 0:23:49.5

Professor: Pardon me? Yeah, this is at H equal to zero so it must be m naught

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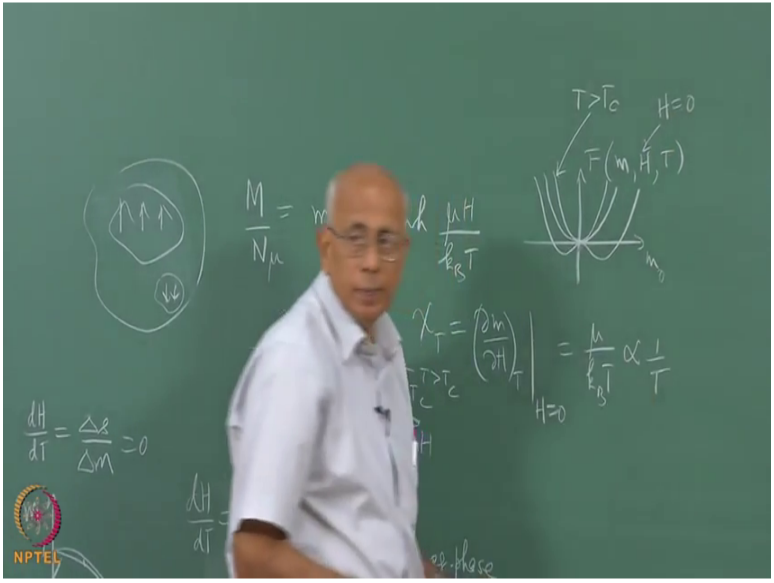

plotted with that...

Student: Minima is the m naught.

Professor: Pardon me? Well, Ok I am being a little loose here. In principle what happens in Landau theory is that you write it as a function of the thermodynamic variable, phenomenon logically and then you say that thermal, thermodynamic equilibrium state corresponds to an extremum of this F. So I have written a functional relat/relation, as a functional of this m and said that the minimum is at the average value which you already have here, Ok.

Student: That means the m naught should not be there in the axis.

Professor: Yeah, that is why I put an m there, right.

Student: No what I meant is that at the minimum you can...

Professor: Ok, I mean

Student: Because you are...

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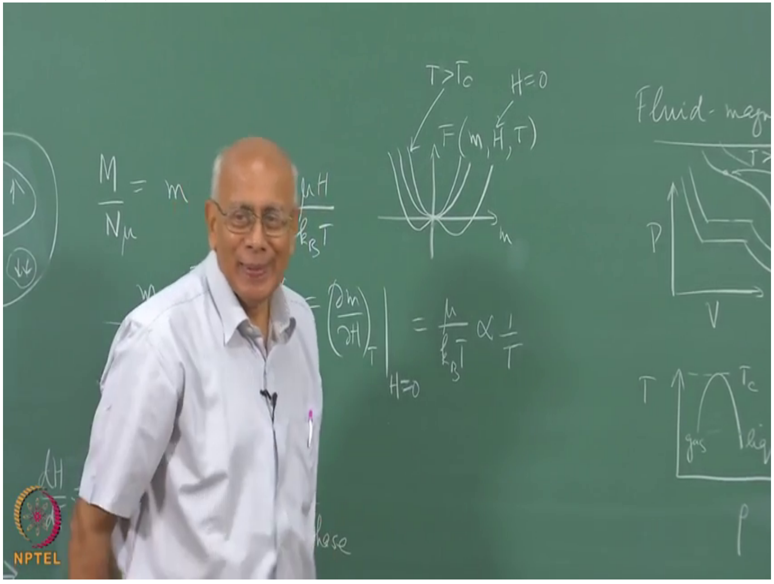

Professor: Yeah I know. I should put expectation value or something. We will do that. When we need it, we will be careful with our notation, Ok, alright. So those 3 figures and these 3 figures are sort of one-to-one correspondence.

(Professor – student conversation ends)

Today what we will do quickly now is I want to show you, this is a square root type of singularity. That is this curve here in this region; the m is like the square root of T minus T c, exactly the same thing in this case, parabolic

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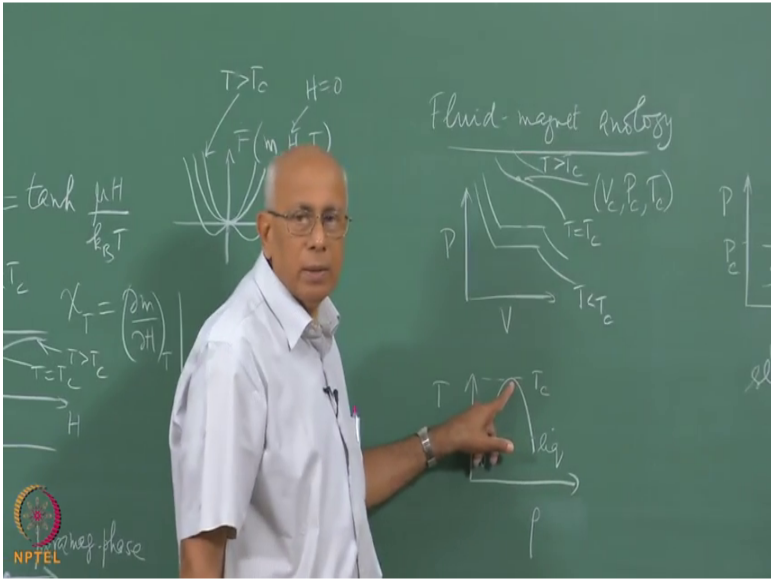

in this case. Then I want to show this graph here

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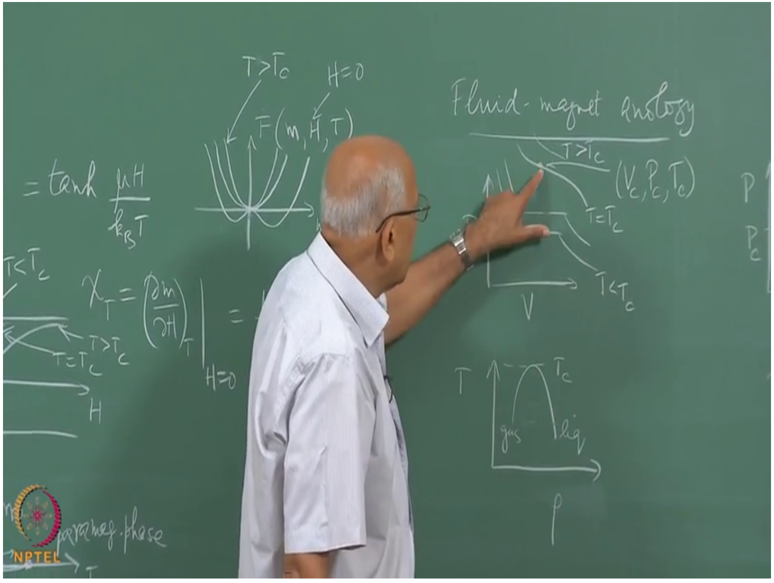

corresponds to the reduced pressure being a cubic function of the reduced volume. So remember I already defined P minus P c over P c equal to p and V minus V c over V equal to v in a little abuse of notation here

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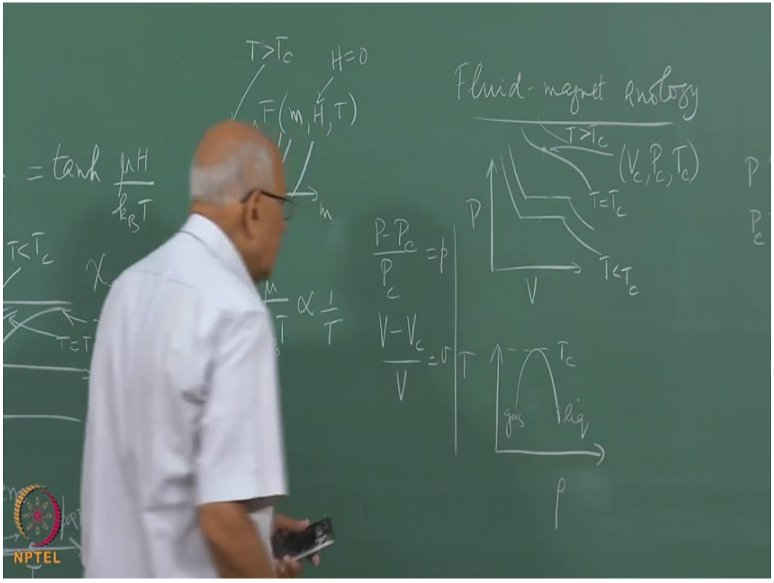

and I said in this region, focusing on this region and expanding it, it looks like this, v p I will just write modulus,

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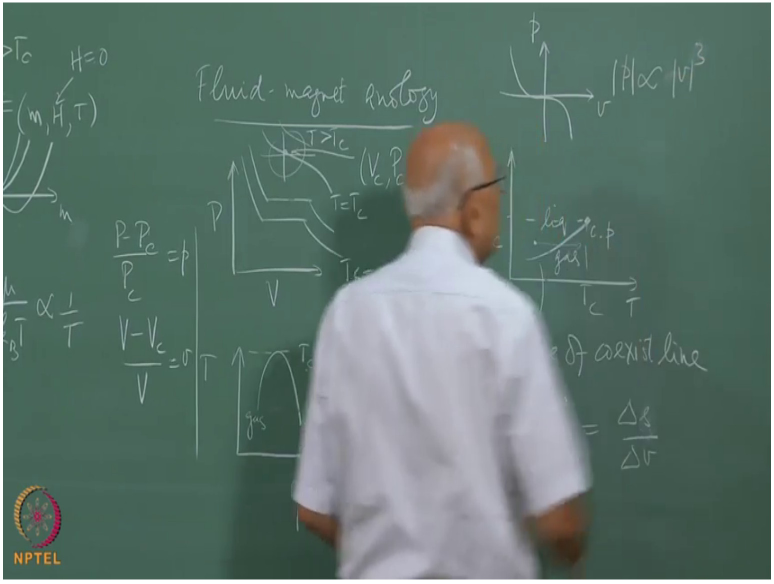

Ok

Cubic curve, I just want to show exactly the same way that at this critical isotherm I would like to show that H is proportional, it goes like n cubed on this critical isotherm, the curve that goes like this.

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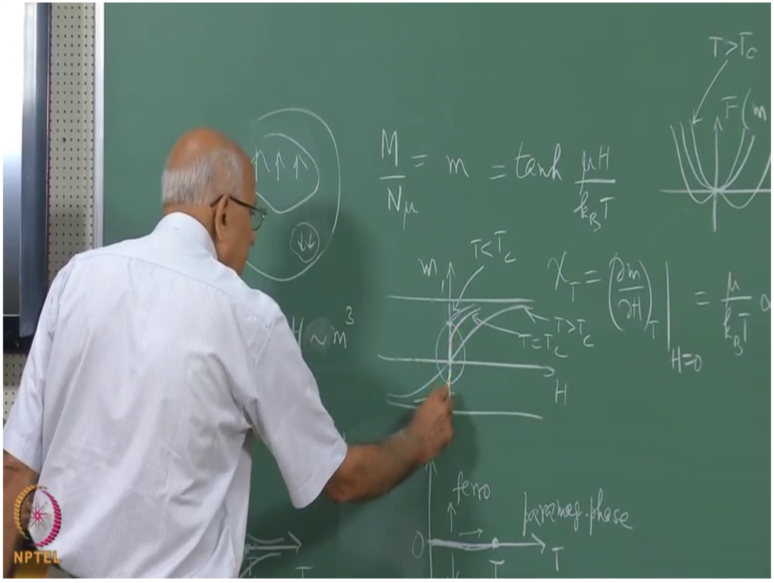

3 is a critical exponent and I want to show that in both cases, Ok, especially in this, alright. Now the point about critical phenomena is that these exponents, this 3, this half which we have still got to establish, we will also establish that this is not the correct formula; that you will get a susceptibility which goes not like 1 over T but 1 over T minus T c. So this is going to be changed to something which goes like

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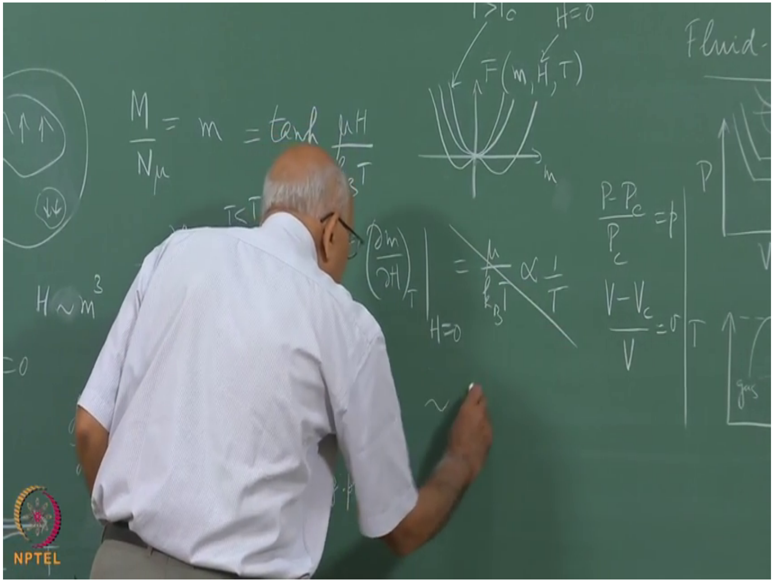

1 over T c minus T, this quantity as you come down

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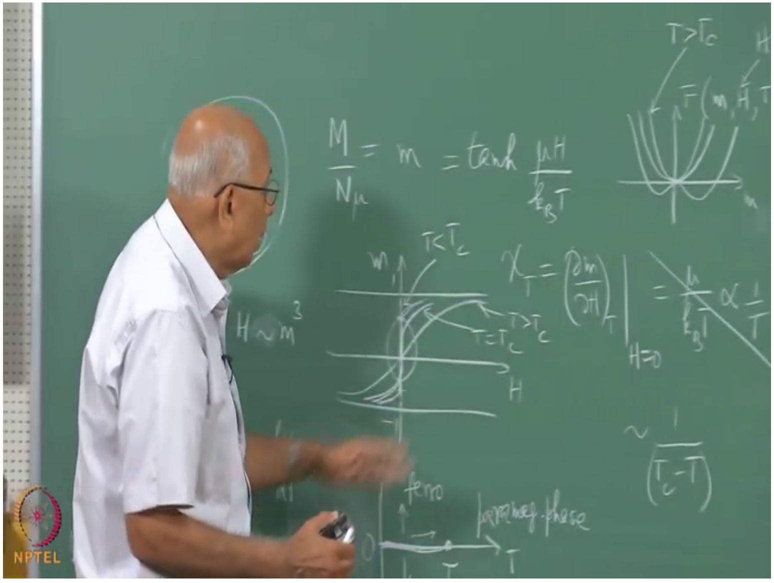

from the paramagnetic region.

There is an exponent 1 here. In exactly the same way, the analog there is the compressibility. That will also diverge on this graph like 1 over T minus T c, Ok in the critical region. Now you could ask one question which is what happens to the compressibility as I go from below the T c, below the critical temperature? Well nothing much happens there, it still diverges at that point. The question is can I define a susceptibility in this case at all?

Can I define, because the function is discontinuous? Remember it is defined in this fashion. So just to get it clear, I just draw 3 graphs now. In the para phase you had no problem. In the critical isotherm, you had this and I said

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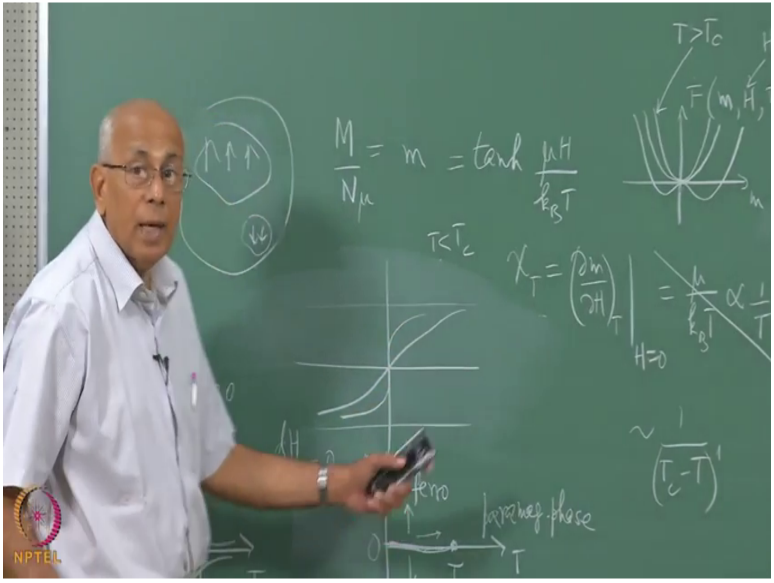

that it diverges, it blows up. It should because this is not linear, it is a cubic curve. So slope becomes infinite at that point. But below the critical point, my isotherms look like this. The function becomes discontinuous.

Can I still define a slope and a susceptibility for it? The function is not continuous. Can I define a susceptibility? Why not?

(Professor – student conversation starts)
Student: You cannot take derivative there
Professor: Why not? I can take a right derivative here; and I can take a left derivative here.

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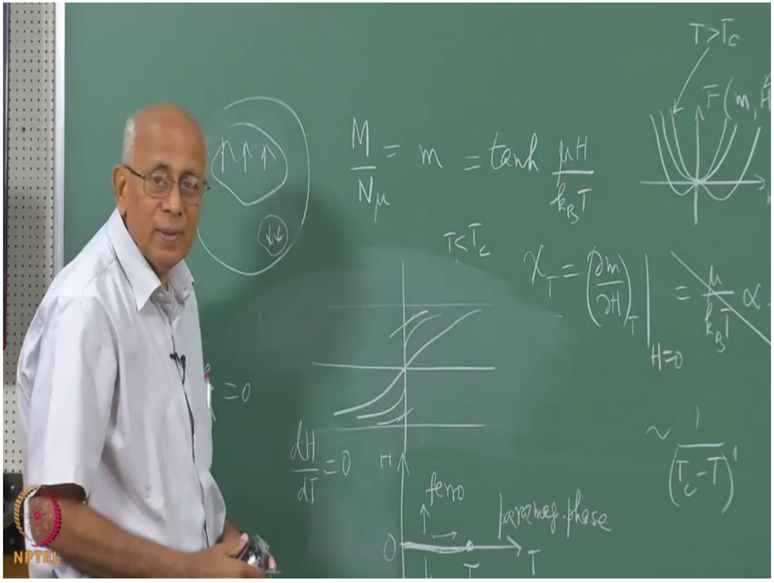

Student: Right and the left derivatives don't match.

Professor: We are not sure they will or not. But if there is symmetry in the problem they do. In this problem, there is, m to m, H to minus H there is, yeah

Student: Inside it is not there.

Professor: Yeah, so it is perfectly alright. You can therefore define a susceptibility. So here is an example for function as a finite discontinuity, it may still have a derivative at that point. They have a left derivative and a right derivative and they may even be equal, Ok. So it is perfectly alright in this case, alright. So this is, these are things which we have to deal with now.

(Professor – student conversation ends)

So what we will do is, while we have a model here we need to improve this model. So that is what I am going to do now. So we are going to take this model here and write in more sensible fashion and see whether we get the results we want or not. Now here goes.

In all of condensed matter physics, magnetism is always served as a kind of paradigm, especially spin models served as a kind of the basic, fundamental paradigm for various modeling all sorts of phenomena. And among them the simplest model is the so-called Ising model which has served as a model for not just magnetism but binary alloys plus a whole lot of, today a huge number of physical phenomena can be modeled by the Ising model, Ok.

But I will try to motivate it in the simplest manner possible. What we start by saying is that, if you apply an external magnetic field on a system of magnetic moments, these moments would try to align along the field. Lowest energy configuration is along the field. But in addition to that, the moments may be interacting between each other, Ok. What is the simplest interaction between 2 magnetic moments?

If I have a moment mu 1 and a moment mu 2 in this fashion, you have to ask

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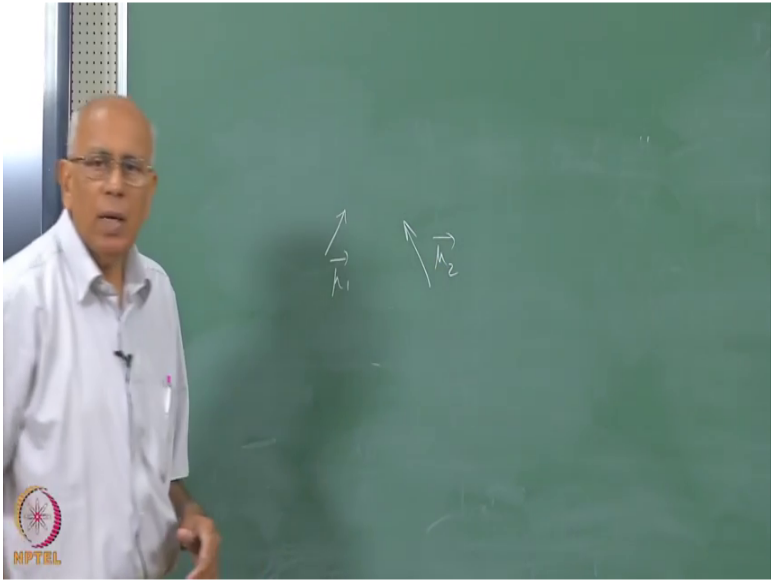

what is the potential energy of interaction between them. Now this is not a trivial matter because this is going to depend on not only the distance between them, these are point dipoles. So you must imagine they are, their physical extent is actually atomic in size so it is very small. But it not only depends on this thing here but it also depends on the relative orientation of mu 1 and mu 2 with respect to the vector joining them. And you got to construct a scalar.

Now one term

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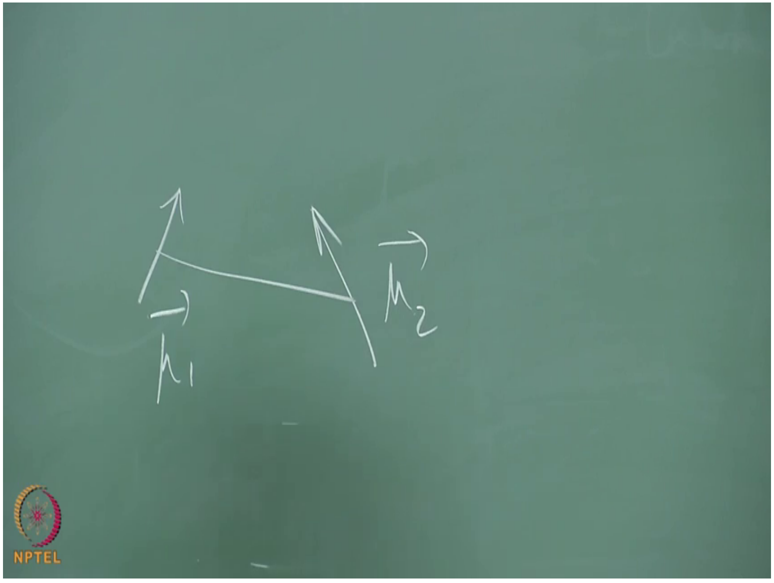

in that scalar if you remember from electrodynamics would be of the form mu 1 dot mu 2 over r cubed. And then there is another term which goes like mu 1 dot r, mu 2 dot r divided by r to the power 5 with a different coefficient. So in general, it is fairly messier quantity. But we are now concerned about situation where you can only point upward down. Nothing more than that. We also want to ask

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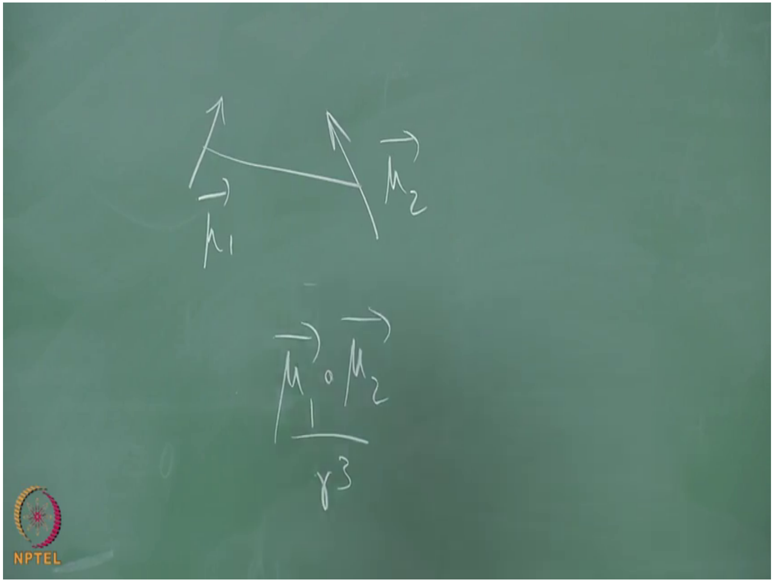

what leads to magnetism, what leads to ferromagnetism.

On the face of it, it looks like this kind of interaction cannot lead to ferromagnetism because when you have 2 dipoles, this and this, then north pole and north pole, they would tend to align like this because that is the lower energy configuration. So on the whole when you average you are going to get zero because plus and minus would just cancel out, right. So dipole ordering generally tends towards the state of zero magnetization, anti-ferro magnetization at best, certainly not ferromagnetism.

The origin of ferromagnetism is much more subtle and this is what Heisenberg discovered, one of his biggest discoveries. He discovered that purely due to the fact that electrons are fermions and obey the Pauli Exclusion Principle, when you bring 2 atoms together their electronic clouds would tend to repel each other. There is a statistical repulsion and you can now ask what happens to the energy levels of this combined system here.

Then you can write down after a lot of rigmarole, an effective Hamiltonian which tends to favor under suitable conditions a parallel alignment. It is purely a quantum mechanical effect. It is called the exchange interaction, Ok and this exchange interaction depends on the overlap of wave functions of electrons belonging to neighboring atoms if you like, and depends on other details of the system. There are situations, there are some substances for which the ordering tends to be anti, the ordering is anti-parallel for lower energy but most of the ferromagnetic substances it is always plus up here.

So essentially an s 1 dot s 2 kind of interaction with a minus coefficient which tells you that the parallel ordering is favored leads to ferromagnetism, Ok. It is not a direct dipole-dipole interaction. Because it is classical, that is long range, that dies down in this fashion etc. This exchange interaction comes about because if you have one atom here with an electron which has got a wave function like that, and another one with an electron like this, this overlap integral leads to that effective so-called exchange constraint.

Since these are bound electrons the wave function dies down exponentially with distance. So this interaction which leads to ferromagnetism is a very short range interaction, essentially acts only on nearest neighbors in a lattice for instance.

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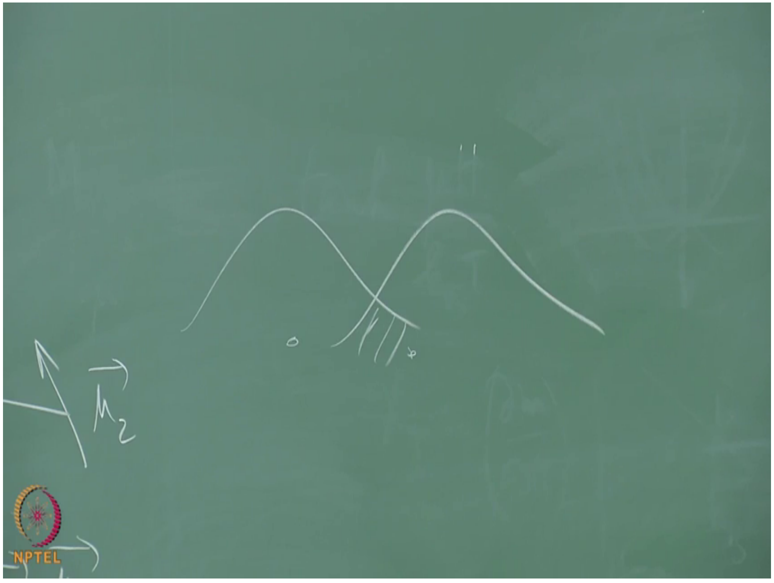

It is exponentially short in range but that suffices to produce long range order. In a big crowd, if each person acts, suddenly one person starts looking in one direction and he or she can only affect his neighbor and he starts looking this way too, then this short range interaction can spread over the entire crowd and soon everyone gets aligned.

So short range interactions, if sufficiently strong, can lead to long range order, Ok. That is an important lesson and that is what happens in ferromagnetism. So to cut a long story short, in our model, in the Ising model, the Hamiltonian looks like this. Now we start writing a Hamiltonian, then we write a free energy and so on and so forth. So we assume that the system is on some kind of lattice for example, so lattice with sites labeled by i, j etc, these are site labels

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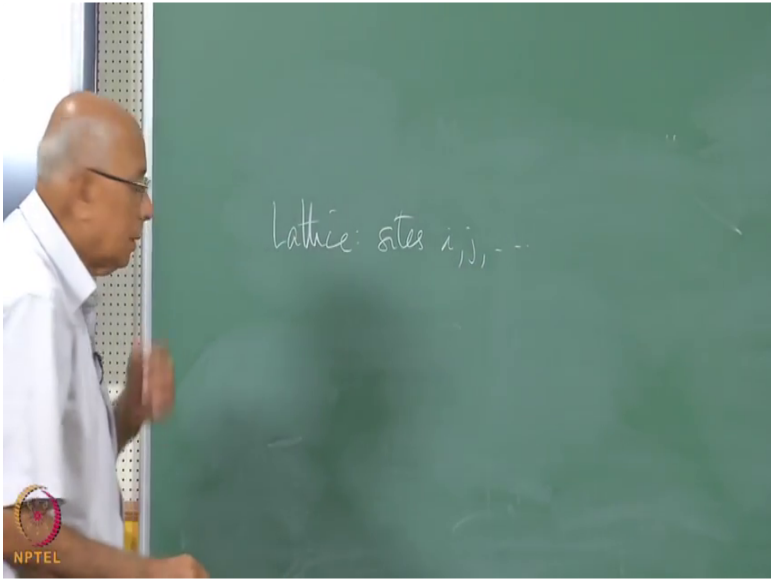

and we are only interested in the magnetic properties so I write down only the magnetic degrees of freedom here.

I write the Hamiltonian only as far as the magnetic moments are concerned. That is all that we are concerned with. So the Hamiltonian H is equal to minus, I put an external field H and then a summation over I, let us say S i is of the spins or moments,

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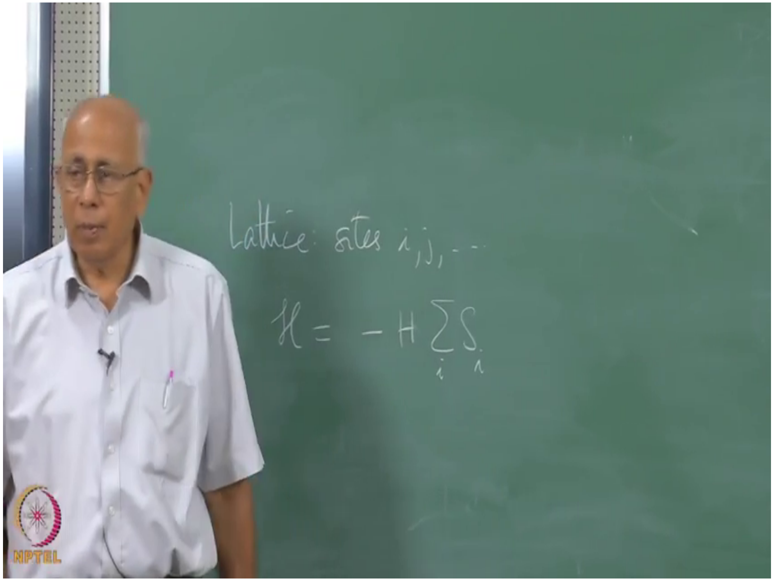

from now on I am going to get rid of this mu, little mu, I am going to subsume it in whatever parameter you, so this is going to be a pure number which I am going to take to be plus or minus 1, the mu has gone into this, out here. The temperature is going to come out, externally, separately.

So all that has been done and I have chosen proper units etc etc. It just makes the notation much simpler. So I have this, minus, this is under a uniform magnetic field H across the entire lattice minus a term which describes the interaction, which is nothing to do with the external field at all. And we are interested in ferromagnetic ordering so that is minus 2 summation j which depends on 2 sites at a Time:, the spins of 2 sites at a Time:, so i and j, j i j, the strength of the interaction or overlap integral depends on this i and j out here and then S i S j.

Since I have assumed only one direction of magnetization, one axis everything is scalar here. Otherwise I would have to put vectors, these would be operators then you have quantum mechanical three dimensional problem for the spins which would be r and s. We are now doing just what is needed; the essential part of the physics as far as magnetism is concerned for our purposes at the moment, Ok.

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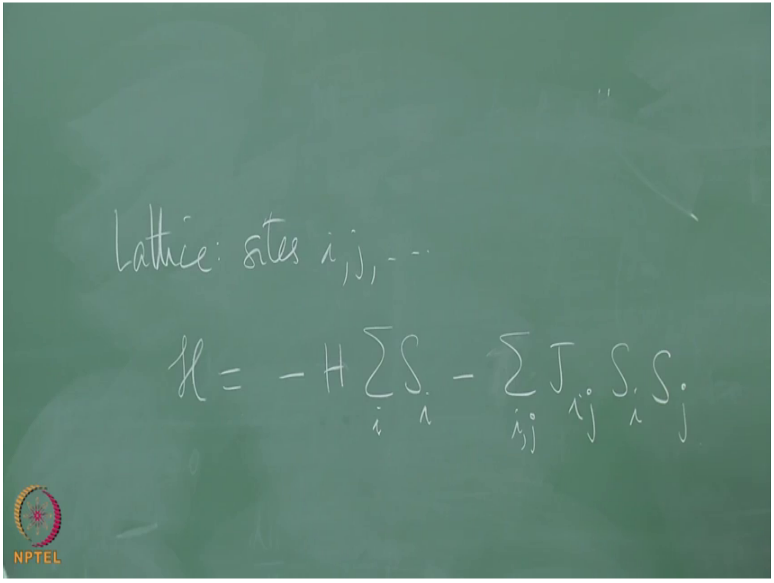

Lattice: sites $i, j, \ldots$

$$\mathcal{H} = -H\sum_i S_i - \sum_{i,j} J_{ij} S_i S_j$$

I will assume that each S i is equal to plus or minus 1,

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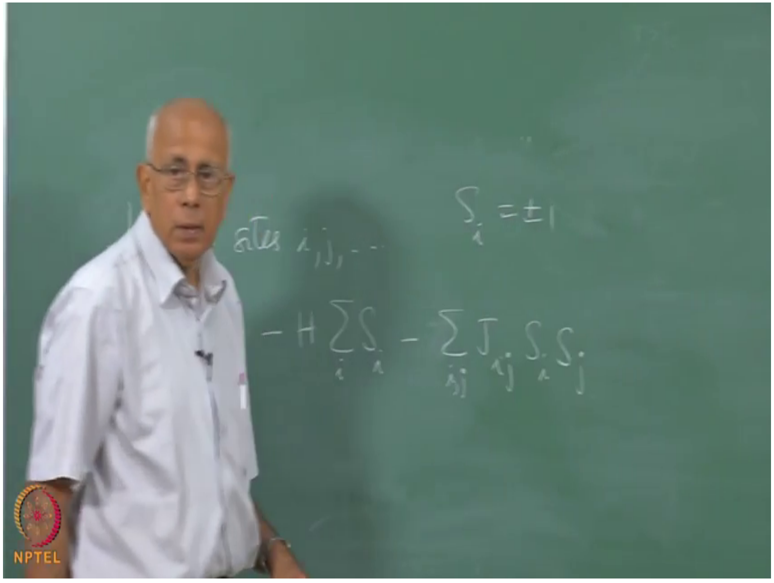

Ok and I am going to take traces. Now if I have n of these spins, then since each S i is plus or minus 1, I have 2 to the n possible microstates. Whenever I say trace, I mean trace over all the possibilities, Ok and of course we will put in the Boltzmann factor very shortly here. This thing is called an exchange integral and I have already said, that since this is a quantum mechanical effect and the interaction range is very short, i and j have to be nearest neighbors otherwise you do not have an interaction at all. That is generally denoted by putting a bracket like this.

This means i…

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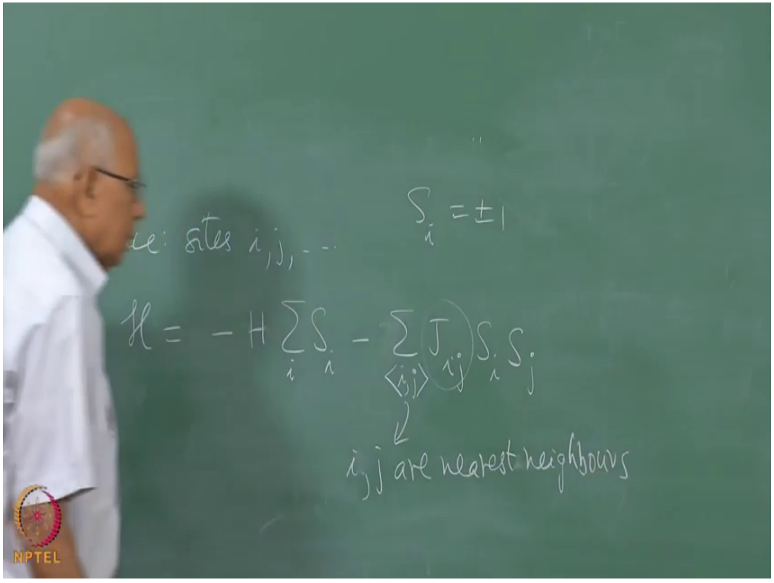

Ok. How many nearest neighbors do you have on a line, on a linear lattice?

(Professor – student conversation starts)

Student: 2

Professor: 2. What about a square lattice? Well a square lattice will have, so if this side has the nearest neighbor 1, 2, 3 and 4.

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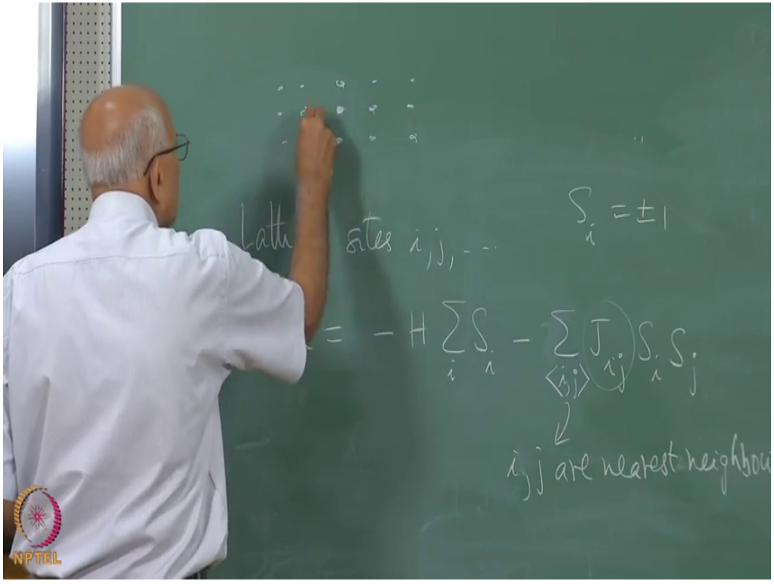

Latt… sites $i, j, \ldots$

$S_i = \pm 1$

$= -H \sum_i S_i - \sum_{\langle ij \rangle} J_{ij} S_i S_j$

i, j are nearest neighbou

A D-dimensional lattice, hypercubic lattice? 2 D. In three dimensions you have one on top, one below, one behind, one front etc

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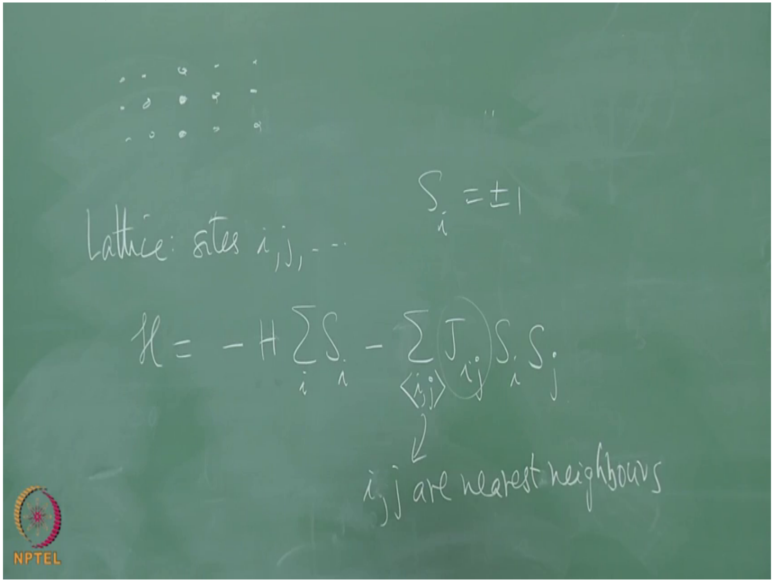

Lattice: sites $i, j, \ldots$

$S_i = \pm 1$

$\mathcal{H} = -H \sum_i S_i - \sum_{\langle ij \rangle} J_{ij} S_i S_j$

i, j are nearest neighbours

so you have 6, so you have 2 D nearest. Whatever it is, it is a finite number, that is very crucial. It is a finite number. Otherwise you are finished. You will see in a second, you will see, if that is not true, you are finished.

(Professor – student conversation ends)

Why do I say that, you are finished if it is not true? Because you take the expectation value of this guy, you have to average over all the sides, over all the possibilities. This fellow will tell you, for a given value of this j, i j, if both are plus it gives you a number 1. If both are minus it gives you 1. If one is plus, the other is minus, it gives you minus 1, etc. And you have to add over all these things. But the point is, if this is summed over all i and j, 1 to N, capital N, and this is the interaction, some fixed number, let us suppose that it is the same number, no matter what the range, what the distance between i and j is.

Then you see it is clear that this term is of order capital N squared. Because I pull out the j outside, it is some finite number. I have to sum over i and j and it is going to be of the order capital N squared. That violates the extensibility. This has got to be linear in the system size, Ok.

So an infinite range interaction, with the same interaction strength is going to finish you off. It has to drop down. It has to drop down. So that is not even a thermodynamic system. Once I put nearest neighbors

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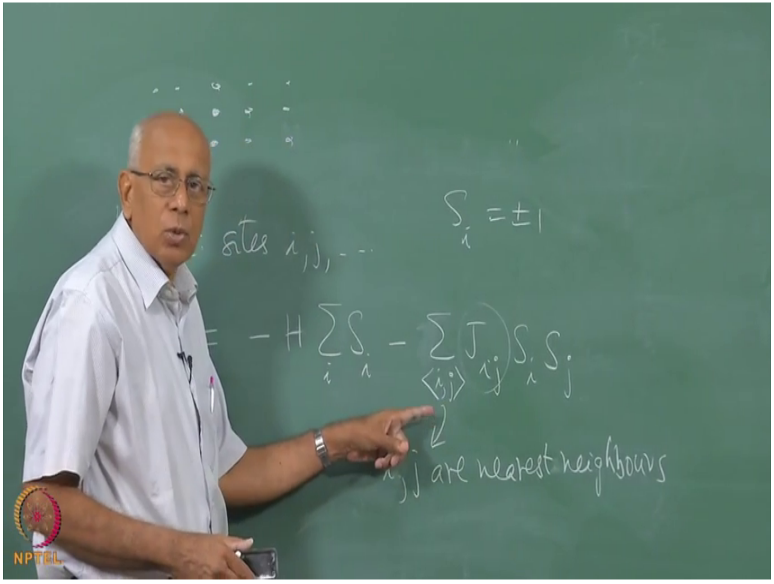

there is no problem. It is a finite number. I go one step further and say this guy is a constant across the nearest neighbors.

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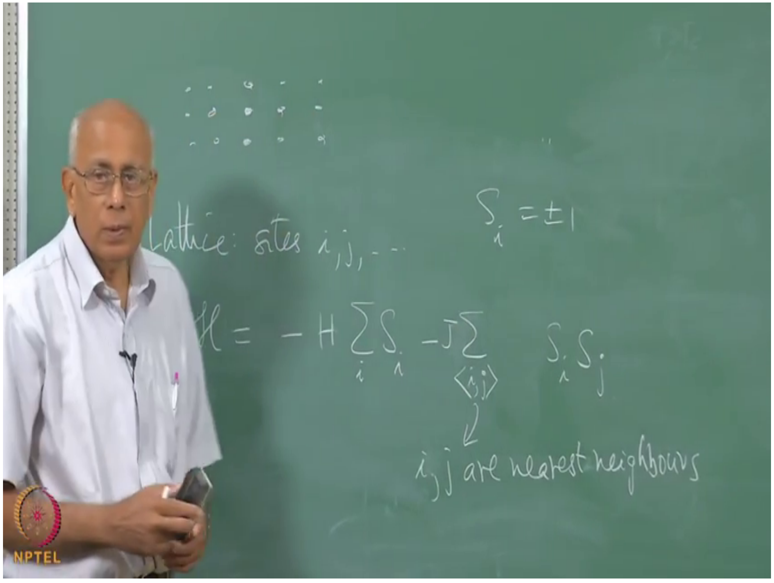

In other words, this interaction is the same as this, is same as this, is same as this, Ok.

And what is the job? The task is to say, Oh this whole thing is put in a heat bath and I am going to find the partition function is trace e to the minus beta H. This trace stands for saying that this set S i is equal to plus or minus 1.

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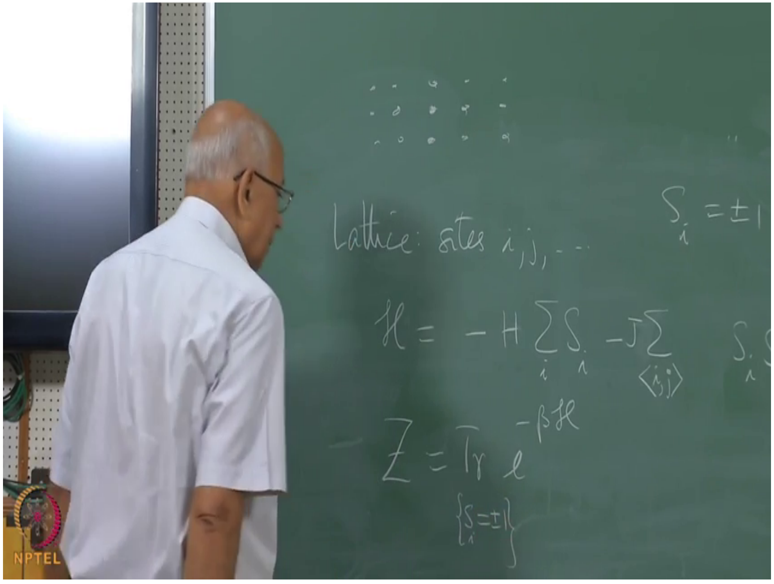

Every S i takes values plus 1or minus 1 and I have to find this whole thing. Now is this such an impossible task? By the way you can do this in one dimension, on a linear chain on suitable boundary conditions but you cannot do this in higher dimension so easily, Ok.

Even if you switched of this field, you still cannot do it in higher than 2 dimensions, Ok. In 1944, Onsager solved this problem of finding this exactly, for this problem without this on the square lattice. It is a brutal force. It is very, very complicated proof but now it is a standard proof. It is a long, it is a long derivation but it can be done explicitly. And then the idea is to find out what happens when the number in infinite, N becomes infinite. As long as N is not infinite, you can't get any singularity because remember, Oh...

By the way I should also say the free energy is minus k Boltzmann T log Z and once you have

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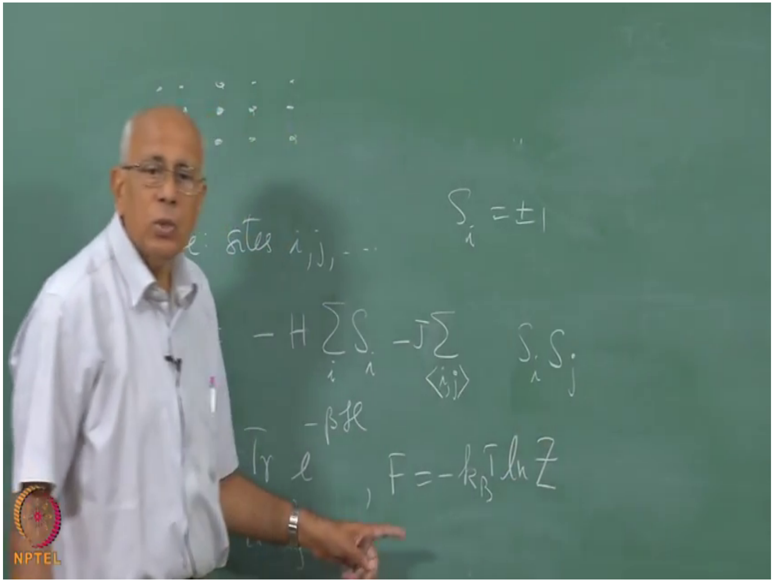

the free energy, all thermodynamic quantities can be computed, their variances everything can be done in equilibrium statistical mechanics. What is the complication? The complication is this. Because you have that interaction, you have to enumerate all possibilities. The possibilities of this will affect what happens here. But the possibilities of this will affect what happens here and therefore here and so on.

So it is an extremely complicated problem and there are very, very few exactly solved statistical mechanical problems. This is called the two dimensional Ising model, Ok. No one has succeeded in doing it with the field present. And no one has succeeded in doing anything beyond two dimensions, in three unknowns no one knows to do this. It is probably not doable in this form anyway, Ok. Now our task is going to be much simpler.

We are simply going to argue that each magnetic moment in the ferromagnetic phase; we will see an effective magnetic field due to the fact that all its neighbors tend to be aligned in the direction of the field, Ok. So this term is going to be replaced by an effective term. We want to eventually find the expectation value of this, to find the internal energy. Because that is the expectation value, the Hamiltonian with respect to this weight function. The expectation value of this is essentially the magnetization what I called m.

So let me write that down.

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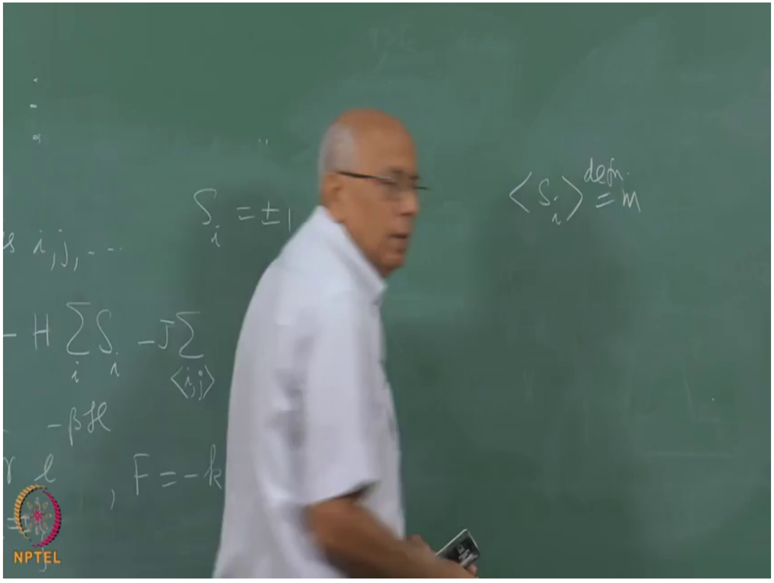

Definition, in the units I have chosen, I have put mu; I have subsumed it inside this cons/constant, inside the field etc. So it is the magnetization per particle which has this quantity here. The internal energy will be the expectation value of this guy, which is expectation here and then expectation here, Ok. So the problem will arise because we do not know how to compute this expectation.

The m comes out here

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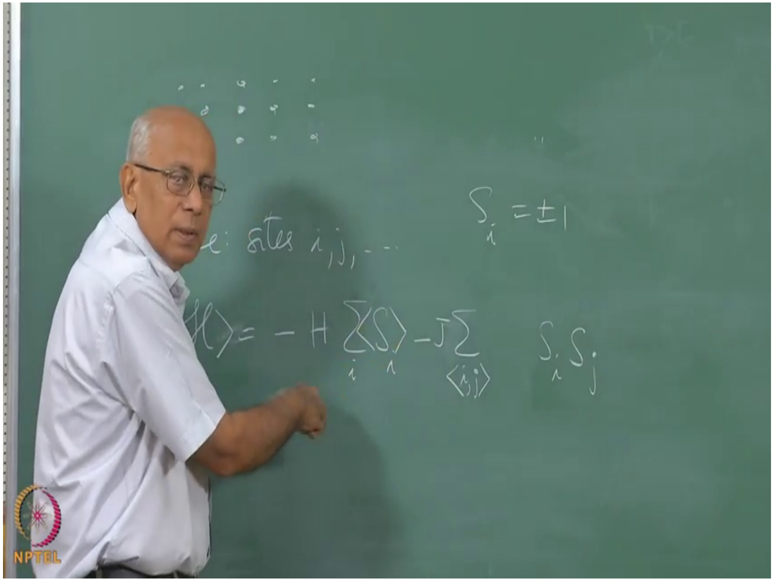

and by the way, it is independent of i because in infinite lattice, every magnetic moment is the same as every other moment on the average. But you have this quantity here and that is bad news. Because it is correlated. So what this assumption of mean field theory does is the same as the van der Waal's approximation in the gas case is to say this thing here, if you look at a particular site i, I look at a particular site i, then j runs over all its nearest neighbors

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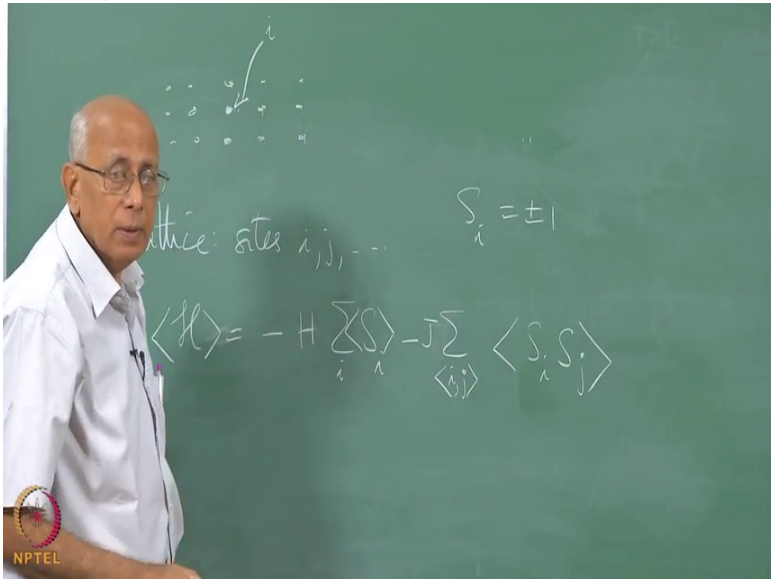

and as far as this is concerned, I can replace this S j here by the expectation value of this S j, Ok.

So if I make that approximation then, and then I take expectation values. But of course there is nothing to take expectation value here because is already an average. It factors out in this fashion. What has been the effect of that? The effect of that is to write this as equal to minus summation i some H i S i, an effective field at every point with a minus sign where H i is equal to external field plus J Time:s summation j equal to nearest neighbor of i S j.

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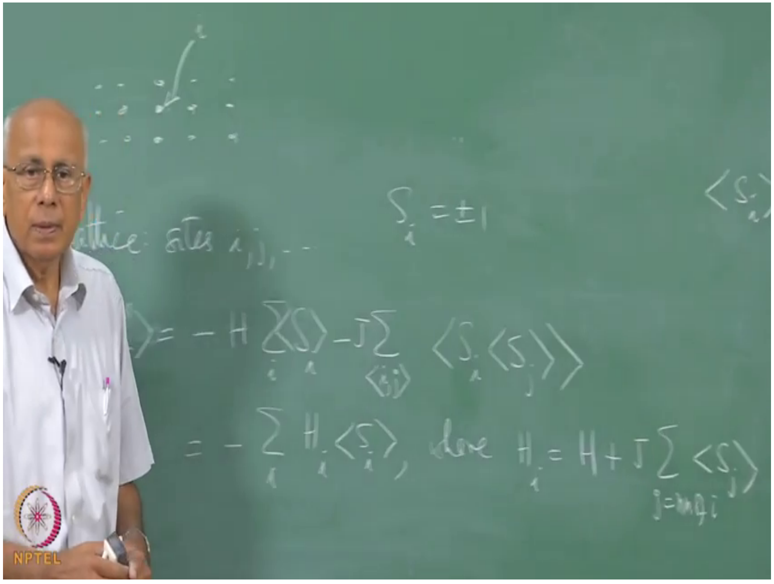

So essentially it is as if I have put this spin here in a field which is not only the external field H but also the internal field due to the fact that these guys have got a magnetization on the average. So it has been put equal to S j here. So it adds to the external field, some effective field, Ok. This is called the

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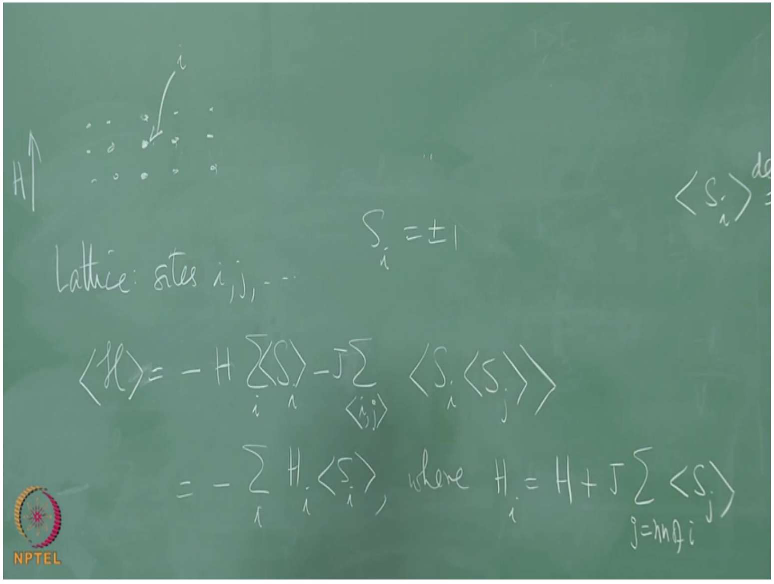

molecular field approximation. This is the old famous Weiss molecular field approximation. It is changed this field to this.

But now we can go further. We have already said j must be nearest neighbor of i. So let us put in, let us say we got, we are working on a D dimensional hypercubic lattice because one of the things we learn in critical phenomena is that things are very sensitively dimension dependent. How many spatial dimensions are you in? The effect of thermal fluctuations depends very sensitively on what dimensionality you are in.

We will see, in one dimension no ordering is possible because thermal fluctuations destroy this order completely. In two dimensions it is possible in this class of models. As you go in higher dimensions, it gets more and more possible, Ok. So the power of thermal fluctuations or quantum fluctuations for that matter to restore symmetry and stop ordering gets lower and lower as you increase the dimensionality. Because the number of neighbors increases and tends to order the system, Ok.

See all our phase transitions is a competition between disordering tendency of entropy and the ordering tendency of the internal energy, Ok. So you have F, this is just a heuristic argument; you have

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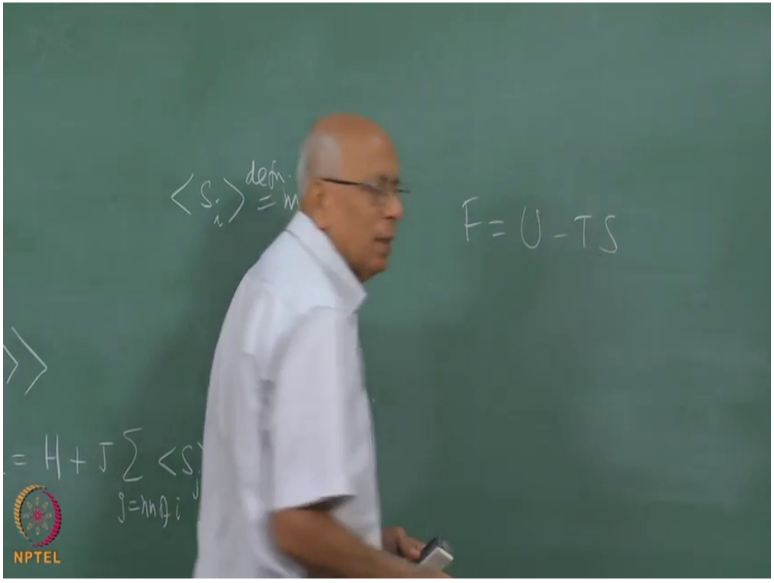

U minus T S. This will tend to order, that will tend to disorder. And the competition between the two will decide the minimum of F that you are at. At low temperatures, sufficiently low temperature, this is low and U win and the system has a minimum controlled by the internal energy.

On the other hand, at sufficiently high temperatures, that wins and you have a disorder state. That is really all that happening but that is saying it in very, very bare bones. The rest of it is matter of details to work things out. But it is worth remembering this. So what does this become? There are 2 D nearest neighbors and each of them has the same exchange constant j and this is m itself. So this is equal to H plus 2 D J m, in this fashion, right?

(Professor – student conversation starts)
Student: Can we call it 0:50:33.8 H effective
Professor: Yeah, but it was for each side, so H effective.
(Professor – student conversation ends)

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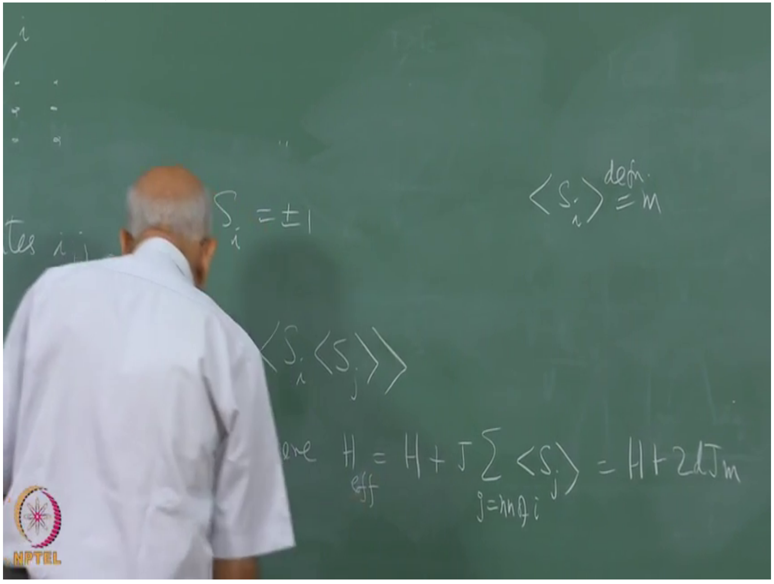

Now let us go back and put this in our model, Ok.
(Professor – student conversation ends)

By the way I wrote our model down by averaging the value of mu up and down using the Boltzmann factor. We could have written a partition function for it. Each of them has a partition function for t. Partition function e to the beta mu H plus e to the minus beta mu H, the partition for the full system is that to the power N, that quantity is twice cosh beta mu H to the power N. And then I can take the free energy minus N k T log Z and found thermodynamic quantities. We could have done that too.

So since you have Z equal to e to the beta mu H cross …

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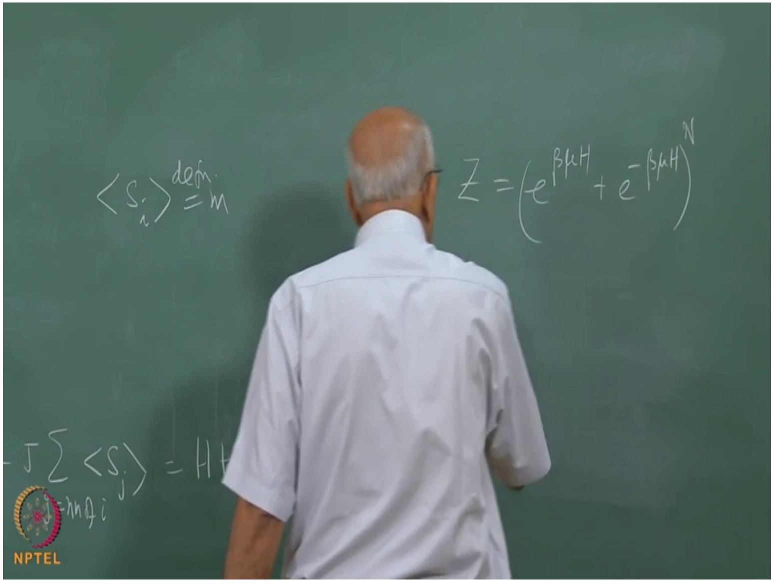

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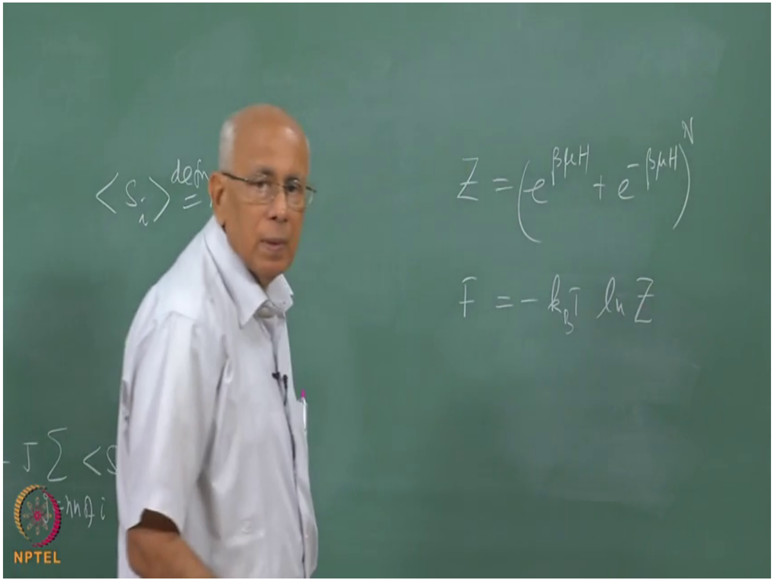

the magnetization equal to, it is after a Legendre transform. Remember that you have the free energy which has S d T minus a V d P or P d V depending on which free energy you are talking about, since we are using H as the control parameter, we are using P which is the analog of the Gibbs free energy, so there is going to be minus m B H. Ok.

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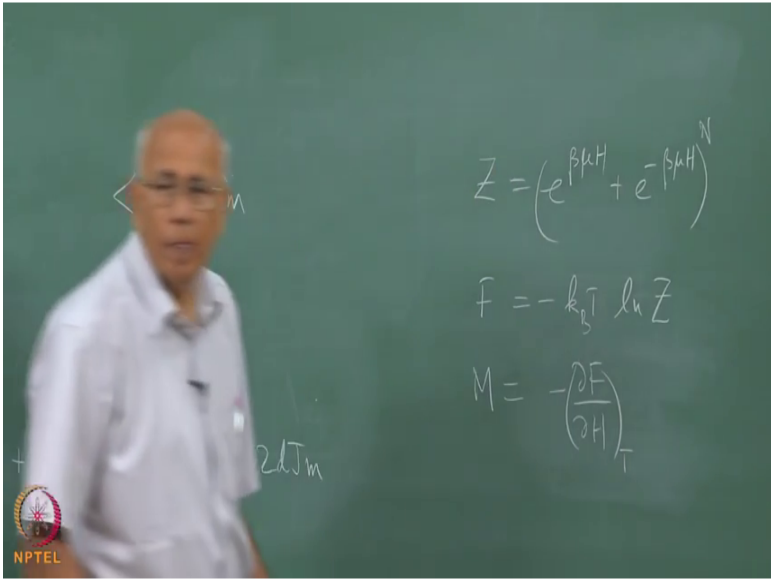

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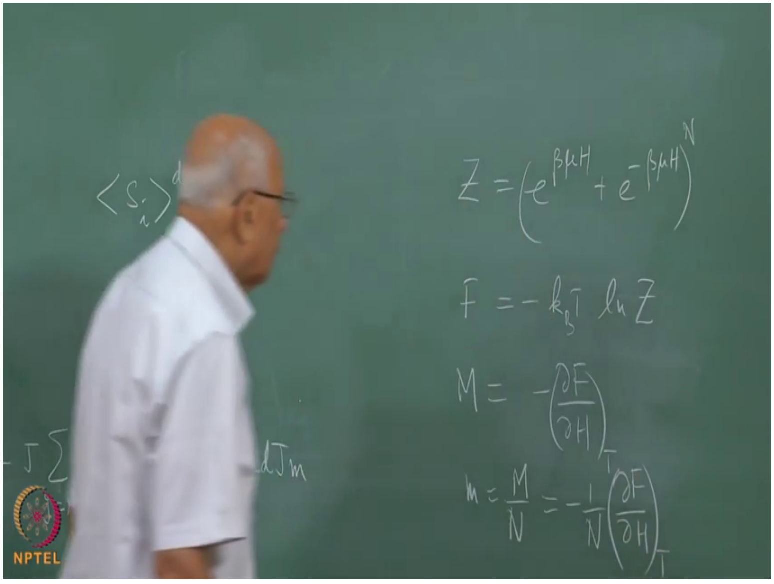

(Professor – student conversation starts)

Student: We have already worked out m equal to tan...

Professor: Precisely, so I want you to take this guy and show that this is, turns out to be tan hyperbolic, exactly

(Professor – student conversation ends)

I just argued saying that if it is up, the magnetic moment is mu, if it is down, it is minus mu. We took the average over the Boltzmann

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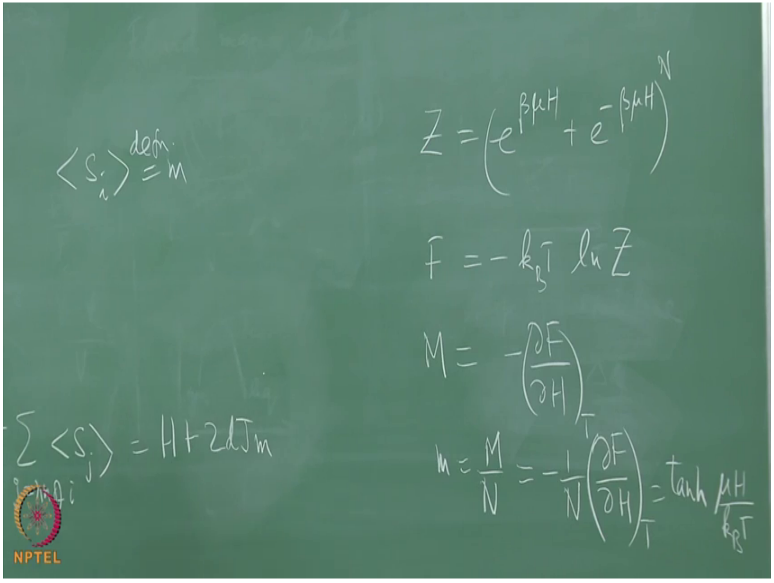

$$\langle S_i \rangle \overset{\text{defn.}}{=} m$$

$$Z = \left(e^{\beta\mu H} + e^{-\beta\mu H}\right)^N$$

$$F = -k_B T \ln Z$$

$$M = -\left(\frac{\partial F}{\partial H}\right)_T$$

$$m = \frac{M}{N} = -\frac{1}{N}\left(\frac{\partial F}{\partial H}\right)_T = \tanh \frac{\mu H}{k_B T}$$

$$\sum_{j} \langle S_j \rangle = H + 2dJm$$

factor and did it directly but you can go through the rigmarole here. By the way this is helpful because we can write a formula for the susceptibility which we are going to use. So kai t equal to delta m over delta H at constant T

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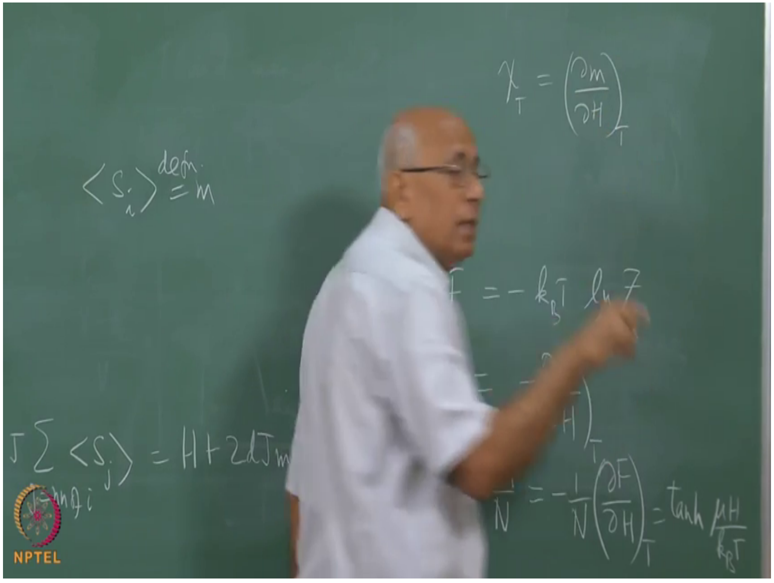

$$\chi_T = \left(\frac{\partial m}{\partial H}\right)_T$$

$$\langle S_i \rangle \overset{\text{defn.}}{=} m$$

$$J\sum_{j} \langle S_j \rangle = H + 2dJm$$

and zero field, and zero field, so we can write it from this. We can write one more thing.

This is equal to minus 1 over N delta m over delta H so, Ok. The susceptibility cannot

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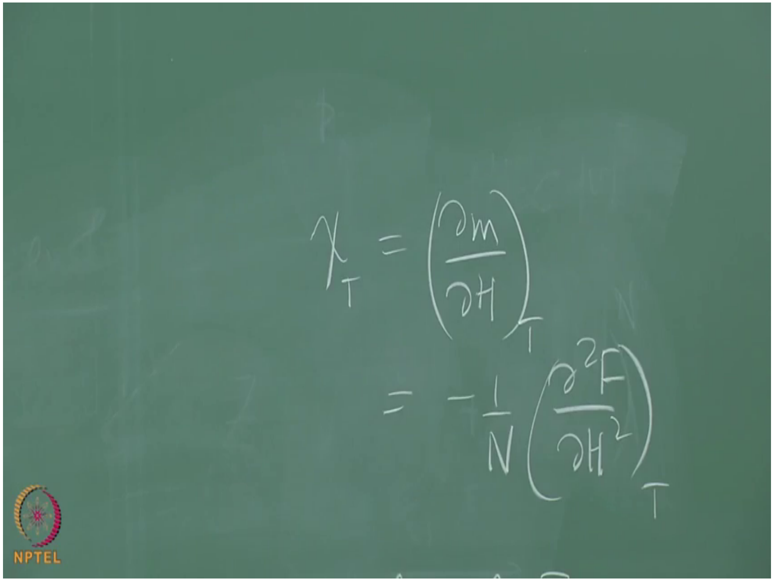

$$\chi_T = \left(\frac{\partial m}{\partial H}\right)_T = -\frac{1}{N}\left(\frac{\partial^2 F}{\partial H^2}\right)_T$$

be an extensive variable, it is intensive, it is like a bulk modulus or something, cannot depend on system size. This fellow is an extensive variable that is why got an m here, this is an intensive variable, definitely. Ok. Like I promised the second derivative of a free energy would always be some response function, in this case susceptibility. We are going to use that in a minute.

So we have this and let us it put back in our equation of motion. So it says, purportedly, m equal to tan hyperbolic mu over k T H plus 2 d J m, Ok. That was our

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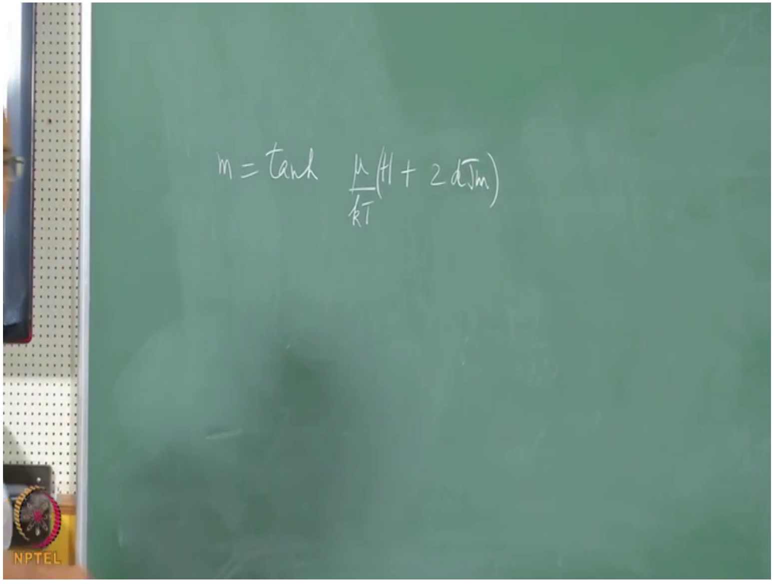

$$m = \tanh \frac{\mu}{kT}(H + 2dJm)$$

magnetic equation of state. This is our van der Waal's equation

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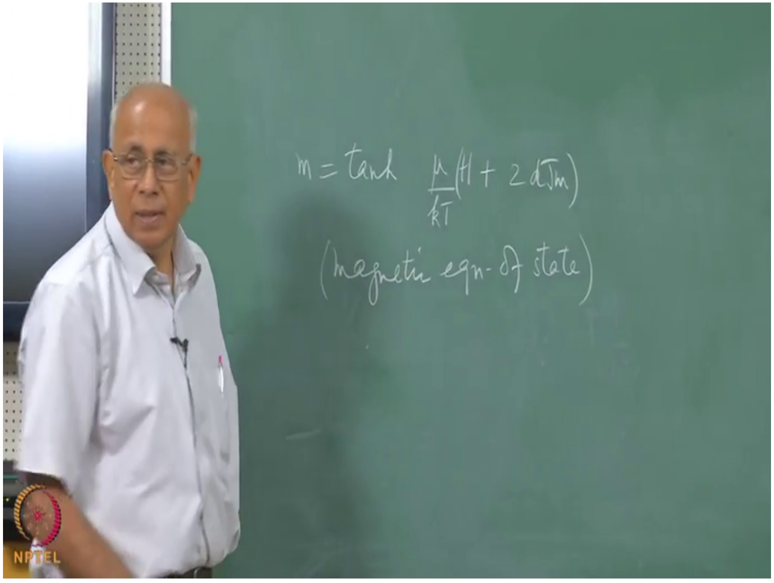

if you like in this mean field theory.

First order of business let us find out what is T c from this. Let us find out what is the spontaneous magnetization. So spontaneous magnetization. What are those intercepts? m naught is given by tan hyperbolic 2 d hey I got rid of this H mu, I subsumed in this in my notation , 2 d contour 0:55:35.9. So let me write that 1 over k T.

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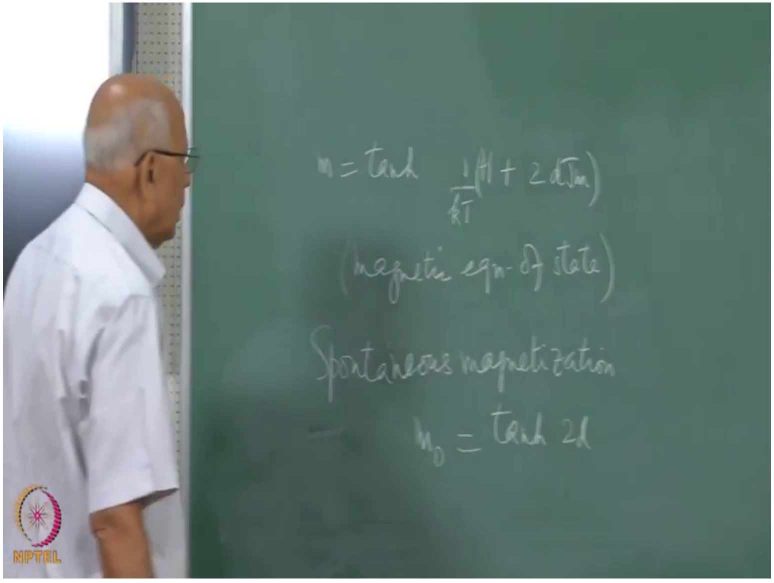

So tan hyperbolic 2 d j m over k Boltzmann T, m naught. I put H equal to zero. By the way

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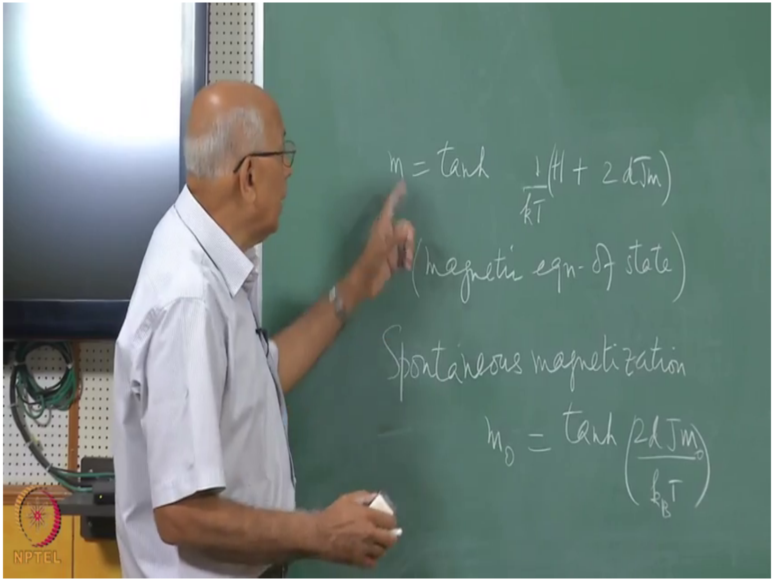

this is an equation which is a transcendental equation. You have a power of m here, and algebraic function, you got a transcendental function. So you cannot solve it in explicit closed 0:56:07.4 form, only numerically you can solve this. But we can solve for what happens when H is zero. That were those intercepts in the m versus H graph. Well, it is equal to this. And the question is, does this thing have any solutions or not?

So let us plot both sides. So I plot m naught. The left hand side is m naught itself, the 45 degree line

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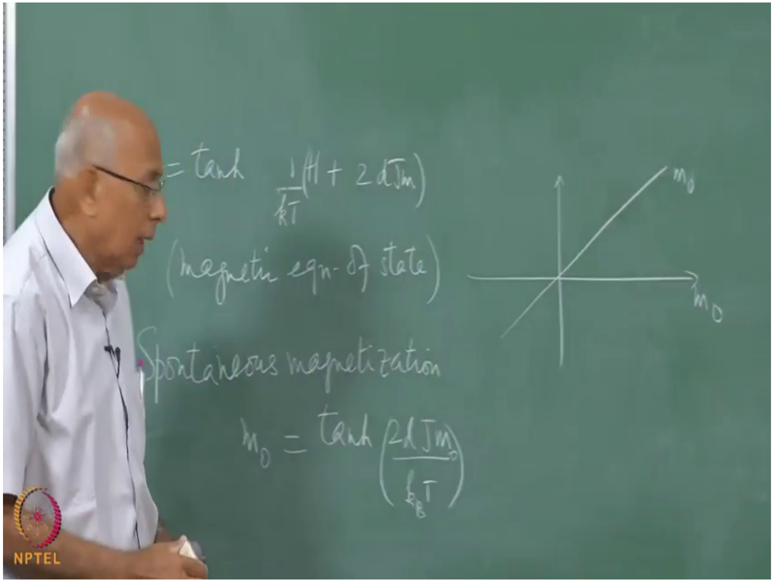

and this one is the tan hyperbolic guy. Now hyperbolic has a slope, tan hyperbolic x goes like x near the origin. So it is linear near the origin, saturates to plus minus 1. And two possibilities arise. One of them is the graph looks like this. Then the only solution is out here.

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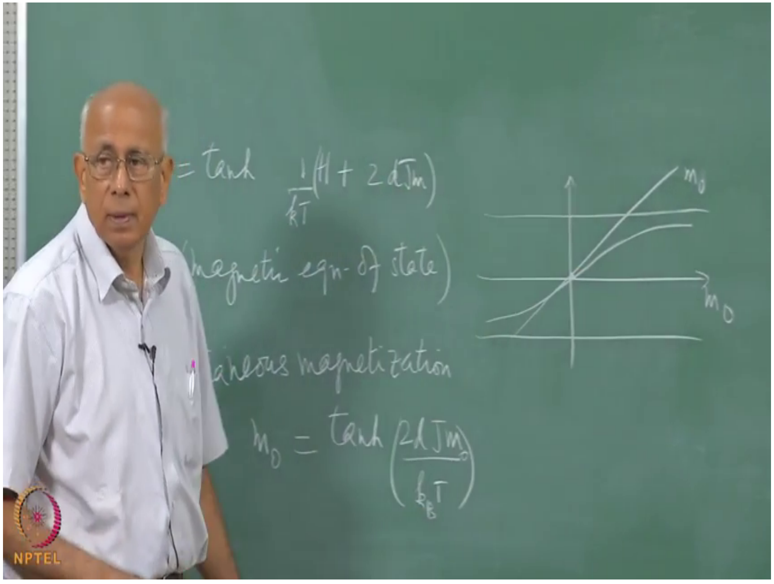

But if the slope at the origin exceeds 1, then the graph would look like this.

And then you have 1 solution, 2 solutions, and 3 solutions

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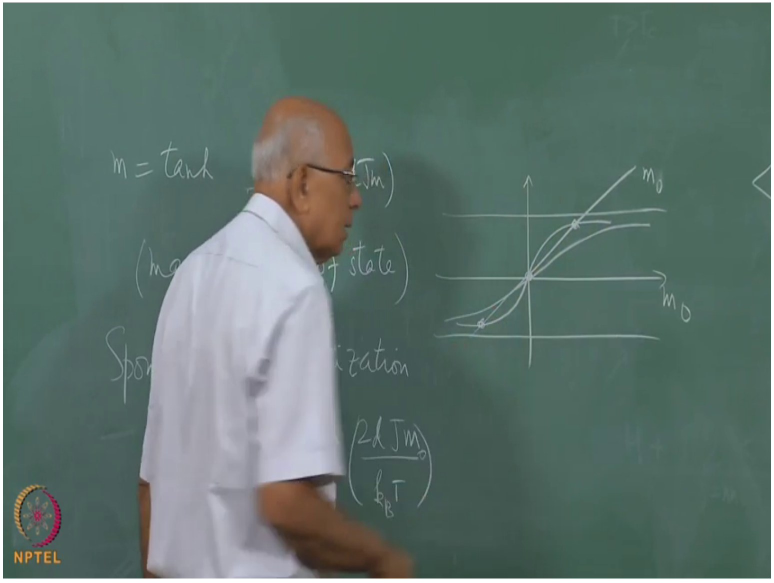

depending on whether the slope is greater, or less than 1. So you have m naught equal to zero is the only solution as long as the slope at the origin which is 2 d J over k Boltzmann T is less than 1.

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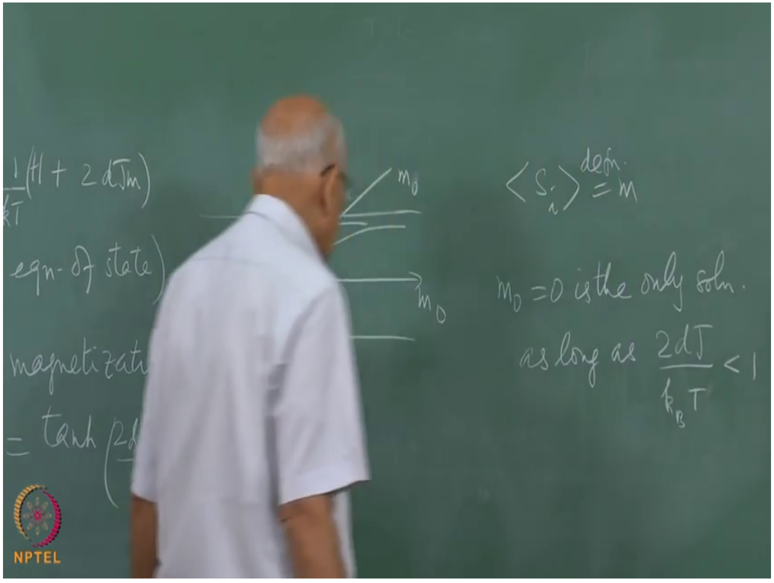

Or T greater than T c which is equal to 2 d J over k Boltzmann T. So we found the critical temperature

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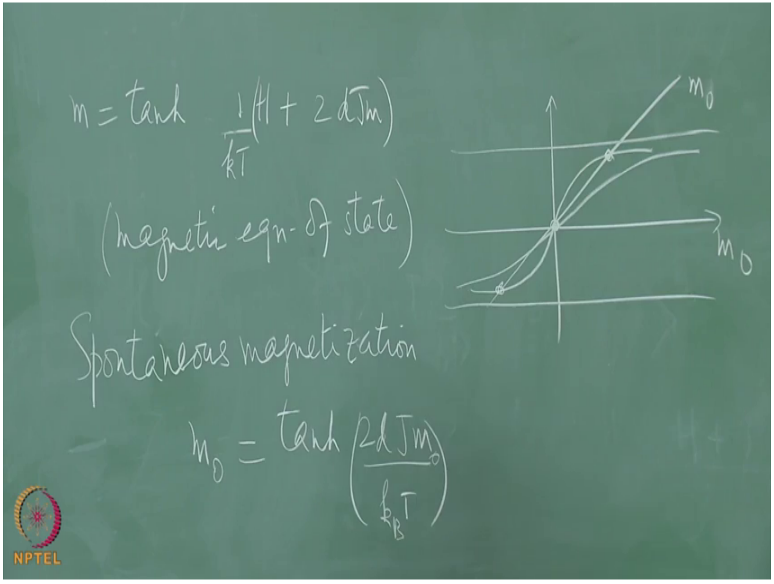

in this problem. As you expect, it has got to be proportional to the exchange constant J, the greater this is, the higher the temperature at which disorder will

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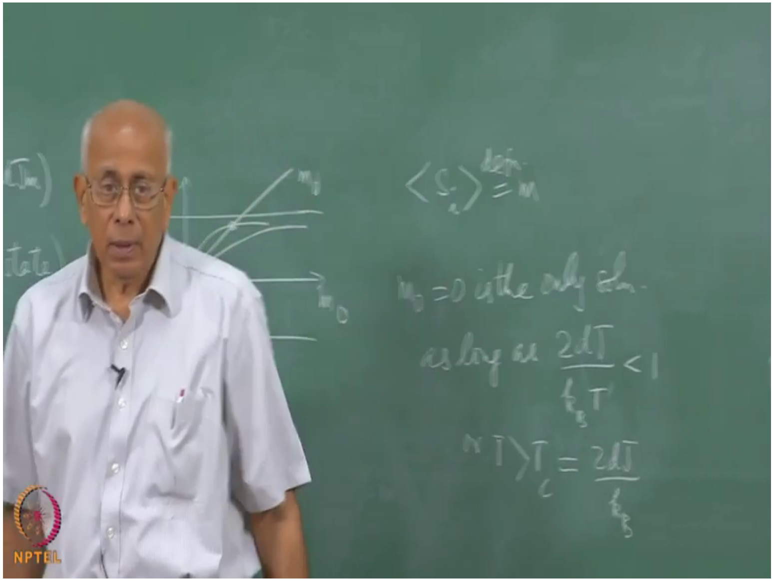

set in, and it depends on the dimensionality through the number of nearest neighbors, Ok.

If you have more of them in the higher dimensional lattice, even at a lower temperature, higher temperature you would still have order, Ok. Now it is easy to show that these two solutions correspond to precisely those minima I showed in the free energy curve and this corresponds to an unstable 0:58:46.9 here, Ok.

So we have our first formula which says T c is this guy here.

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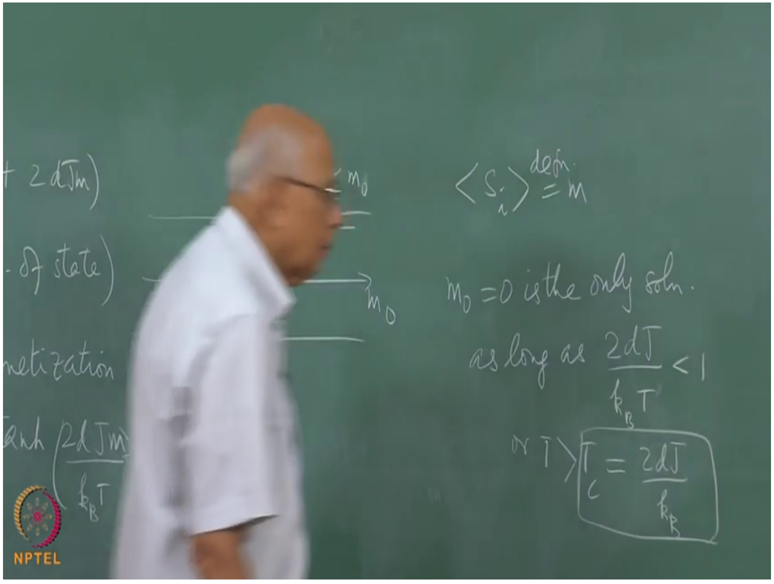

Now we put that back here. So I can write this magnetic equation of state as equal to tan hyperbolic h over k Boltzmann T plus, now 2 d J over k Boltzmann is T c so m T c over T.

And now you see what happens when you lower the temperature further and further. This as I lower the temperature this slope guy goes up more and more and these points are moving further and further away and that is exactly what happens when you plot this m naught versus T.

They start at this point but they move further and further out towards plus 1 and minus 1.

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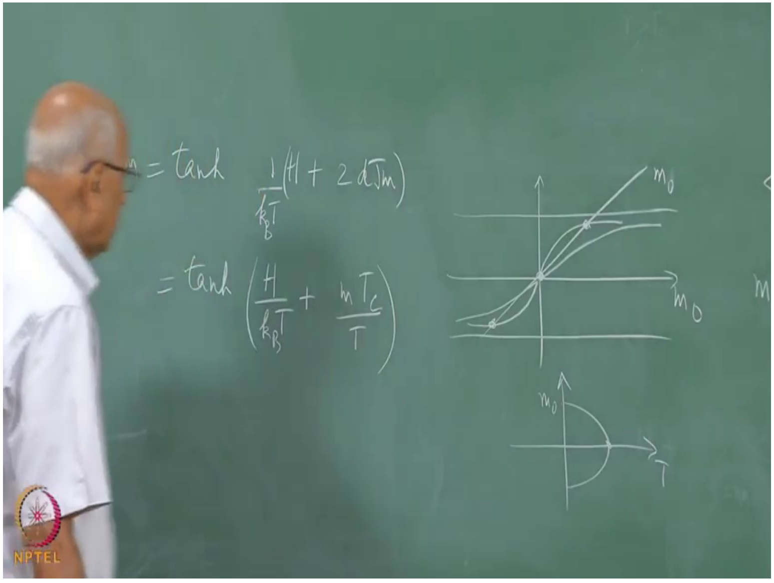

So this is our equation of state, this is like volume, this is like, this is temperature and this is like pressure. But equation of state

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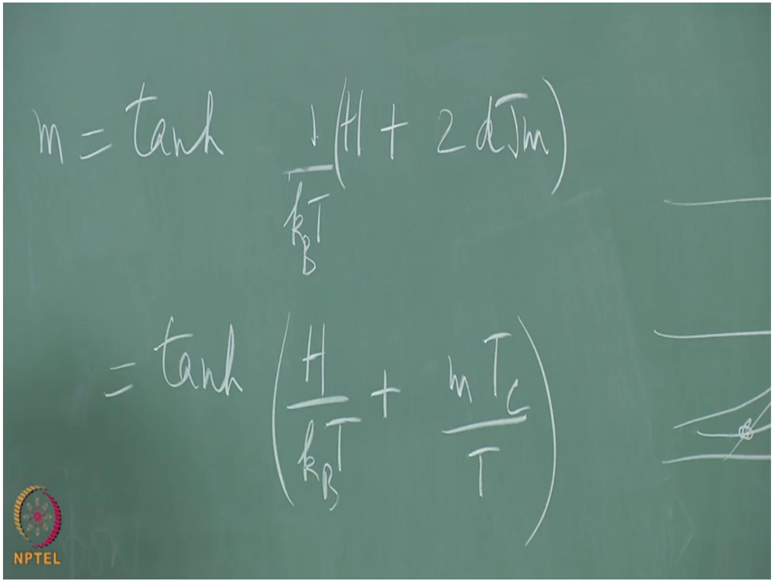

says p equal to something or the other. So we would like to solve for this and see what does this thing look like. I want H as a function of m and T. That is not hard to do. So let us write that. There is just one more line.

Then we will resume from here. So this is equal to tan hyperbolic H over, or beta let us just call it tan hyperbolic H over k Boltzmann T plus tan hyperbolic m T c over T divided by 1 plus tan hyperbolic H over ...

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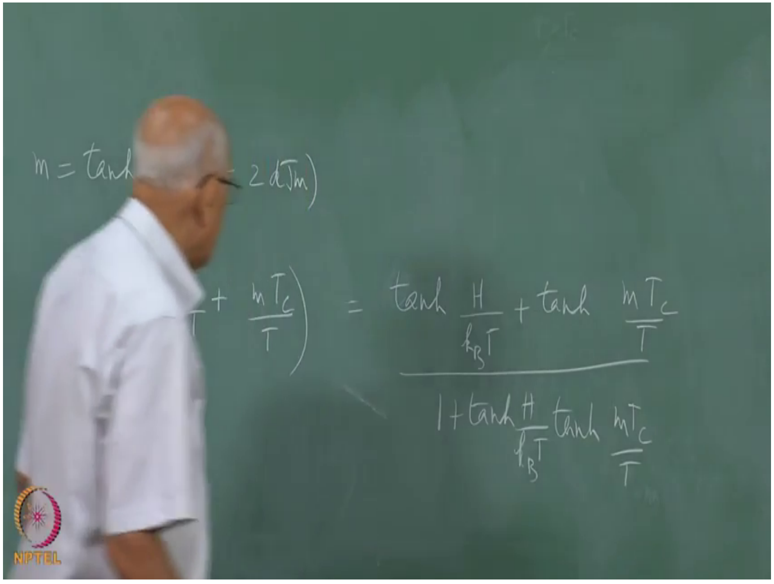

Just use the formula

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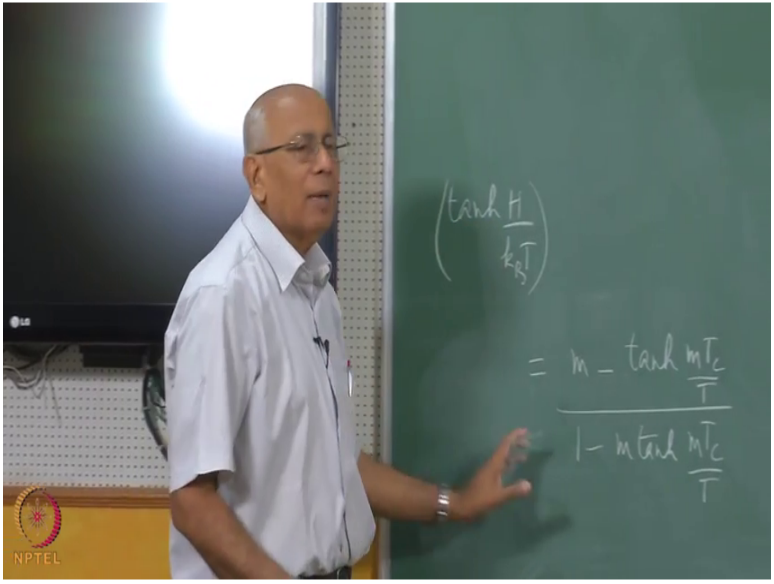

for the tan hyperbolic of a plus b and so this is equal to m, so let us solve for this. It says tan hyperbolic H over k Boltzmann T Time:s 1 minus m tan hyperbolic m T c over T, this guy moved this there is equal to m minus tan hyperbolic...Ok, that is the solution. So you have, H is tan hyperbolic inverse of this whole mess. Ok

When you see that is correct, 1 minus m tan hyperbolic, that is Ok. And that is equal to m minus tan hyperbolic, yeah that is Ok, alright. So now we are interested in the critical region. We have just seen exactly what happens at the critical point. Now I am going to stop here but I am going to, tomorrow I assume that you have already done this algebra, so show that...
As you approach T c from below,

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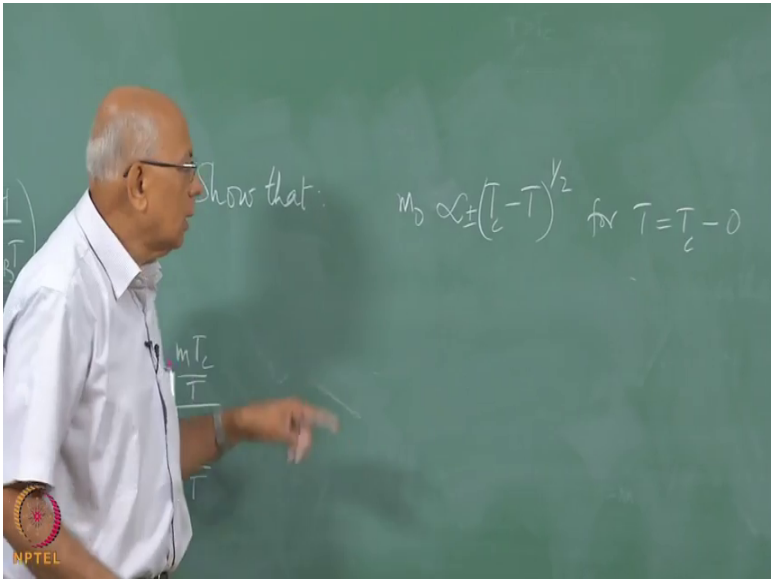

show that it goes like a square root, Ok. Then...

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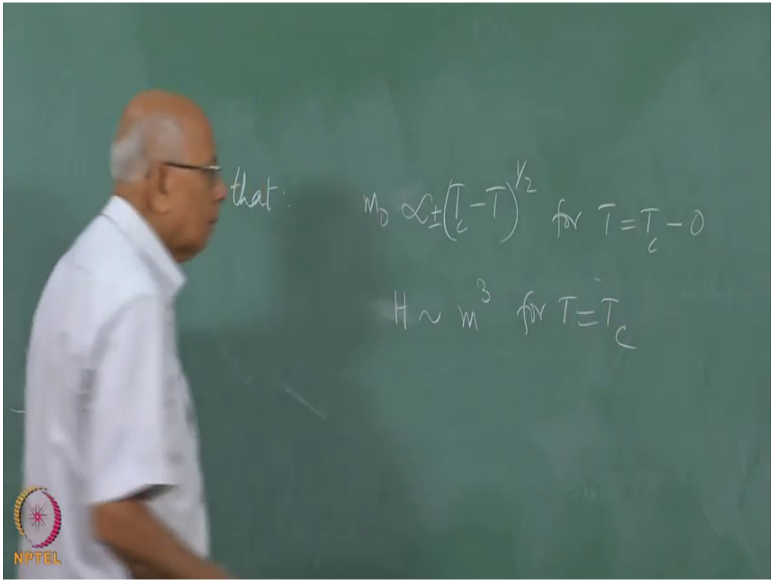

On the critical isotherm, that you can do very easily by putting T c equal to T. Then all this goes away and you have to show that H is proportional to T cubed, m cubed, the second...

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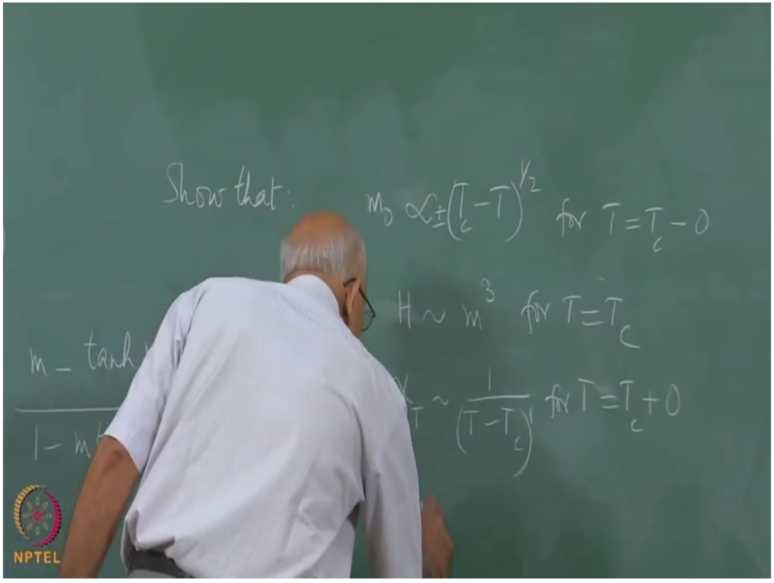

Show that: $m_0 \propto \pm (T_c - T)^{1/2}$ for $T = T_c - 0$

$H \sim m^3$ for $T = T_c$

$\chi_T \sim \frac{1}{(T-T_c)}$ for $T = T_c + 0$

m _ tanh

1 - m

with may be a different coefficient.

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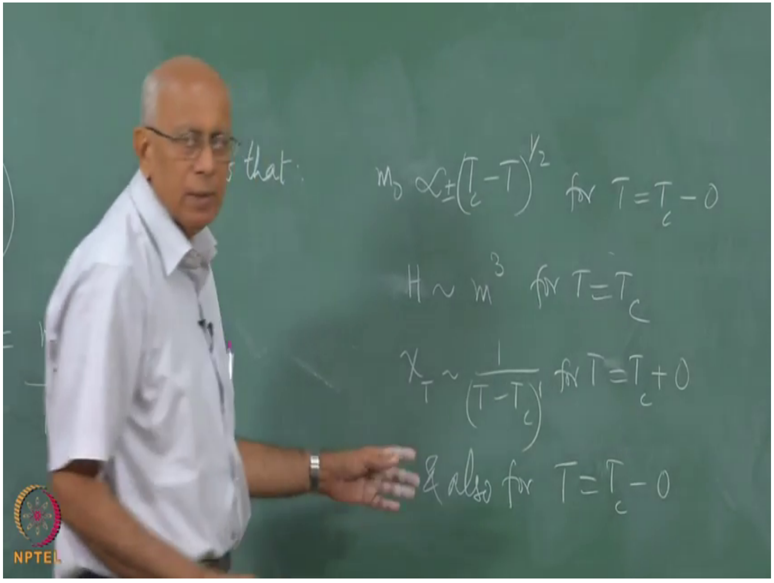

that: $m_0 \propto \pm (T_c - T)^{1/2}$ for $T = T_c - 0$

$H \sim m^3$ for $T = T_c$

$\chi_T \sim \frac{1}{(T-T_c)}$ for $T = T_c + 0$

& also for $T = T_c - 0$

So all you have to do is to solve this self consistently in each case, Ok.

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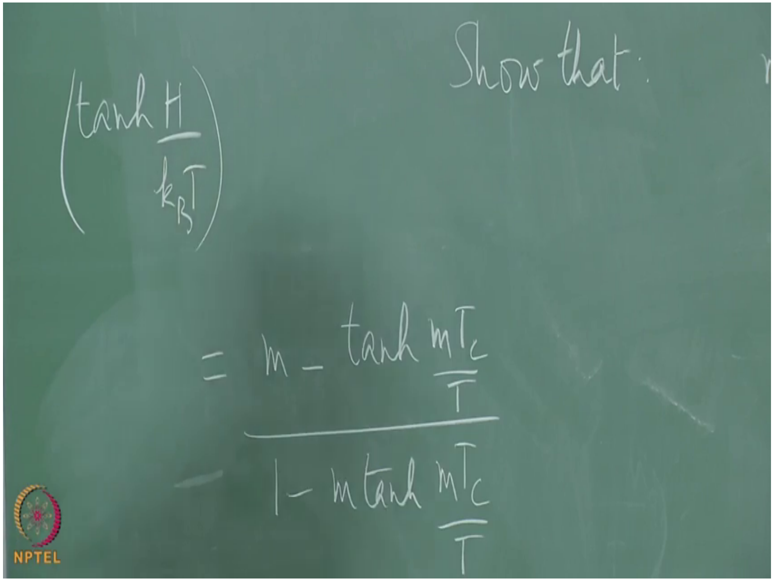

$$\left(\tanh \frac{H}{k_B T}\right)$$

Show that:

$$= \frac{m - \tanh \frac{mT_c}{T}}{1 - m \tanh \frac{mT_c}{T}}$$

Now what this means is that the susceptibility will look like this, kai T versus T and this is T c, this is susceptibility, it diverges at T c, Ok.

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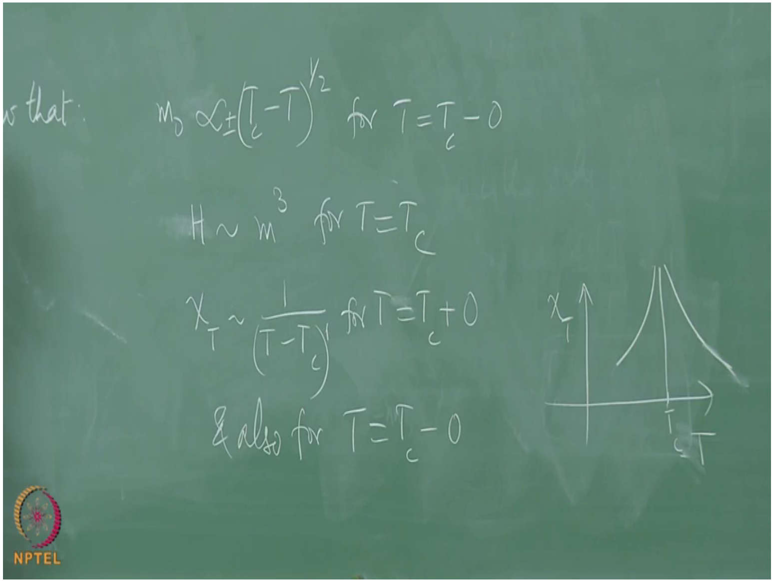

that:

$m_0 \propto \pm(T_c - T)^{1/2}$ for $T = T_c - 0$

$H \sim m^3$ for $T = T_c$

$\chi_T \sim \frac{1}{(T - T_c)^1}$ for $T = T_c + 0$

& also for $T = T_c - 0$

So we will start from this point.