Non-equilibrium Statistical Mechanics
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Module No 01
Lecture 17: Fokker-Planck equations (Part 1)

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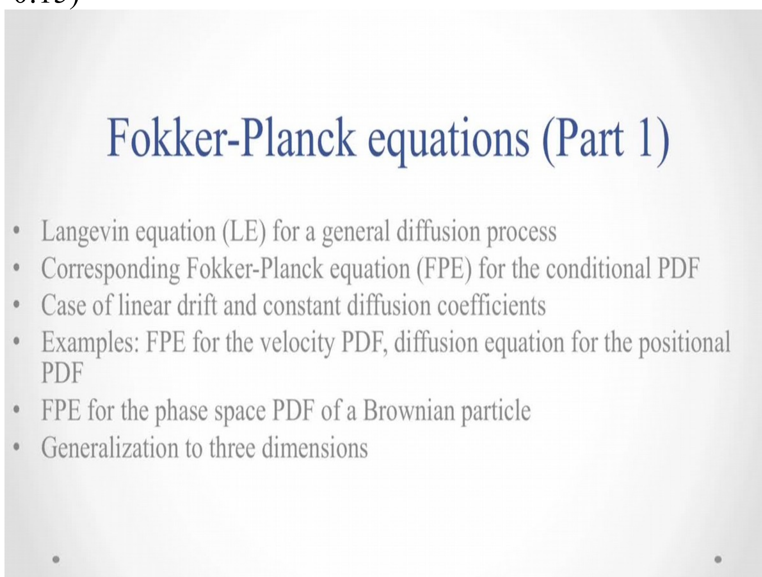

We had just started talking about the Fokker–Planck equation associated with the stochastic differential equation. So let us do this systematically and let us proceed as follows.

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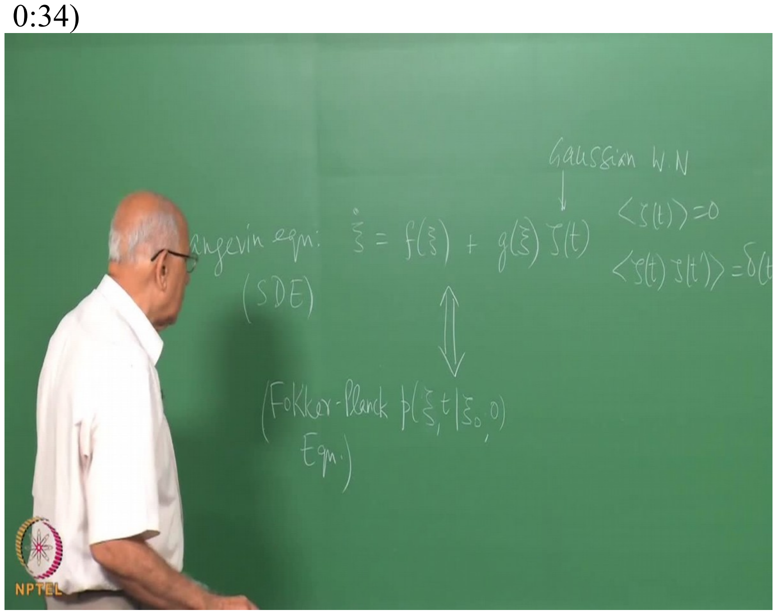

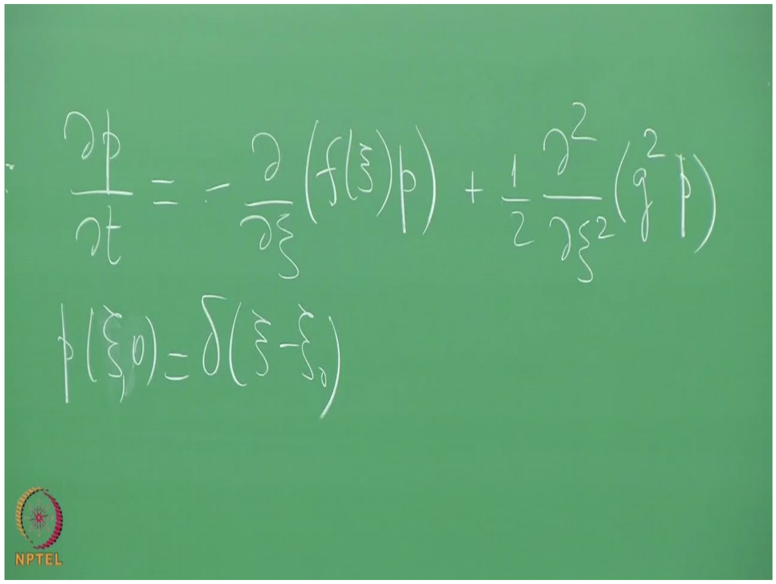

So 1st we have an equation which is like the langevin equation, kind of generalisation of it and let us continue to call it the langevin equation for a random variable psi which is a 1st order differential equation of the form psi dot equal to some function of psi, let us look at stationary or nonstationary processes but no explicit time dependency. We could include it if necessary but let us look at the case without it because all the cases we are going to look at will be without it.

+ the term G of psi times white noise, a Gaussian white noise which has got 0 mean and a Delta function correlation. So psi of T equal to 0 as zeta of T zeta of T zeta of T prime equal to Delta function of T - T prime with unit strength. And any strength in the noise is subsumed in this quantity, G which could be a constant for instance. Okay. Now technically, this noise is a Markov process, assumed to be a Markov process, it is Gaussian in the sense that all its probability distributions, joint distributions and so on are all Gaussian functions.

And it is a stationary noise as you can see here, the level of the autocorrelation and it is a Delta corelated noise so that the power spectrum is flat in this case. Then this implies and in turn this is implied by an equation for the conditional probability density of this variable psi. So for this quantity, P of psi T psi not 0, so this quantity here which is also guaranteed to be a Markov process in the sense that this to point this conditional density determines all joint densities, this quantity P satisfies an equation called the Fokker–Planck equation and the equation reads as follows.

So this is the stochastic differential equation for which I will use the abbreviation SDE and that implies for this P, a Fokker Planck equation or SDE in short for this quantity which is as follows. Delta P over Delta T equal to this side, a term which is - Delta over Delta psi F of psi times P itself. That is the deterministic part, the drift part. Without this, you have a normally differential equation, it is deterministic + a term called the diffusion term which is one half G square times P. G square is a square of this function. That is the Fokker Planck equation for this quantity.

Of course it has got to be in this case since the initial conditions are psi not, at T equal to 0, the value is psi not, the initial condition on this is simply a Delta function. This P at T equal to 0 is Delta of psi - psi not. So P of psi 0 equal to Delta of psi not. So very often, I am going to suppress this initial condition. Just call it, P of psi, T and P of psi, 0 is some prescribed initial condition. And this is what you have to solve this equation with. We are not going to derive this. It is not very hard to derive. It is actually quite straightforward. I have to introduce movements of something called a transition rate or a probability per unit time and then using thing that, make an expansion, etc, etc and show that this is what is obtained in the case where this is a Gaussian white noise.

Student: (())(5:14)

Professor: Pardon me.

Student: (())(5:17) discretise time?

I do not discretise I do not have to discretise time but it is convenient to do so when you are deriving the equation. If I am going to start from what is called the chain equation for Markov processes, go to what is called a master equation and then a Kramers model expansion and so on and finally we show that for Gaussian white noise, it reduces to this function. And there is also the question of the interpretation of this equation because there is this noise here multiply, this deterministic term multiplies this noise. So this is called multiplicative noise whose amplitude depends on the value of the variable at that instant of time through this function here.

And for multiplicative noise, you have different ways of interpreting this equation and we are using what is called the itoim interpretation. There is a Stratonovich interpretation and there are several other ways of interpreting this equation which gets us into stochastic calculus which I do not want to do here. So we will take equation for granted because we are going to actually look at it in the simplest of instances, in the simplest cases and our focus is only physical processes. Like motion of a Brownian particle or three-dimensional motion and the potential, et cetera et cetera. Okay.

So this equation is not very simple to start with because it is a second-order equation in in the random variable in the variable psi and it is a first-order equation in time, so it is a complicated partial differential equation. The solution is also very complicated. Now the cases we have already looked at fall squarely in this. You can see what is going to happen immediately.

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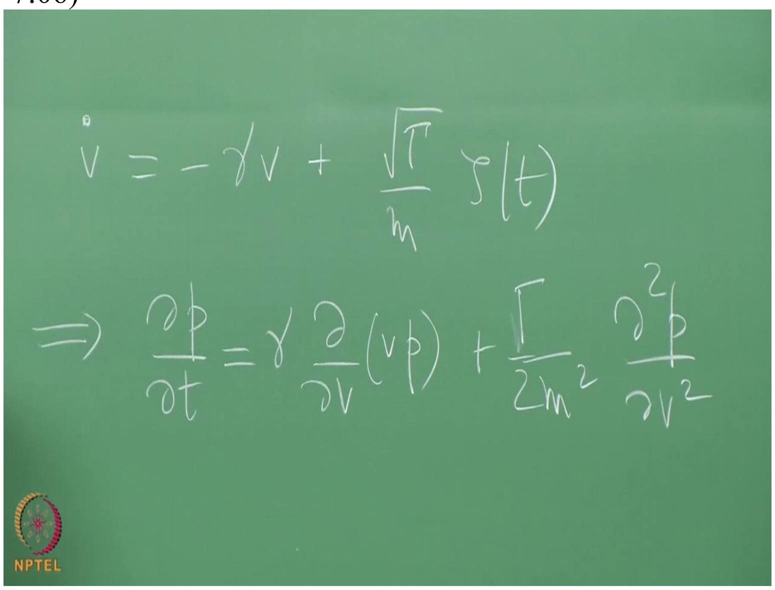

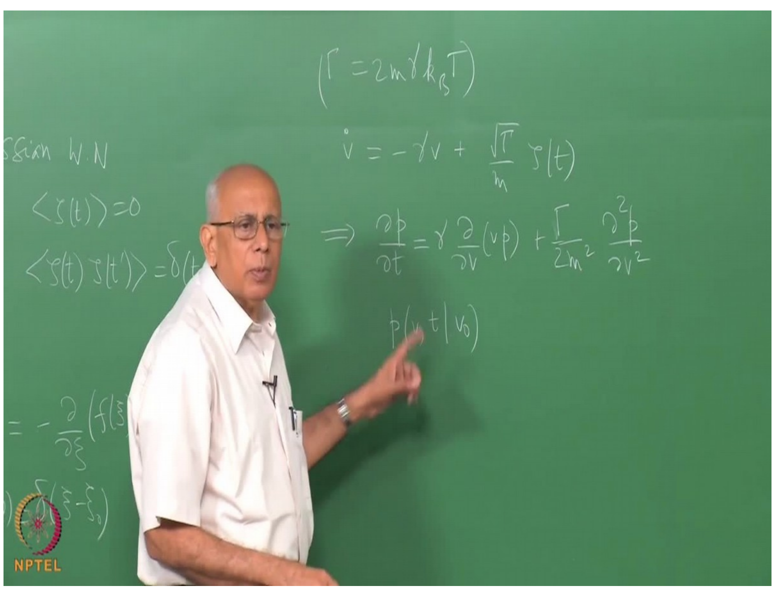

For instance, we considered the velocity process, we said one Cartesian component of the velocity of Brownian particle was satisfied in equation - Gamma V + square root of gamma over M times zeta of T. I call square root of gamma zeta of T is what I had called eta of T earlier in noise because I said the correlation function is gamma times the Delta function. I got now a unit strength Delta function here for the 2 point correlation. So therefore I put the gamma outside in this one.

So in this problem, psi is V. F of psi is a linear function. It is - Gamma V and G of psi is a constant and that is called additive noise. Because it just says to the deterministic part, you are adding noise whose strength is independent of the value of the variable of the random variable. This thing is a pure constant. Now look at what is going to happen immediately here. This will imply as we have seen, if you apply the general theorem out here, this implies that Delta P over Delta T equal to gamma times Delta over Delta V V times P because that is - F of psi which is - gamma V - sign cancels, + one half and then G square which comes out as a constant.

So it is just gamma over 2M square D2P over DV. So without proof, using this general correspondence, I am asserting that given this langevin equation, this is the equation satisfied with a conditional density of the velocity. Now of course, we know from the langevin equation, we already proved the fluctuation dissipation theorem. We know that gamma is 2M gamma K Boltzmann T. We already know that. For consistency, saying that the system remains in equilibrium, thermal equilibrium, when you take the full average over all realisations of the noise and then over all initial velocities, we know that you should get the Maximilian distribution finally, equilibrium distribution should remain Maximillian. That happens only if this is equal to that and that is what we argued, what is the fluctuation dissipation theorem in this instance? I said, the strength of the dissipation and the strength of the noise must be related to each other in this consistent fashion.

But that follows from here too immediately. You can see that if you ask what happens as T tends to infinity? This, what we are doing, is starting with P of VT V not because I know this is a stationary process, so I know this is a stationary process. So I do not write the 0 there. It does not matter in this case. Or just a side remark, the fact that this equation has a noise which is a stationary random process, does not guarantee that the driven process psi is stationary. It guarantees it is Markov but it does not guarantee that it is either stationary or Gaussian because this could be all kinds of functions out here.

What is guaranteed is that it remains Markov and in general will have a finite correlation time, not 0 as the noise. So what is carried over is a Markov property. Neither the stationarity, nor the Gaussianity is carried over to the driven process in general. But we know, we have already computed the velocity correlation function in this case and we know it is E to the - Gamma T apart from KT over M and we know that it is a stationary random process. So I do not bother to write the 0 there. Whenever it is a stationery process, I will not write the initial time explicitly because we can always shift it out okay.

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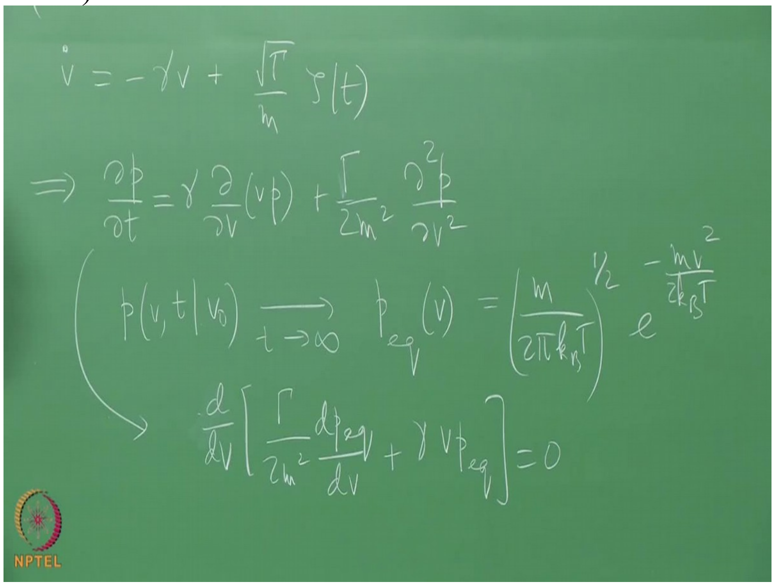

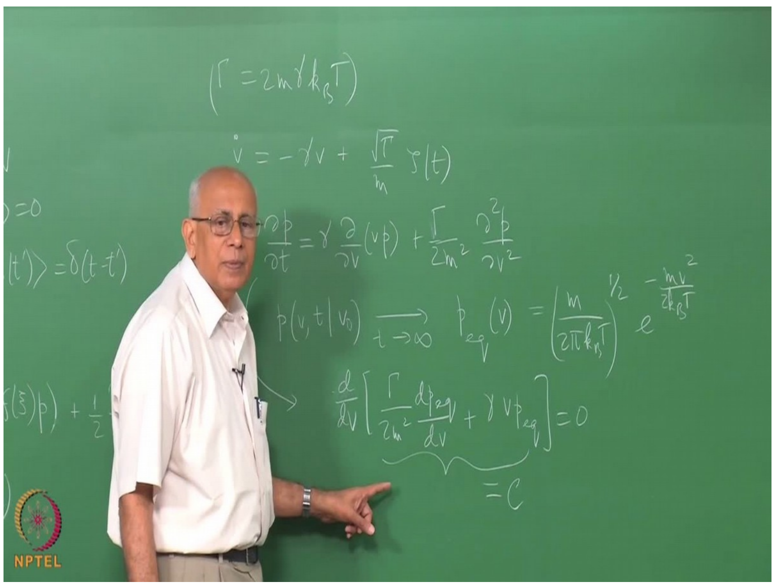

The This quantity here we know that it should become as T tends to infinity, it should go to the equilibrium distribution, the Maximilian distribution, right? So this should tend to be equilibrium of V because it should forget the memory of the initial velocity and go to the final velocity and be in equilibrium distribution. As T tends to infinity, it should be the stationary distribution. But we know what this is. We know this is M over 2 pi K Boltzmann T to the power half. It is a single component we are talking about, E to the - MV square over 2K Boltzmann T.

We know that is what it should become right. On the other hand, if I examine the equation itself and ask, what happens to this as T tends to infinity? If there is a stationary equilibrium distribution, this must become 0 because it cannot depend on time. And then as we have already seen before, this will imply that the equilibrium distribution will satisfy an ordinary differential equation, no time dependence there which will look like this. This will look like gamma, it will look like D over DV of gamma over 2M square DP equilibrium over DV. That is this term here + gamma P times P equilibrium is equal to 0 because this side is 0.

Therefore the bracket is a constant. Therefore this implies that this fellow is equal to a constant independent of V. But we also want an equilibrium distribution which is normalisable. So this means P equilibrium as V tends to infinity must go to 0. It must have finite moments, mean velocity, mean square and so on, which means that this fellow should also go to 0 or derivatives should also go to 0 as V tends to + or - infinity ok which implies that the constant must be 0. Because this thing is equal to a constant and we say that at V equal to infinity, this constant is 0. So since it is a constant, it is 0 for all V. But then that is a 1st order equation and we know what the solution of it is. It is trivial.

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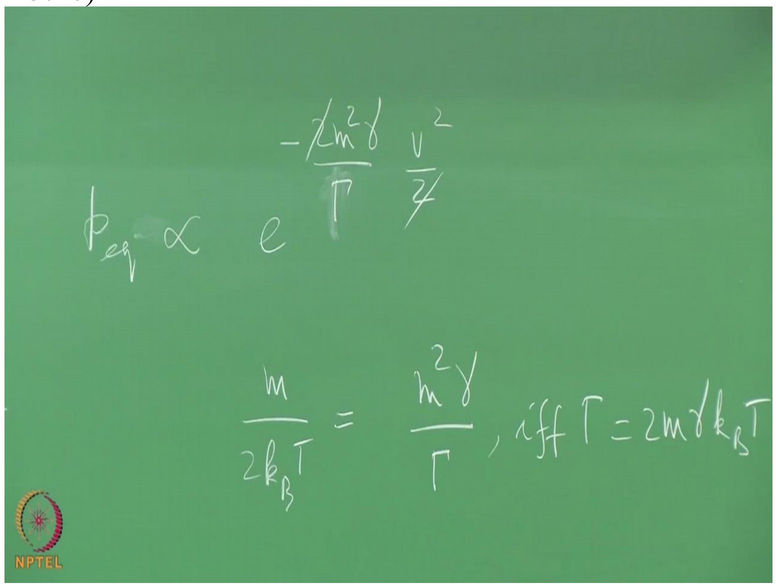

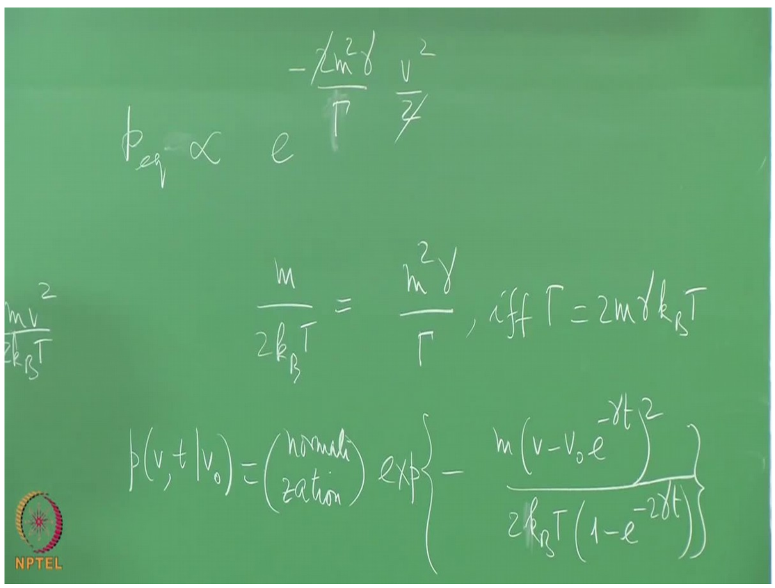

So this thing here will imply that P equilibrium is equal to, is proportional to apart from constant of proportionality, E to the power - 2M square gamma, I will multiply through by this guy. You should be careful with all the 2s and so on. Yes 2M squared gamma over capital gamma V square over 2 because I integrate VDV and that gives me a V squared or 2 and that is this ok. So cancelling out these 2 fellows, you get this. So this matches, this thing here if and only if, M over 2K Boltzmann T equal to M squared gamma over gamma. So you are back to this.

You are back to this. So that is one more way of saying that this must be the fluctuation dissipation theorem. If and only if gamma equal to 2 M. So it is just saying it in in that in terms of the probability distribution or probability distribution or (den) probability density function rather than the stochastic equation itself. It is the same fluctuation dissipation theorem you are going to get okay. Now what is the general solution of it? Well, now we need to examine this a little more carefully. This equation is not too hard to solve. It is not trivial though because this differential operator is not self adjoined. It is this 1st order term here, it is a bit of a mess.

There are several ways of solving this. It turns out that the solution is precisely the Ornstein–Uhlenbeck distribution. This turns out to continue to be a Gaussian process in this case and it is the Ornstein–Uhlenbeck distribution. So we know what that is. We know that E of V in V not equal to an exponential apart from normalisation, - M V - V not E to the - Gamma T whole square. That is the mean value, the peak shifts to the left, divided by 2M 2KT and then the variance and then a normalisation factor which is essentially M over 2 pi KT times these times square root of that whole thing.

So the mean is this value which goes to 0 as T tends to infinity, starting at V not and the variance is a delta function at T equal to 0. Variance vanishes at T equal to (zee) 0 and then broadens out it hits the value given by the Maximilian distribution, depends on the temperature.

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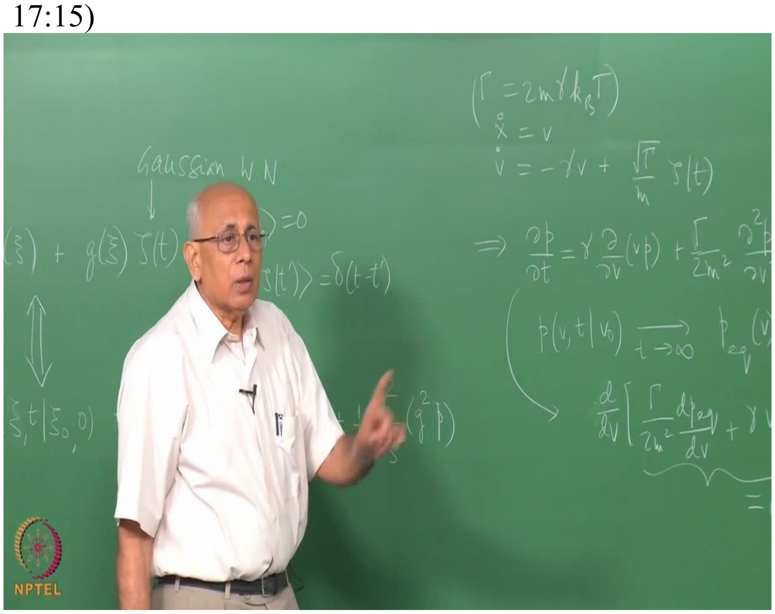

And this is the OU, Ornstein–Uhlenbeck distribution. Next question. What about the phase space distribution? What about the distribution of X, V and T? We need to know whether it is a joint distribution. Is it possible to get an answer for that? And the answer is yes. We can see how that is going to come about because I should really augment this equation with another equation which says X dot equal to V. We are talking about one cartesian components, so it is just an X and a V. So X dot is V and V dot is this out here.

In this case, there is no external force and therefore there is no F of X which you might get from a potential maybe but you do have this systematic part and then you have the random force here. So one should be able to join to write down of P of X, V, T given X not, V not and 0 and an equation for it, a corresponding equation. For that, you need a generalisation of this to higher dimensions, to moment, to 2 dimensions. We will come to that in a minute but before that, let us settle this other question. We know there is a diffusion limit and in the diffusion regime, we know that the mean square displacement goes like 2DT and we know that D is KT over M gamma.

Can we get that from this? Does it follow up from this thing at all? Does it follow from this equation? And the answer is yes, it indeed does. Because what we need is an equation which says that the velocity is delta co-related. Because when you are in the diffusion region, the velocity correlation time is gamma inverse. And now you are saying, you have T much much greater than gamma inverse. So one way of implementing that is to say, I take gamma to be so large that gamma inverse is negligible and then at all T practically, you have only diffusion regime. So one could look at the high friction limit of this equation and ask whether that is going to work or not. Okay? There are several ways of implementing this but let us do it that way and see what we get.

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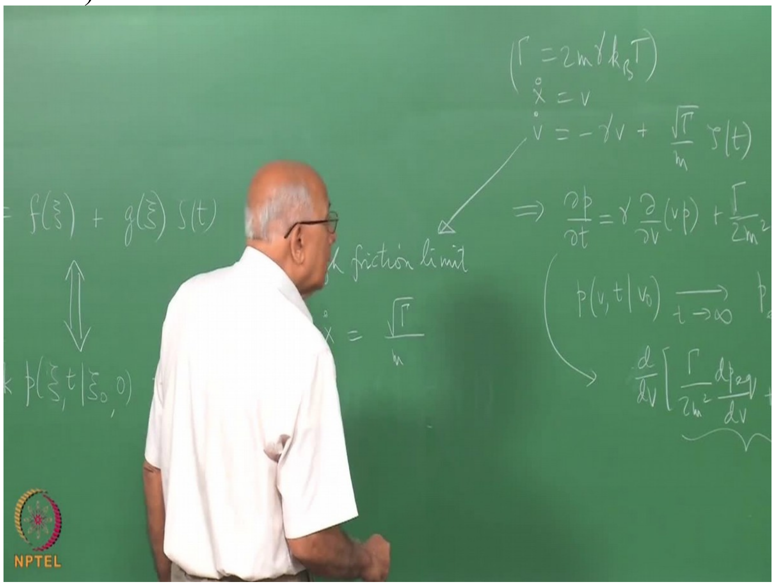

So a high friction limit of this system is going to be gamma tends to infinite gamma high friction. High compared to what? Well, the statement is that gamma is so large that all 10 times that you look at are such that T is much much greater than gamma inverse. So I drop the inertia term. This came from M times V dot, mass times acceleration. I drop that term, I retain this term and look at it as a stochastic equation. So now my stochastic equation says X dot that is V is equal to root gamma over just one second. Before I do that, there is one thing I want you to bear in mind which is since we need this, we have imposed this fluctuation dissipation theorem for consistency, we can rewrite this equation a little bit and this is going to be useful to do so.

Student: (())(21:04)

Professors: Pardon me.

Student: Large gamma limit.

So that I can take the large gamma limit, yes. But this is also going to be needed in another context, we will see in a minute.

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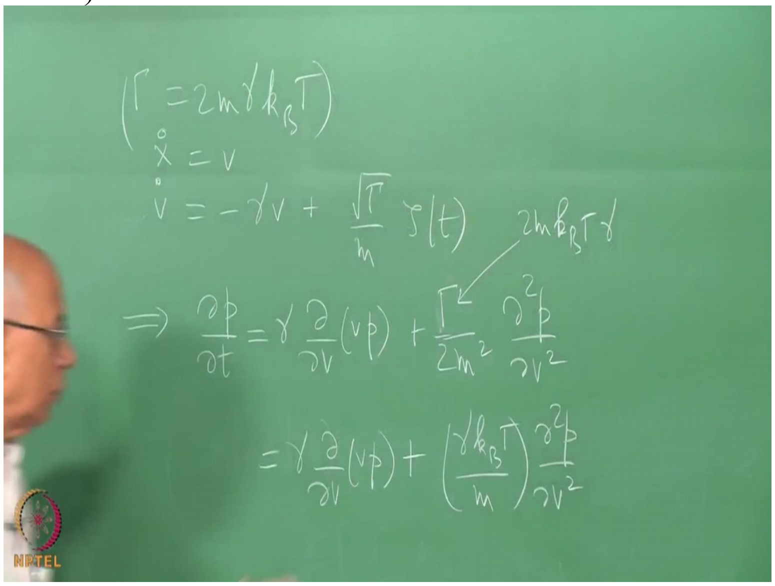

So this is really gamma delta over delta E VP + for this I substitute 2 M KT gamma and then the 2 cancels, the so this is gamma K Boltzmann T over M D2P over delta. So the diffusion coefficient if you like, in velocity space is, KT over M. In position space, it is already what we know. It is KT over M gamma but in velocity space, it is turning out to be gamma KT over M. We will keep that aside for a moment. So let us look at the high friction limit of this.

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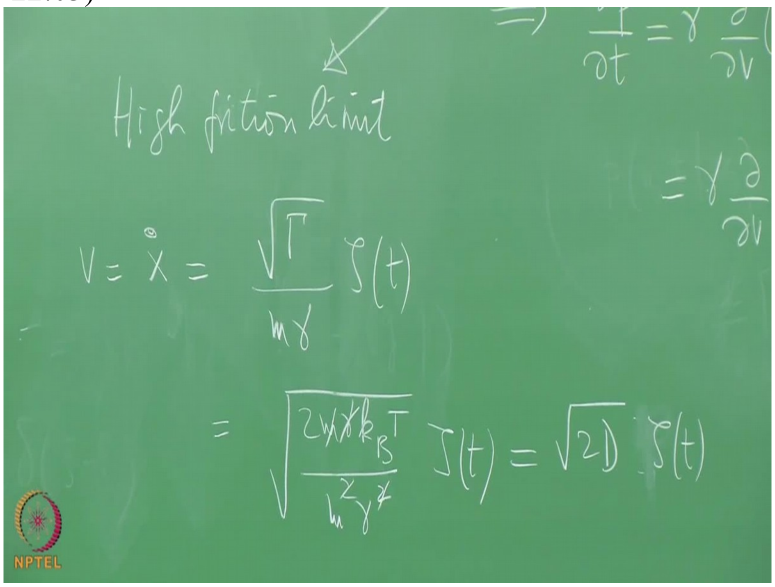

So let us look at the high friction limit of this and that reads, so I drop the V dot term, this term and retain this, bring it to the left. So it says V equal to root gamma over M gamma Z of T. So you are really saying, sorry V which is X dot which is what it is in the diffusion regime because it says the velocity is uncorrelated. For our practical purposes, it is a delta function correlation but we can simplify this right? So this is equal to square root of or put that inside. So 2M gamma K Boltzmann T over M square gamma square Zeta of T equal to the M cancels, one of them, the gamma cancels. Twice KT over M gamma. But remember, we had set D equal to K Boltzmann T over M gamma.

We discovered that the mean square displacement in the diffusion limit was actually 2 DT where D was given to be KT over gamma. That is what we found. So let us put that in. This says this is square root of 2D Zeta of T. And now I go back and appeal to this general Fokker Planck correspondence between the stochastic differential equation and the corresponding Fokker Planck equation. I stare at this and I say look, X is Psi now, Psi is X now. There is no F of psi now of zeta the or psi. That term is missing. There is only a noise term. And G is square root of 2D. It is a constant.

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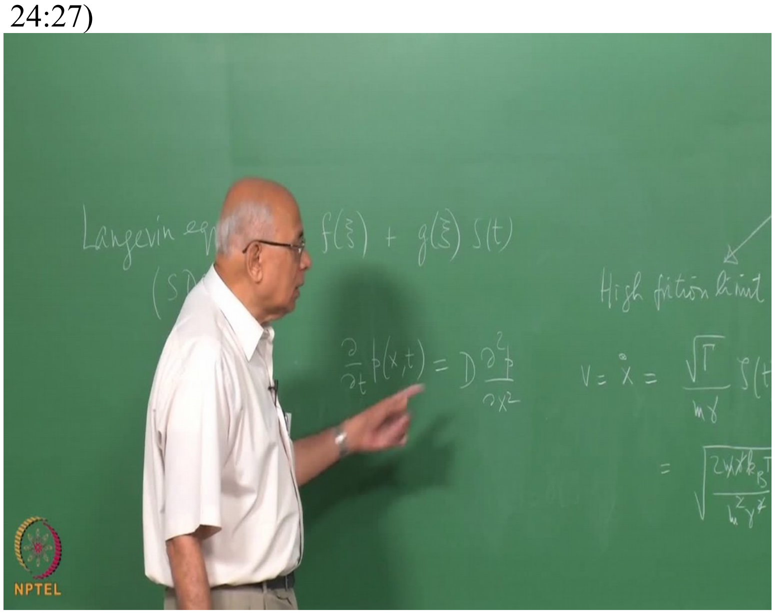

So this tells us that delta over delta T P of X, T for some given X not which I will impose as an initial condition right is equal to one half G square. But G square is 2D and half of that gives me D. So this becomes DD2P over DXT. That is the diffusion equation. That is precisely the diffusion equation okay. So you see the origin of the diffusion equation from this language, from this point of view, it is precisely the fact that you are working in a regime where the velocity correlation time is essentially 0.

And therefore, the velocity is a white noise. And when you integrate it, you get an equation for take this equation and write it for as a density in X position you have this diffusion equation. So this emerges from that correspondence. So it is a part of that correspondence. There is no drift term in this equation. It would be there if I put an external force. If I said I am looking at diffusion in a potential, even gravity, there will be an extra term, there will be that 1st, the F of psi term would be present. But that's completely missing here.

In the velocity case, there was a friction term which was proportional to the velocity. It was linear in the velocity that made life a little easy out here. In the diffusion case, that's in that's missing. Now if I look at sedimentation, namely diffusion of a molecule in a vertical column under gravity for instance then there would be a constant force and a constant force would lead to just P. There will be no V here, just P on this side. So you would have a 1st order term + a 2nd order term on this side.

And that would lead to an extra contribution. It would change the solution here considerably. It would lead for instance if you ask in a finite column if you ask or even infinite, semi-infinite column under gravity if you asked, is there a study distribution, the answer is yes, there will be one. But is there a steady distribution for this on the infinite line? Is there an equilibrium distribution for this? If there were, then this should be 0 and then you get D2P equilibrium over DX to be equal to 0 and the solution to that is P equilibrium is AX + V but that is not normalisable, it is not normalisable. And therefore there is no such equation, there is no such distribution.

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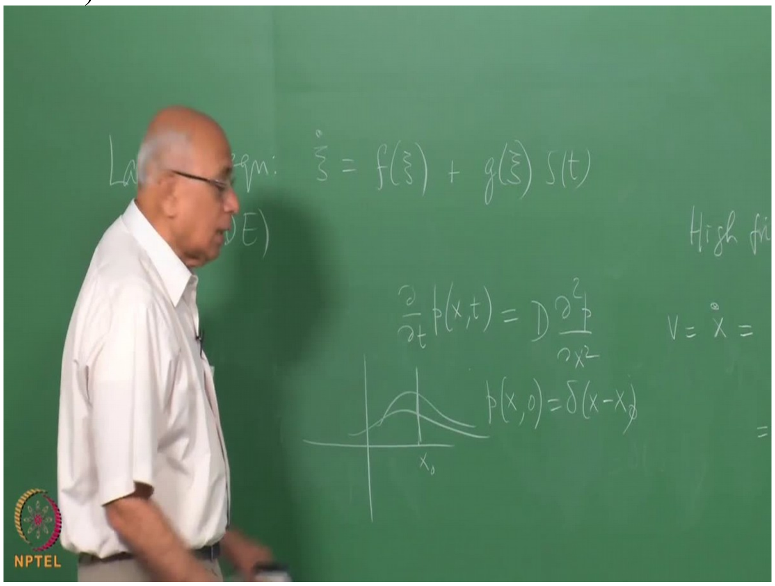

And you know that in this case what happens is that if you start at delta function at X not, you could impose a condition, P of X0 equal to delta of X - X not. If you start at this delta function distribution and you look at this P, as T changes, it does this, et cetera, whatever. It doesn't even drift.

It doesn't because there is no drift. Because there is no drift at all, what it does is to start at X not, it broadens out and broadens further and decays to 0 such that the total area under the curve remains 1.

It is normalised in one. No material is going away. So this thing does not have an equilibrium distribution. If you put gravity, we will do that later on, we will do this sedimentation problems later on. You will discover it and to the bottom, everything goes to the bottom. So there is an equilibrium distribution under certain cases but in general, there is (())(28:27) for this. So this is how you get the diffusion equation okay? Well now you could ask, what about the phase space distribution? What does that look like? So now we have to be a little more careful.

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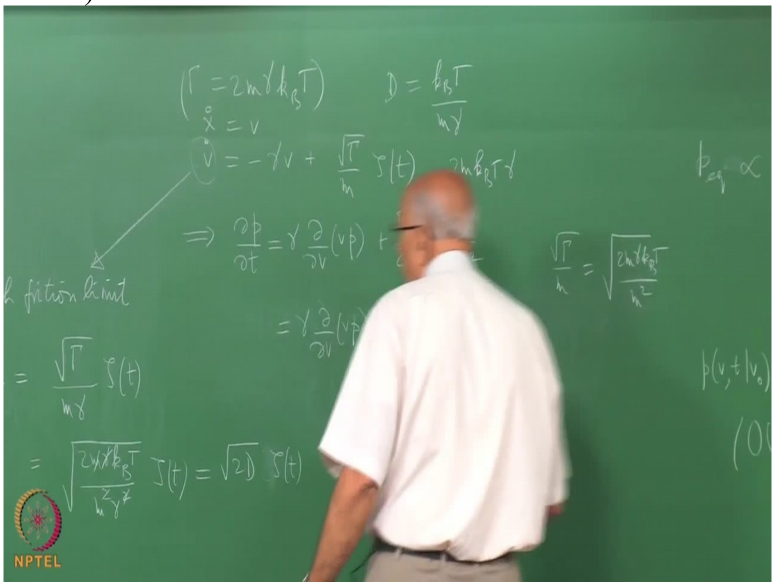

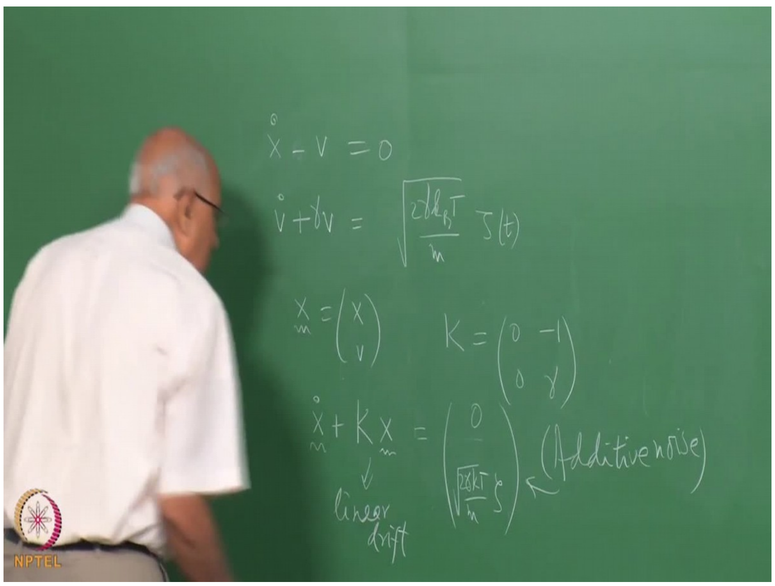

A stochastic equation is a pair of equations. So we have X dot + V equal to 0. Sorry, - V equal to 0 and you have V dot + gamma V equal to whatever this fellow was. Yes, root gamma over M. Let us let us change this thing here. This is equal to root gamma over M equal to root 2M gamma K Boltzmann T over M square. So 2 gamma KT over M, let us just put it that way. This fellow here times zeta of T. So it is a pair of coupled equations.

Now let me introduce a vector X which is XV, put this in matrix, column matrix here. I am going to write this as a single equation in a vector form and let us introduce a matrix K, a drift matrix which is equal to there is a - 1 so it is 0, - 1. There is + gamma here, so it is 0, gamma. So introduce that matrix. Then the left-hand side for these 2 together becomes X dot + KX gives you the left-hand side equal to on the right-hand side, a noise which is essentially this fellow here. 0, root 2 gamma KT over M times zeta. I can give it a symbol, some vector noise, it does not matter. Okay.

Now the question is what is the corresponding Fokker Planck equation corresponding to this. And it turns out that what you have is a special case of a much more general case. In a case where the noise is additive, this is additive noise because there is no X or V dependence here. There is no G. It is only constants. Moreover, the drift is linear. So this is a linear drift. So you have addictive noise and linear drift. This makes life easier. The general case is also something we are going to write down but the expression for the Fokker Planck equation for the linear case with linear drift and additive noise is very straightforward and it is very natural. Let me write it down.

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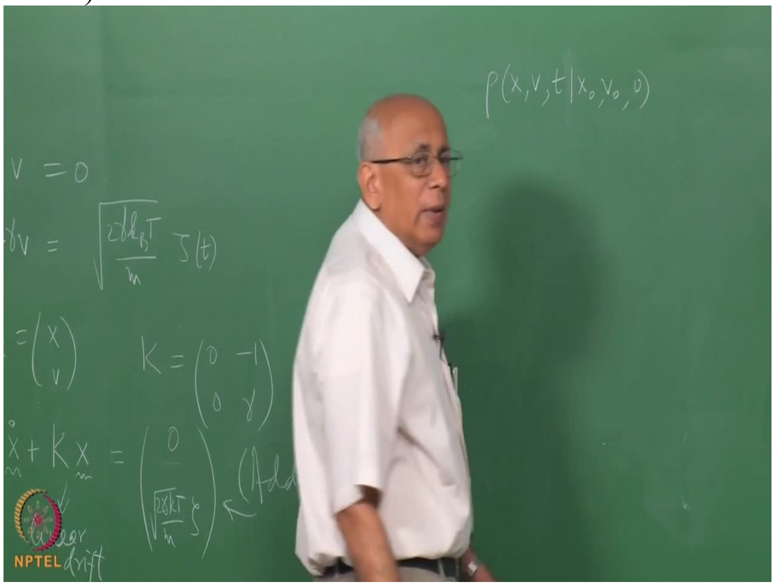

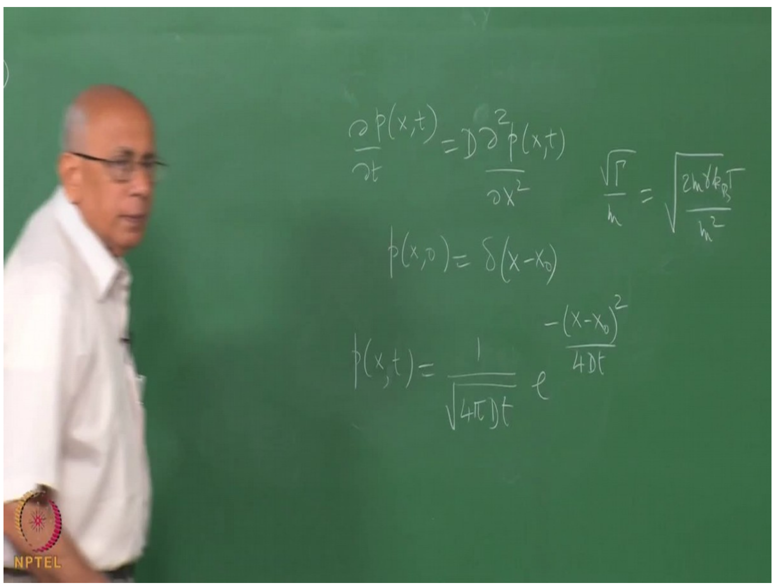

It says so now we are going to look at the density rho of X, V, T given X not, Vnot, et cetera. So let us put X not, V not, 0. Oh, by the way, when we looked at the high friction limit of the original langevin equation and got to the diffusion equation for the positional probability distribution function, was X of T a stationary process or not? So we had this quantity, P of XT delta over delta T equal to D D2P of X, T over DX2 with initial condition X not, et cetera. So we had P of X0 equal to delta function of X - X not and we know how to solve this equation.

You do Fourier transform with respect to space, Laplace with respect to time, et cetera, many ways of solving this. These are fundamental Gaussian solutions with natural boundary conditions. Namely P is 0 as X tends to + - infinity. Then the solution of this equation is worth remembering. It is a fundamental property. It is equal to 1 over square root of 4 pi DT E to the power - X - X not whole square over 4 DT. That is the fundamental Gaussian solution to the diffusion equation.

The peak remains at X not out here and the peak actually goes the the width of this peak goes to infinity linearly with time. That is why you have diffusion. So in this case, the average value of X - X not square that actually diverges. While the probability density itself decreases at all points. That is why one overthrew T. It is normalised to unity. The integral for - infinity to infinity in X for all time is finite is one. Okay. Is this a stationary process? No, imminently no because the various changes with time.

So it is not stationary. It is Gaussian. It is Markov but nonstationary okay. Earlier, the velocity process alone, the noise is Gaussian, stationary, Markov and delta correlated. The velocity process is stationary, Gaussian, Markov but not delta correlated, exponentially correlated. The position in the friction limit is nonstationary. It continues to be Gaussian, Markov.

Professor-student conversation begins:

Professor: Yes?

Student: (())(35:04).

Professor: Yes, exactly.

Professor-student conversation ends.

So this is not, yes I emphasise once again. This is not the exact equation for the positional density at all. How would you get that? Well, that would get you get from here, you would get from this quantity.

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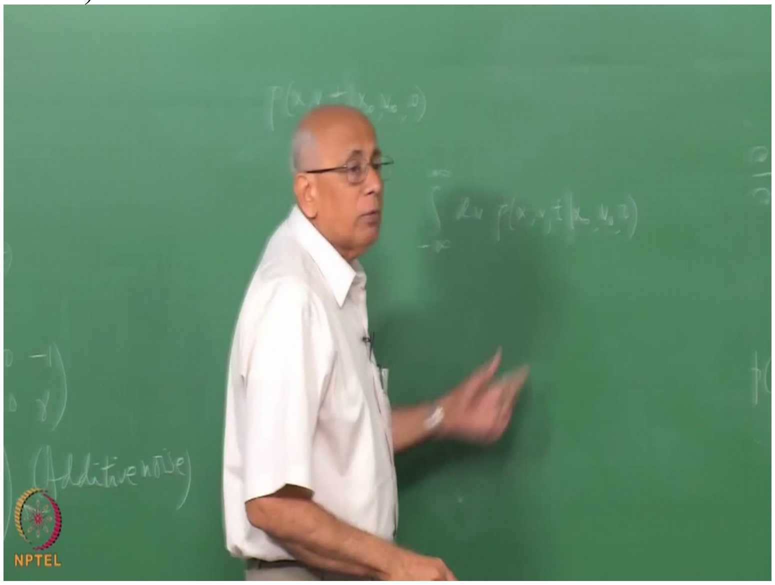

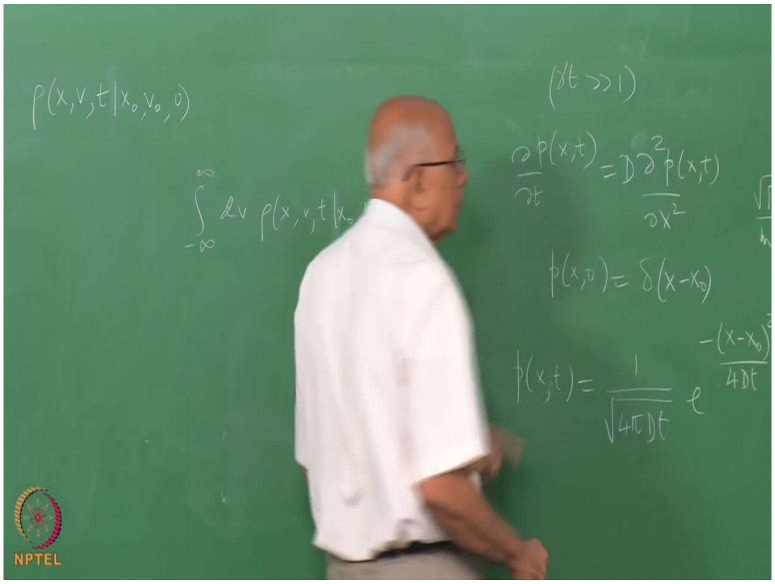

So if you did this, if you integrated - infinity to infinity DV rho of XVT, if you did with those initial conditions of course, X not, V not, 0 so you integrate over the other variable, velocity, you would then get a distribution in X, the conditional density for X, the exact conditional density for X. That is not this because as he points out, this is true only in the lesion when gamma D much much greater than 1. Only there is this true. You really have to go back here and do this.

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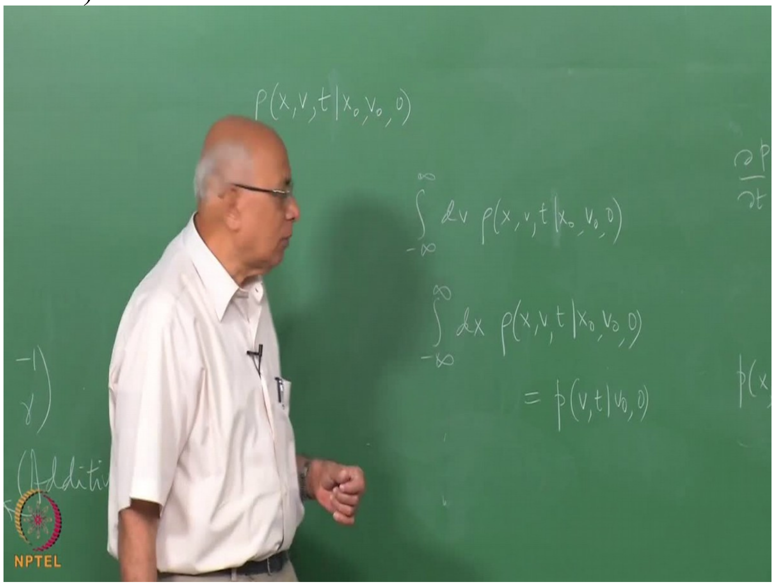

Similarly if you got the exact answer here and you did this, - infinity to infinity DX rho of XVT X not, V not, 0, you would expect to get. What would you expect to get here? You would expect to get, this should be equal to P of VT V not. You would expect to get this because you would get the conditional density in the velocity now and you do. And you do in this case. The X not dependence must somehow disappear and you should get this. But what happens when I am anticipating myself a bit?

What happens is you did this? You integrate over the velocity. Is that, you get an answer for P of XVT which involves both X not and V not which immediately tells you that it is not a stationary process. X alone is not a stationary process. Even worse, we know that this quantity satisfies a Fokker Planck equation the velocity variable alone, this quantity does not satisfy anything of that kind. It does not satisfy any simple master equation. So that is the problem. This is the problem. It is a highly nonstationary process even in the diffusion limit.

Student: (())(37:37) velocity also, there will be X not, V not?

There is not. There is V not dependence of course, we saw that. There is V not dependence. Certainly, there is V not dependence but the point is, when you integrate here, this quantity, there should be no V not dependence. There should be X not yes, but there is V not dependence as well. So you cannot decouple the velocity completely showing that this is retaining in the correlations, memory, et cetera okay. We will see, we will see what the solution looks like and then we will be able to examine this.

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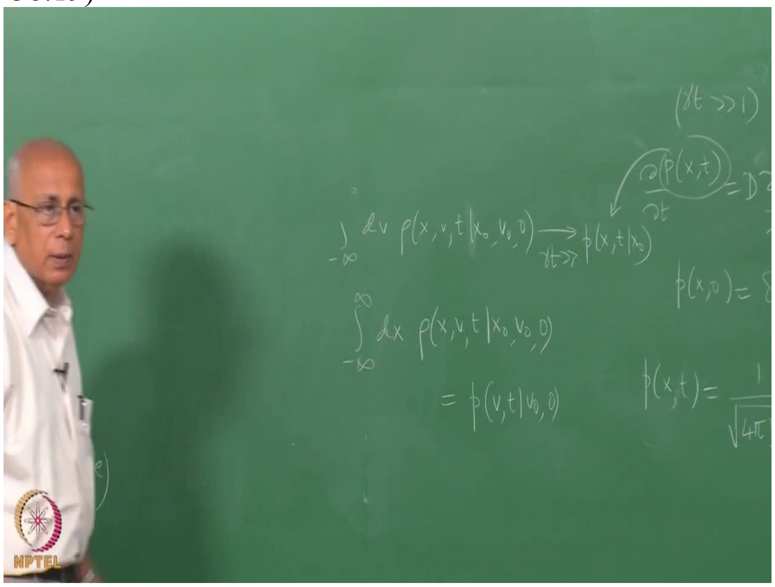

So that is a good point that this quantity is equal to this only in the limit. So as gamma T is much much greater than 1, goes to this P of XT, this P of XT. Goes to that but not for all T. Okay so the question is what is the Fokker Planck equation here? And the answer is the following.

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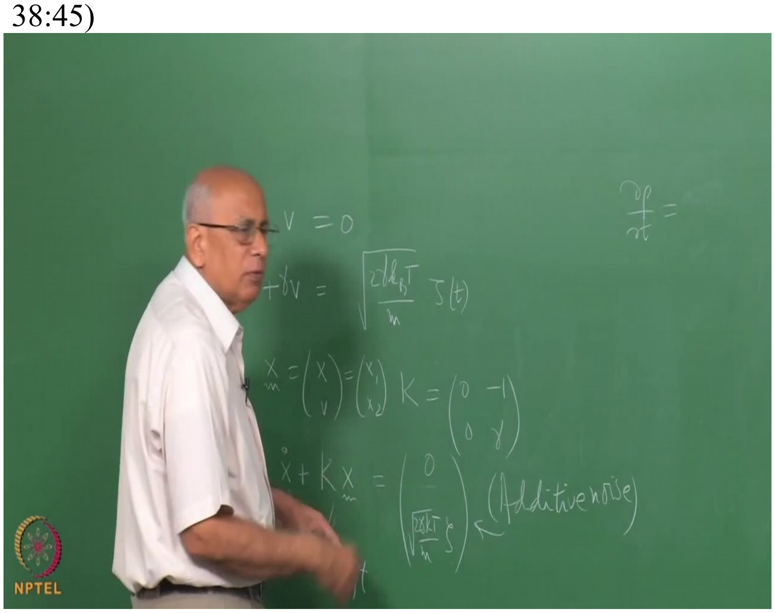

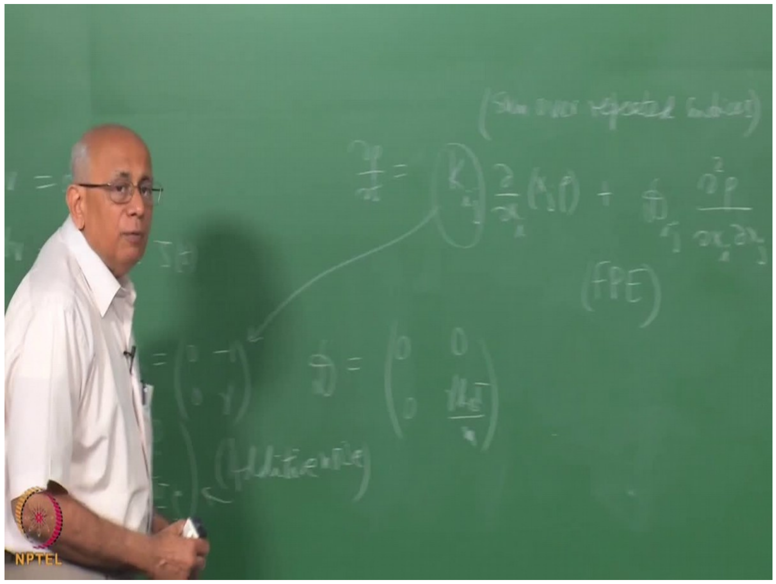

You have Delta rho over delta T equal to in such a case, let me call momentarily let us call this equal to X1, X2 in index notation, 1st component, 2nd component. Then this is equal to delta over delta XI, sorry it is equal to KIJ delta over delta XI XJ rho where a summation over repeated indices apply + the term which is the diffusion term. It will look like some generalised diffusion matrix here, DIJ times D2 rho over DXI delta XJ. We already know this, we already know what is KIJ is. It is this. I have to write down what is DIJ, the diffusion matrix. Let me use another symbol for it. Let us use DIJ.

So this D in this case is 0 0 0, that is not surprising because this is essentially root twice the diffusion constant in velocity space. That is what this is. So that is the Fokker Planck equation. In phase space all we have to do is to substitute. For this case, substitute for this D and we are done. So let us write it out.

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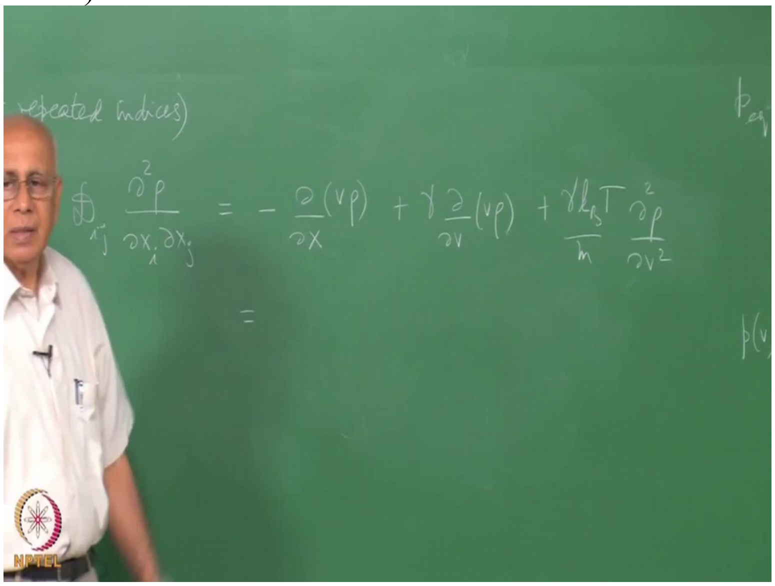

So this becomes equal to, the first-term that is going to contribute is K12 out here and that is going to have a - sign delta over delta X1 that is X, 2 is 0. And then K22 is going to contribute. So that is going to be + gamma delta over delta V V rho, that is it. + this fellow, the only term that contributes is D22. So + gamma K Boltzmann T over M D2 rho over delta V2. Okay. Pardon me. Well X2 is V. X2 is V and we are looking at only D22 because all the other terms are 0. But this is precisely we what we got in the Fokker Planck equation for the velocity. And this too but we have an extra term here. But you see, this term can be simplified a bit because X and V are independent terms in phase space.

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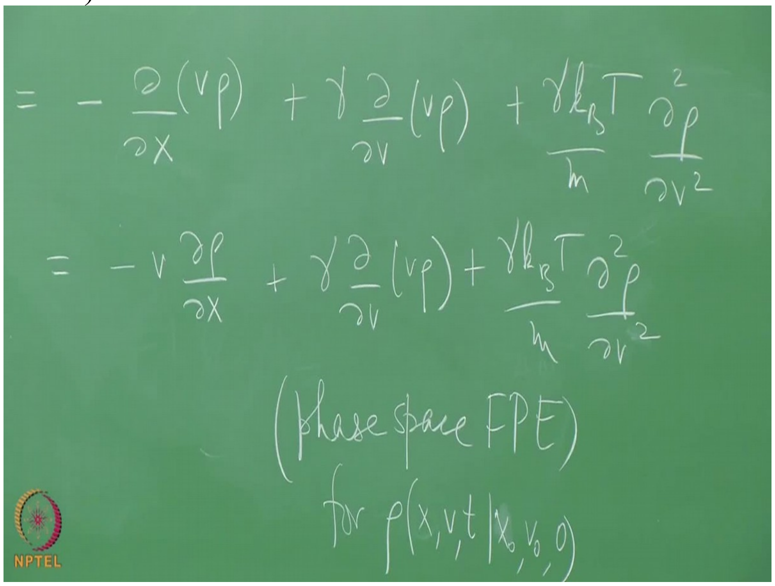

So this is equal to - V delta rho over delta X + gamma delta over delta V 0 + gamma K Boltzmann T over M D2rho over DV2. This is the phase space Fokker Planck equation for the phase space conditional density rho. Remember this is an equation for rho of XVT given X not V not at t equal to 0. We essentially have Delta of V - V not, delta of X - X not and with those boundary (con) initial conditions, you have to solve this equation here. Does this remind you of something?

Well if you bring it to this side, what is going to happen? Yes, it looks like total derivative. This looks like that convective derivative. It is indeed that. It is indeed that, it is just the convective derivative which is sitting here. No external force present, no velocity dependent force, no magnetic field, none of those. Then this looks like a convective derivative. It even has the right sign. You bring it to this side, it is convective, precisely the convective derivative which is what you kind of expect basically.

So can we write the 3 dimensional generalisation of this? That will be a horrible thing but anyways before we do that, I should tell you what the solution to this equation is. You can again solve it with delta function initial conditions, you can give an exact solution. It is a two-dimensional Gaussian, it is a joint Gaussian in both X or X - X not and V - V not E to the - gamma T. It is a joint Gaussian. So it will have an exponential which will involve - X square - V square, it will also involve an XV term in between.

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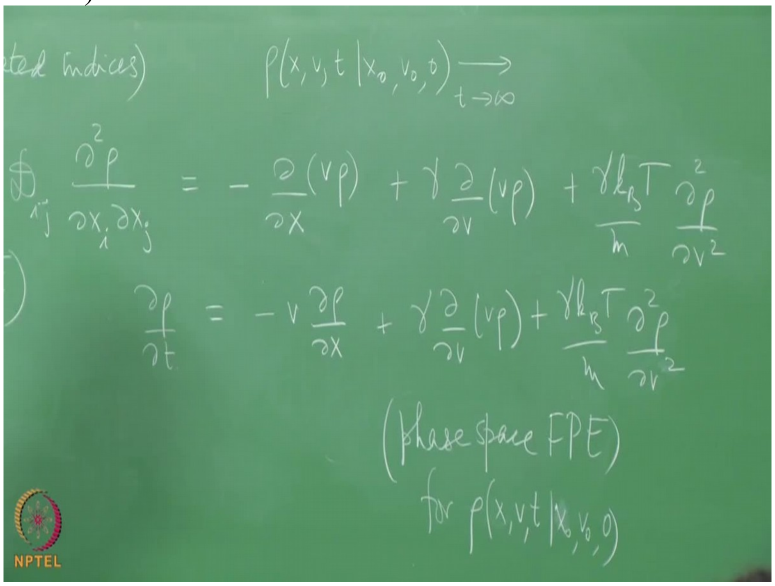

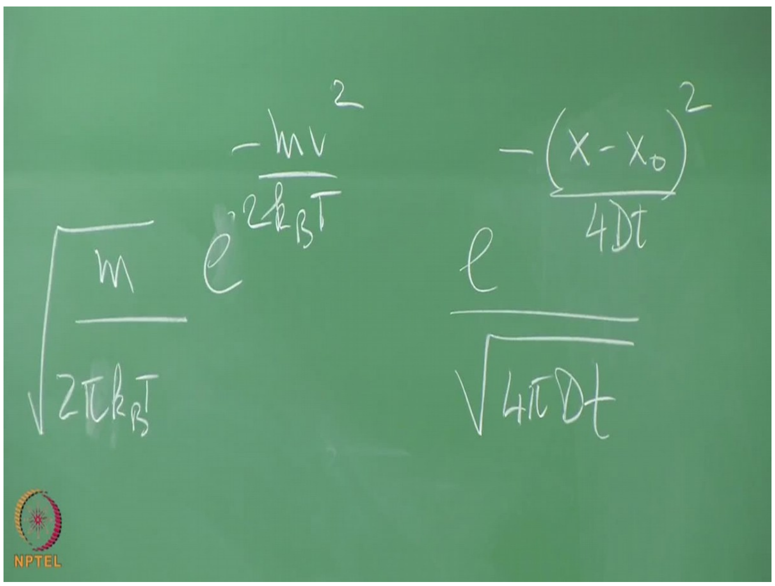

Such that asymptotically so the solution to this asymptotically what would you expect it to do for T tending to infinity? Well, the solution will actually vanish because if we look at the ordinary diffusion equation, the probability density vanishes but but we are not asking as mathematically T becoming infinite, we are not saying that. We are saying, when T becomes much larger than all the timescales in the problem, gamma T much much greater than 1 for instance, what would happen? So let us write that.

That is much more reasonable right? What would you expect it to become? The velocity term arises. In other words, it loses track of its initial value and gets into the Maximillian distribution.

So you would certainly expect that to happen. The position in the diffusion limit will have a diffusion equation solution, Gaussian. So I expect that this is going to become E to the - MV square over 2K Boltzmann T times E to the - X - X not square over 4DT times the normalisation factors. So this fellow is divided by root 4pi DT and this fellow is divided by M over 2K Boltzmann 2 pi K Boltzmann T times square root.

So I expect that to happen. And that is what? That is that should be your check. That in (()) (46:44). But as I said, if you integrate this exact expression for V, you get a very complicated thing for X which will involve V not and X not. But you integrate over X from - infinity to infinity, you will get the Ornstein–Uhlenbeck distribution with initial value V not with no reference to X not for the velocity distribution alone. What would you expect would happen in this three-dimensional case?

So we could write a generalisation of this, we could make these things vectors, here in 3 dimensions. So what would happen to this quantity? Everything else remains the same but of course, different Cartesian components of the noise would be uncorrelated to each other.

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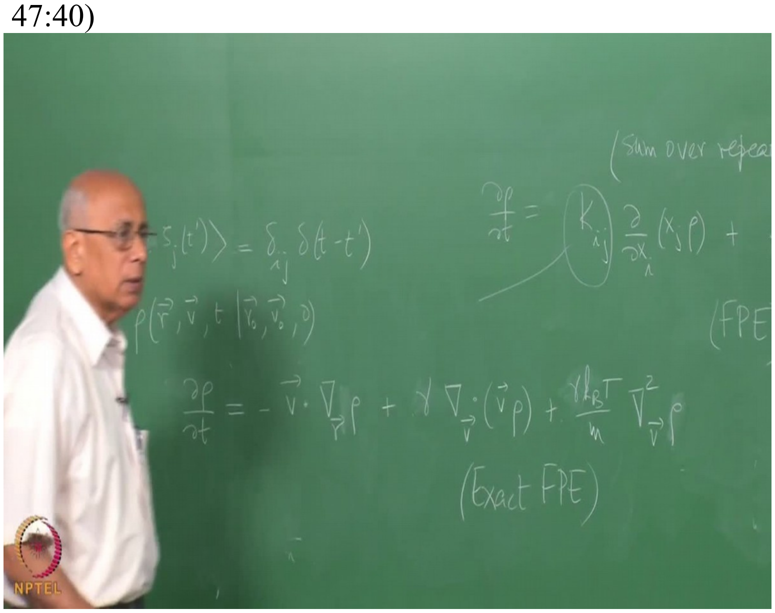

You have to assume that. So you would certainly have to assume that you have a zeta I of T, zeta J of T prime. Average value is equal to delta IJ delta of T - T prime. You have to assume that. That is certainly true. And then what this quantity does rho of R VT given R not V not at 0, we can write down a Fokker Planck equation for this rho for this rho, phase space density in 3 dimensions. We can read it out from here. Just the vector form of it. So we will again get delta rho over delta T is equal to what would happen to the 1^{st} term? - V not grand rho grand with respect to R, with respect to the components of the coordinate + what is the next term going to be?

So gamma times, grand with respect to V dot V rho right + gamma K Boltzmann T over M grand with respect to V squared, this is the exact Fokker Planck equation for the phase space density in 3 dimensions. Just a straightforward extension of this. And the solution is a generalised Gaussian in all 6 variables with the same sort of properties once again. So once you have this correspondence between the stochastic differential equation and the Fokker Planck equation then the matters, writing down the Fokker Planck equation is very straightforward okay?

Now we have to go to the next stage where will get cases where you have multiplicative noise and then the question is what happens if you have a higher dimensional case? We should be able to write generalisation there too and we will do that and we will apply at least in the simplest instances, we will apply to a couple of examples so that you see what the use of this (())(50:05) is. But this is a fairly intricate problem. At the same time, it is amenable to exact solution in this case. Okay, let me stop here and we will take it up.