Mechanics, Heat, Oscillations and Waves
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Lecture – 09
Kinematics in A Plane

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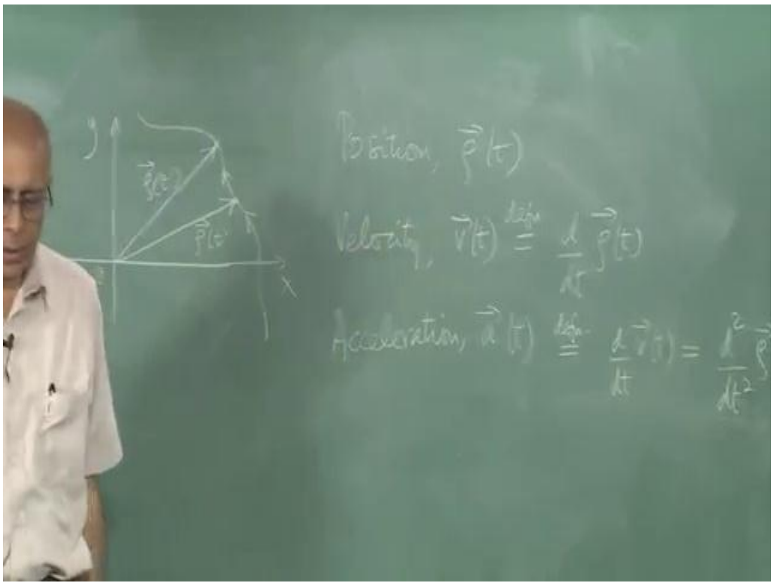

So, now let us look at the kinematics of a particle, whose position instantaneous position is given by this radius vector rho of t.

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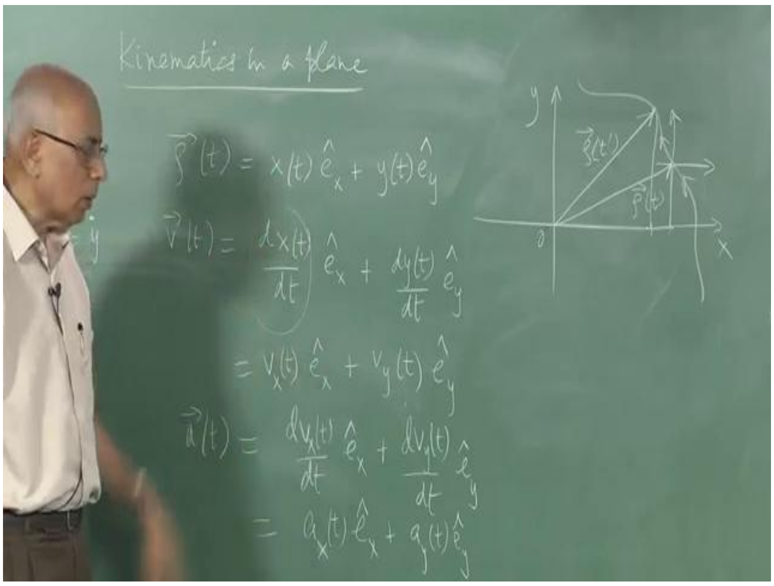

And what we would like to do is the following, give a rho of t; you can always decompose it along the x direction and the y direction, wherever you are, like this. For example, sometime t and we can write this as x which changes with time e x plus y of t, e y. Notice that, I do not put any time or coordinate dependence here for e x and e y, because they are constants.

They are constant in magnitude as well as direction once in for all, and therefore there is no argument here for e x and e y. Unlike, what happens to x and y, because at this particular instant of time, this was the x coordinate and that was the y coordinate. At a later instant of time, the new x coordinate is here and the new y coordinate is there and so on. So, these are functions of time whereas, these are absolute constants.

The velocity then v of t is the time derivative of this quantity and what to need to do is to differentiate x of t with respect to the d t times e x. By the Chain rule, librates Chain rule, you also have x of t times the derivative of e x, but that is 0. Because, e x is a constant plus similarly d y t over d t times e y, once again the derivative of e y with respect to t 0. So, you only have the derivative of y of t with respect to time and of course, this quantity is precisely, what you called the x component of the velocity times e x plus the y component of the velocity times e y.

And incidentally, if this particle is moving arbitrarily with some arbitrary acceleration, then there is no reason, why the velocity should itself be constant, and therefore in general you have v x as a function of time plus v y as a function of times e y. Very often one uses a dot, overhead a dot to indicate the time derivative. So, very often one uses a short hand notation in which this is x dot and the y over d t is written as y dot and x dot is in this case a function of time and y dot itself is a function of time.

The next derivative, the second derivative will involve the double dot there and so on. This becomes very combustion to use, this was the Newton's original notation, putting dots for derivative with respect to time, he called it fluxions. But, as you can see the notation is rather combustion, it helps in the first couple of derivatives, but after that it becomes difficult. Most simply for us in Mechanics, it turns out by Newton's laws that the acceleration is what is described by the external force by whatever force you have applied.

So, one does not generally need to go beyond the second derivative, and therefore two dots is something we can manage. But, there are cases where you need to put more derivatives, and then it is not a very good notation, it has be v x and v y in this fashion. Now, what about the acceleration, a of t? This is the derivative of this quantity. So, it is d v x t over d t, e x once again, there is no derivative of this, because it is a constant vector plus d v y of t over d t times e y.

And these in general, could also be functions of time, there is no reason why the acceleration should be uniform acceleration, all though that is a special case. So, in general, this is equal to a x of t e x plus a y of t e y and so on. So, in principle the kinematics at this level in Cartesian coordinates is very straight forward, no problem at all. And, therefore in this matter is simply a matter of finding a x and a y given the force on the object, and then working back to find x of t and y of t.

So, much for kinematics in plane polar coordinates, this may really nothing to it at all. But, if you go to polar coordinates in Cartesian coordinates, but if you go to polar coordinates, then the matter becomes more interesting and a little more identical.

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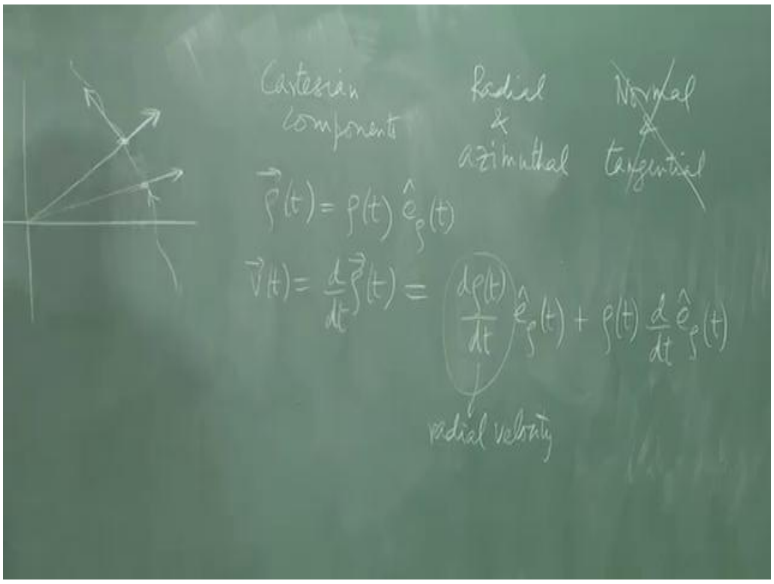

What is it, we are trying to do? We are trying to ask, given the fact that the position of the particle is some given radius vector, some radius vector in the initial time, we are trying to find, if it is path is like this. We are trying to find at every instance of time, we are trying to find, what is the radial velocity for example or the radial acceleration and what is the acceleration which is in the azimuthal direction will be…

Notice it not necessarily the tangential acceleration, because that would be tangential to the path here. Here, we are not saying that, we simply saying, there is an acceleration or the velocity in the radial direction moving away from the origin, and then in a direction azimuthal direction perpendicular to the radial direction. So, we have three different kinds of components, we have x and y components, Cartesian components, and then we have polar coordinates.

So, this is radial and azimuthal, you could also say what is the component of the velocity along the tangent, the velocity is always along the tangent. You would also ask, what is it normal to the path of the particle at any given instant of time. So, you would also have for example, normal and tangential components, we are not going to do this at the moment. Since, we are focusing on plane polar coordinates, but they can also be found at any instant of time, when useful we will do.

So, the immediate problem is the following, given this rho of t, what is the velocity equal to. So, let us write rho of t equal to rho times, it is a function of times e rho, which is also a function of time. Because, if it is at any instant of time, it is at this point, this is e rho at that instant of time, a little later e rho point since some other direction. So, we have rho of t by definition equal to magnitude times the direction unit vector, and then the velocity v of t equal to d over d t rho of t.

Now, comes interesting part, first to differentiate this leaving that alone and then you differentiate that leaving this alone. So, by the Chain rule, it is d rho t over d t, e rho of t plus rho of t, d over d t e rho of t. We need to find, therefore this quantity here, how does this unit vector change as a function of time. Now, what is the physical meaning of this rho of t, d rho over d t? It says, if the distance from the origin was this much at time t and that is the radius vector. A little later, it could have come either closer or further away from the origin or remain, exactly where it is and that is the radial velocity.

So, this thing here is the radial velocity, because it is the component of the velocity in the direction of the radius vector from the point. It could be positive or negative, it could be increasing the distance could be increasing as a function of time or decreasing as a function of time. Increasing would mean, this is positive instantaneously, decreasing means it is negative instantaneously, but that is the radial velocity.

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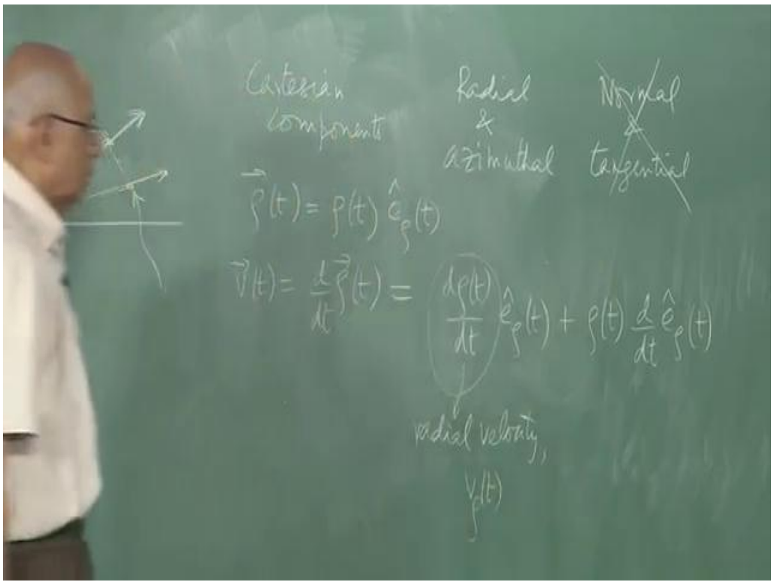

And in analogy with what we have here, I should really call it, v subscript rho, v subscript rho of t. But, now we are left with the question of finding out, what is the time derivative of this quantity here? However, we going to do this?

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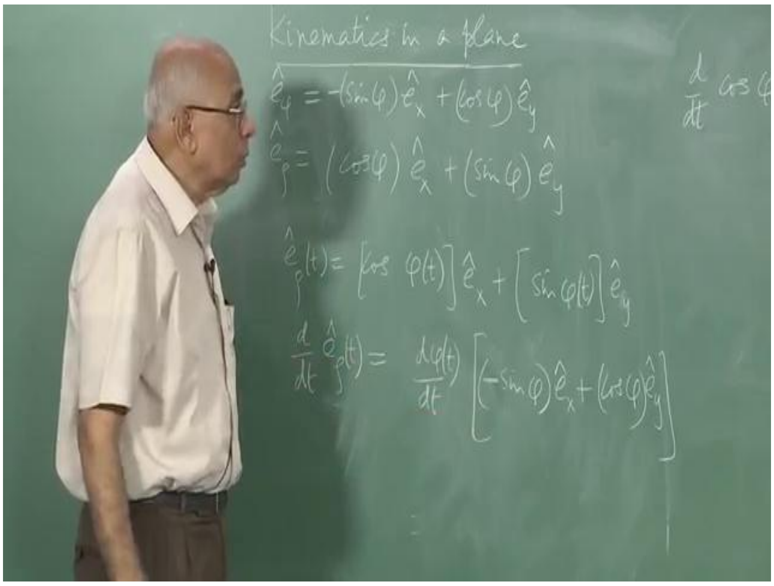

Well for that, we need to take recursive to the fact that this radius vector, the unit vector in the radial direction e rho, we wrote it down in Cartesian component. So, we had a cos phi times e x plus sin phi times e y and now, we have a particle moving, which means the rho and phi for this particle changing as a function of time. So, we have e rho of t equal to cos phi, which is a function of time multiplying a constant unit vector plus sin phi as a function of time, multiplying the constant unit vector in the y direction.

Therefore, d over d t e rho of t is equal to, with derivative of this function of time with respect to time, times the same unit vector plus the derivative of this function times the same unit vector, but the derivative of phi of t, of course, cos of phi of t. Again, it is a function of a function, you have t, and then you have phi of t which is a function of t, and then you have cos phi of t, which is a function of phi itself. So, we know once again from the rule of calculus is the derivative of this function with respect to phi.

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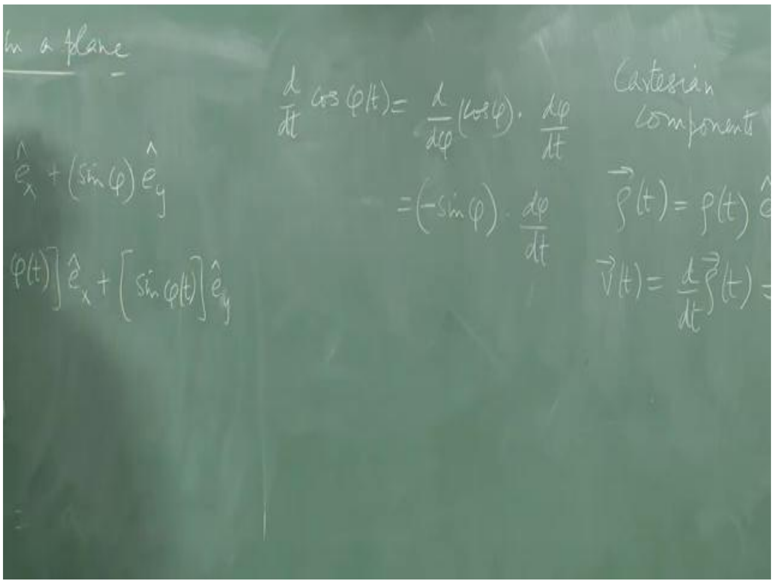

So, let me write that down separately d over d t cos phi of t equal to d over d t d over d phi cos phi multiplied by d phi over d t by the chain rule once again. But, this is equal to minus sin phi, because the derivative of the cosine is minus sin phi multiplied by the d phi over d t.

If I use that here ((Refer Time: 11:56)), this becomes minus sin phi times d phi over d t times e x, so let us write it with d phi over d t coming out. So, this is equal to d phi over d t and that is the function of time, times the derivative of this function with respect to phi, which is minus sin phi, the derivative of this function with respect to phi, which is plus cos phi. So, now, this becomes minus sin phi times e x plus cos phi times e y.

But, if you recall, what we said about vector that is normal to this e rho, the unit vector in the azimuthal direction, if you recall the formula for that, it was precisely minus sin phi times e x plus cos phi times e y. So, that is exactly the combinations.

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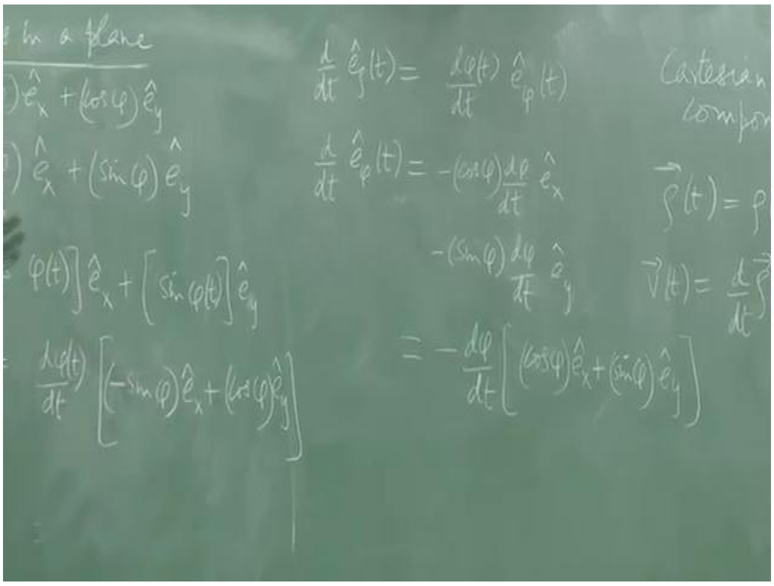

So, we have a very useful formula with says d over d t of the unit vector in the radial direction as a function of time. In general is equal to this quantity here; that is e phi. So, we have d phi over d t and e phi, which is of course a function of time in this fashion, a very useful formula indeed. In fact, while we were at it, we may as well find out, what is the derivative of this function ((Refer Time: 13:44)) with respect to time. So, let us also write d over d t e phi of t equal to the same thing as before we differentiate this with respect to time.

Since, phi is a function of time, the derivative of sin phi is cos phi, the minus sin here, and then the derivative of phi itself. So, this is going to become minus cos phi d phi over d t times e x. So, this is minus cos phi d phi over d t e x, and then this term here ((Refer Time: 14:25)), the derivative of cos phi is minus sin phi. So, there is a minus sin phi, and then a d phi over d t to differentiate this function here times e y.

But, that is equal to a minus sign outside d phi over d t outside, and then inside the bracket you have cos phi times e x plus sin phi times e y. We go back to this equation and realize that combination in this square bracket is just e sub rho.

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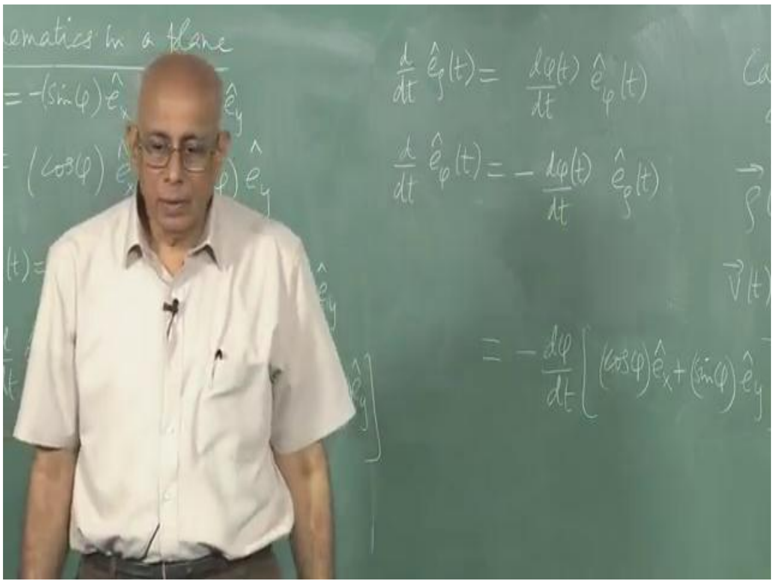

So, we have the compliment of this formula which says, this is equal to the minus d phi over d t of t times e rho of t. So, to sum up, the rate of change of the radial vector, unit vector in the radial direction is along a unit vector, there is a unit vector here along the azimuthal direction. The rate of change of the azimuthal unit vector is along the radial direction with appropriate signs here and this quantity d phi over d t could be positive or negative.

But, whatever it is, it multiplies these unit vectors. So, the whole point is, the rate of change of this vector, unit vector is along the other unit vector and the rate of change of the other unit vector is along the first unit vector, where these two unit vectors form a basis for all points in the plane. So, once we have that in place, you can now write down these formulas properly.

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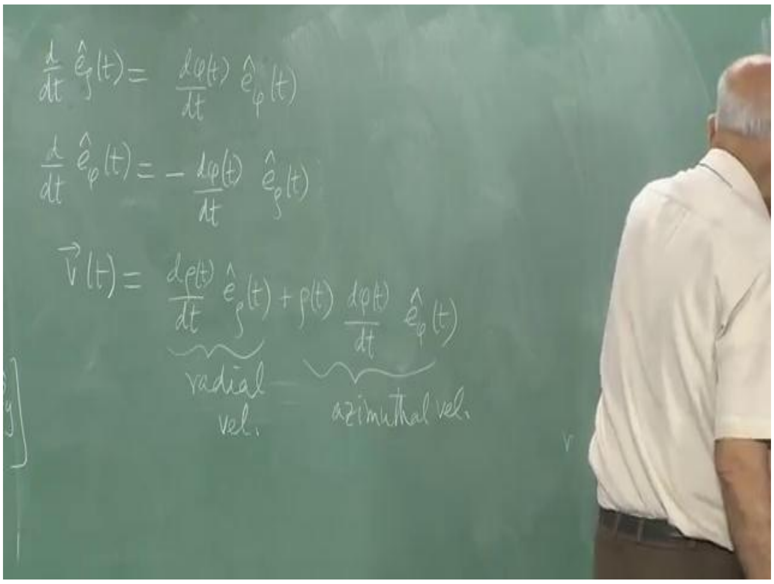

This therefore becomes v of t is equal to this quantity here is d rho of t over d t times e rho of t here, and then for e rho of t over d t e rho of t, I need to substitute this. So, this is a term here, which is plus rho of t, d phi over d t multiplied by d phi of t. So, just as the original rho of t had only a radial component e rho of t times rho. The velocity now has however, both a radial component as well as azimuthal component. So, this is the radial velocity and this is the azimuthal velocity.

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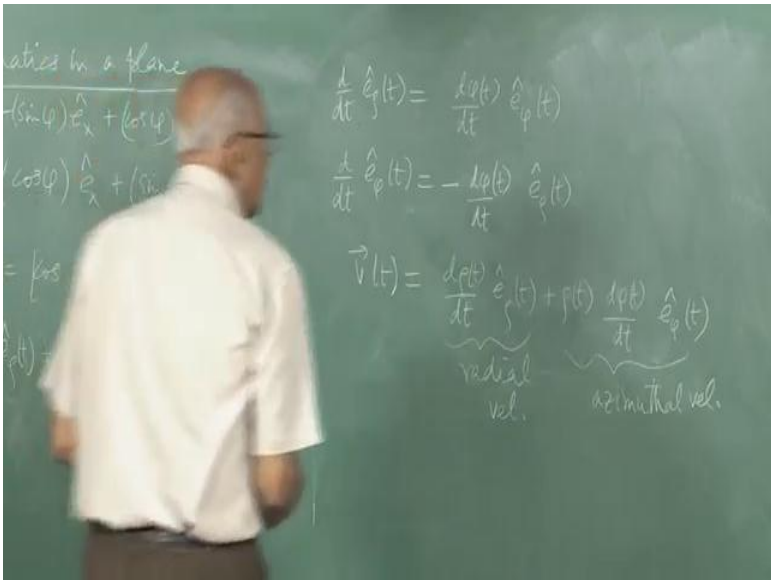

But, we can do a little more physical identification here; we can see that what is really happening is that, if I call this v sub rho, it says v of t equal to let us put dots to tells us, where the time derivatives are going in. So, there is a rho dot as a function of time of course, and then e rho of t plus rho of t phi dot of t, e phi of t and this quantity here is what I called the radial velocity. Strictly speaking, I should multiplied by the unit vector, then call it a radial velocity.

But, you know this rules terminology, very often, when I resolve a vector into components, even without the unit vector, I still call it a component. Whereas, a vector must be a sum of vectors cannot be a sum of functions.

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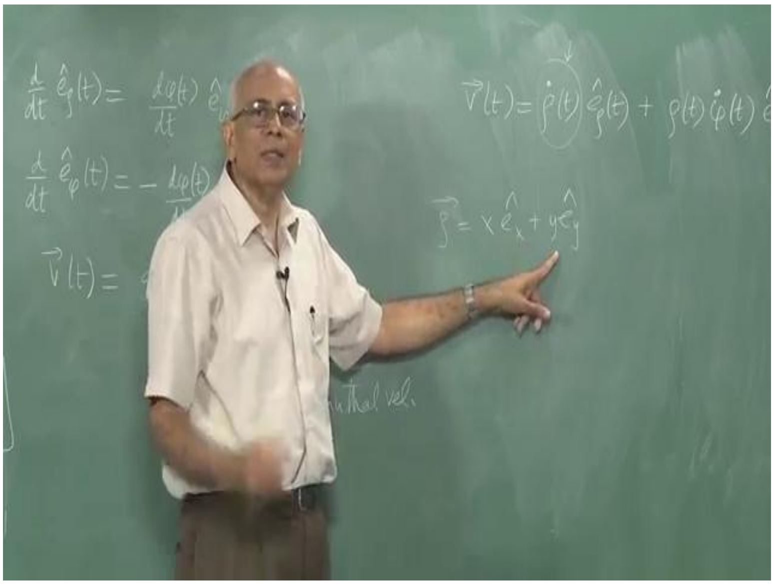

What I am saying is, if I write rho equal to x e x plus y e y, strictly speaking, I should call this the x component and this the y component of the vector, because component means part. But, in rules language, I often say x is the x component, y is the vertical component of the vector. So, in the same sense, one often calls this the radial velocity is actually the radial speed.

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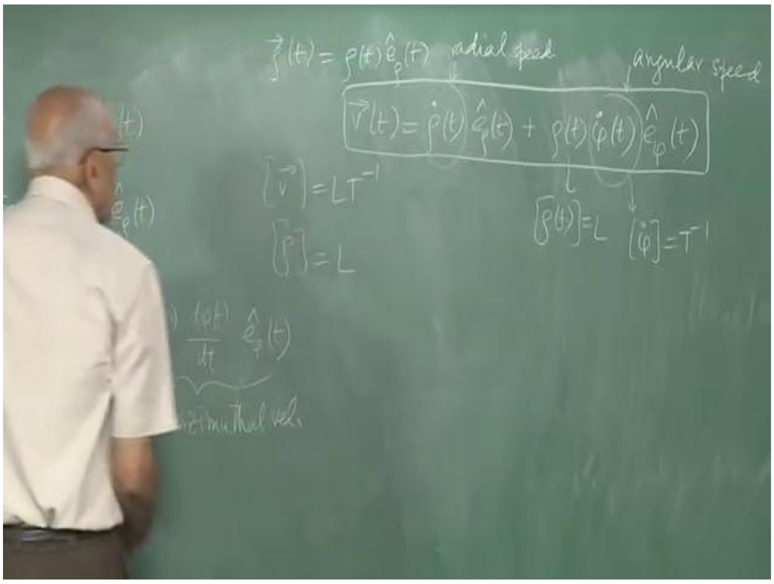

So, this is the radial part, it comes whenever the distance from the origin changes as a function of time, if it does not is moving in a circle that part in the 0 identically. And this part, you can now identify, what is happening, this quantity here is obviously the angular velocity, because angular speed. Because, this thing here, tells you how the angle, the azimuthal angle changes is a function of time. So, this is the angular speed, let us be technically correct was radial speed.

Now, you should check physical dimensions all the time, the distance of course, are the displacement is got dimensions of line, physical dimensions of line. Velocity has physical dimensions L T inverse in this fashion and we must make sure; that is happening everywhere. This rho has dimensions of a length and rho dot has dimensions of L T inverse. So, that is okay, this part is okay, that is the unit vector, it may change it is direction which respect to time, but it does not matter, it still a unit vector and it is dimensional less, no dimensions at all.

So, this part is all right, it is got dimensions of velocity, what about this, well, this is dimensions of length and what about the dimensions of angular velocity, angular speed? Well, angle has no dimension at all, it measured as R divided by radius as you know, it is dimensional less quantity, but that as the dot sitting there. So, there is a 1 over t sitting here, and therefore this is dimensions phi dot as dimensions t inverse.

And again as before e phi of t has no dimension at all, it is a unit vector, and therefore is the dimensionless quantity. So, we are okay dimensionally, the velocity now has been decomposed in to a radial part and an azimuhal part this fashion. Starting with the fact that the position itself had only a radial part, so we started with that and we derived this fundamental relation here, this tells you that the velocity would in general have two components.

One, because the distance from the origin is increasing or decreasing and the other because the azimuthal angle in the path is changing, but a point in the path is changing and these are the two component. This is the analog of the Cartesian quantity we wrote down, we wrote v of t is v x times e x plus v y times e y. Now, we going to straightly more complicated quantity is does not angular speed coming in the radial speed coming on and so on.

Well, at a next stage and we do need to go the next stage, because as you know Newton's law is going to prescribe the force which will control the acceleration. So, we need a formula for the acceleration.

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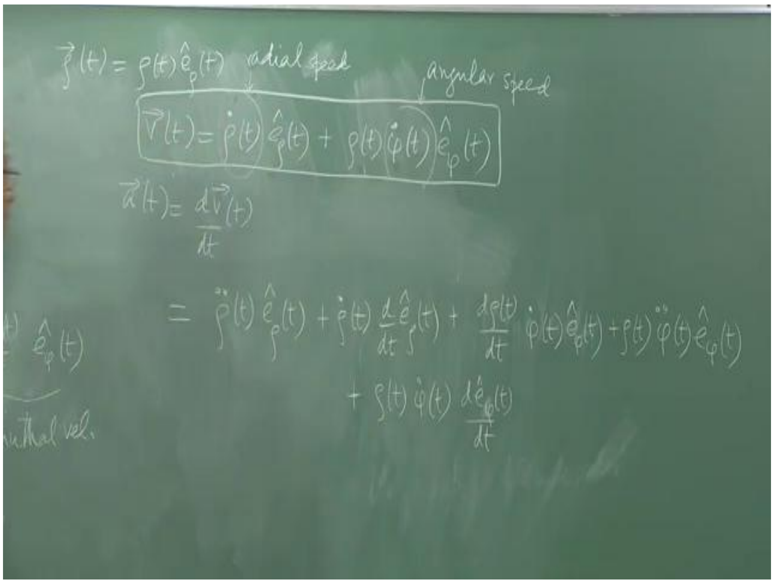

The next thing we have to do is to find a of t equal to d v d t and notice this going to be equal to. Well, again I use the chain rule if I differentiate this, which is all ready d rho over d t and I am going to get d 2 rho over d t 2, which is rho double dot of t, and then this part remains and change. So, that is the derivative of this term times that by the librates Chain rule, and then there is a rho dot times the derivative of that. So, this is rho dot of t times d over d t of e rho of t; that is the next term plus I need to differentiate this.

So, there is a d rho over d t and this fashion and the rest of it remains unchanged phi dot of t d phi of t plus again by the Chain rule, there is the rho of t, I differentiate this that gives me a phi double dot of t, d phi of t. And finally, this one more term, which is rho of t, if I dot of t, d e phi of t over d t. So, you really have large number of terms, it differentiate this, you get two terms, it differentiate this, you get three terms, and therefore you have phi terms in that and we have to add ((Refer Time: 23:58)), well, let us do that.

A target again is to realize is to see that is to exploit the fact that, finally the acceleration again being the vector in the plane should be decomposable into portion, which is proportional to e sub rho of t and another portion, which is propositional to e sub phi of t. So, we need to find out, what is the radial component and what is the azimuthal component for the acceleration and that is where we heard it for.

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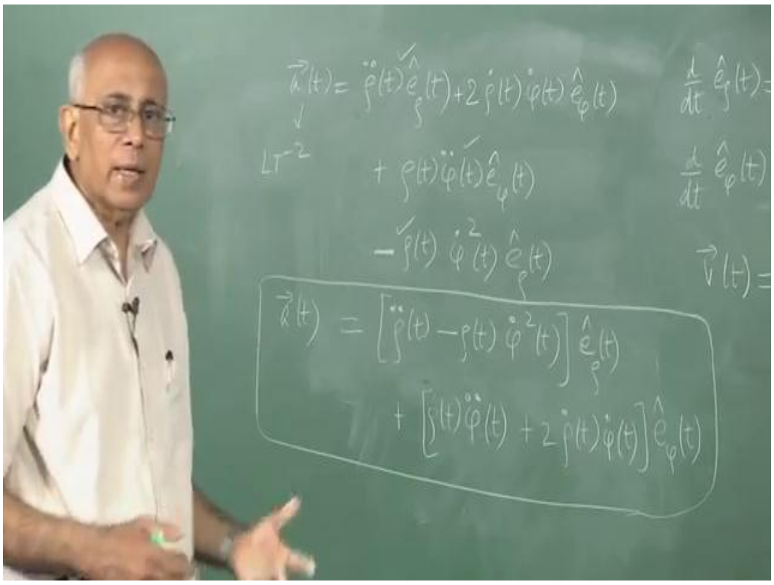

So, we have a of t equal to the first term is rho double dot of t e rho of t and you will recognize, what this term physical meaning is very simply, because just as this portion of it was the radial speed, this part is the radial acceleration. If I need that is the first term, that part is taken care of plus rho dot of t, and then that is find out, what this quantity is. But, we already have a formula for that, that says d over d t d e rho of t is phi dot times e for phi of t. So, phi dot of t, e phi of t, just reduce are the rotational a little bit, I have call d phi over d t phi dot there.

So, that takes care of this term and this term ((Refer Time: 25:29)), and then we have a rho dot phi dot e phi of t. So, we need to put that in rho dot phi e phi of t, it is the same as the terms, so you really have to twice that and that has taken care of this 3rd term. And the 4th term is rho times phi double dot, e phi of t, leave it as it is plus rho of t phi double dot of t e phi of t one more choice in the matter, this is gone.

And finally, you have rho phi dot d e phi over d t, which is equal to phi dot times e rho. So, this is minus sin there and there is a phi dot appearing once again. So, this is equal to plus rho of t, first part, and then phi dot squared of t. So, you differentiate, and then you square it some phi dot squared of t times e rho of t. So, check dimension once again, this is dimensions, physical dimension of l t to the minus 2, because it is an acceleration.

This is dimension l t to the minus 2 from here, so that is okay, this is l, t inverse t inverse, so that is okay, this is l, t to the minus 2, this dimension that is okay, this is l, this is dimension is, and then it is dot squared. So, that is theta minus 2 and that is okay too. So, we are dimensionally all right and all way need to do finally is therefore is to collect portions prepositions to e rho and collect portions proportional to e phi and the discover that this little minus symbol.

The last term was rho of t phi dot of d e phi over d t, but the d e phi over d t as minus sin. So, this is minus. So, this says rho double dot of t minus rho of t phi dots squared of t, this fashion that multiplies e's are rho of t. So, takes care of this term and this term, and then we have terms which look like plus rho of t phi double dot of t, that takes care of this term plus twice rho dot of t phi dot of t multiplying phi of t.

So, that is our grant finale that is our final formula for the general acceleration on in a plane in plane polar coordinates. A radial portion and an azimuthal portion, surprisingly the radial portion and unlike the velocity, we just had a radial speed how here and this was rho times the angular speed if you like. And the acceleration, some new terms of appeared, precisely because these unit vectors change the time.

So, what is happen is that the radial acceleration, which is appeared here, but there is an extra term which is appeared here, which depends on the square of the angular speed, out here multiplied by a radius vector. Similarly, in the case of the azimuthal component, we got this for dimensional reasons times and angular acceleration, because is the second derivative of the change of the azimuthal angle. The first derivative is angular speed and this is the rate of change of the angular speeds.

So, it is the angular acceleration, just as this is the radial acceleration, but there is an extra contribution, which is dependent on a combination of both the radial speed as well as the angular speed. So, you have this extra term here and it has immerse completely naturally simply by differentiating these unit vectors carefully. But, these terms will turn out to have physical significant, this is the most general formula that you can have for the acceleration for a particle moving in a plane.

Everything is already here in here, they could be cases depending on the special motion for instance of the particle is moving on a circle, then the distance from the origin is fixed. And rho, therefore is constant, this becomes the radius of the path, this is identically 0, this term is then identically 0, does not change at all. On the other hand, once you have a variable rho, these terms are non-zero and they exist, you cannot help it.

So, that is exactly what this kinematics does, it is gives you all possible terms, then says and special cases things may reduce may becomes simpler. Later on, will see that this term this term here is related to what is call the Centrifugal acceleration in a rotating frame and this term here is related to what is call the Coriolis acceleration. So, these terms are actually related to what are called non inertial forces in a suitable contacts and we will talk about that and I get to it.

But, I want it to appreciate right away that this general formula arises for using no more than very simple vector analysis and using the Chain rule for differentiation of functions of time. And this all this be done and all these things are already popped out automatically and this is in a plane, moment you go to 3rd dimension, things can get a little more complicated. But, this already beings to tell you that, when you write this problems down in coordinates other than Cartesian coordinates, then various physical of x can be seen to immerse directly.

And we will see how this a place of a roll, then the analysis motion for instance, the non uniform circle motion in a plane and so on. If you have uniform circle a motion, then the angular speed is also constant and this term vanishes identically and this is replace by some omega squared, where omega is a constant, uniform angular speed. So, all special cases can be write out from this, but this is the general form.

Of course, you can go into differentiate once more get an even more complicated formula, like as we said when we do dynamics because of Newton's law is saying that the force will prescribe the acceleration, you need in go beyond this in general. So, much for kinematics in a plane and the next thing we need to do is to go on to three dimension motion, the first we need to talk about vectors in three dimensions, and then we introduce our coordinate systems.