Mechanics, Heat, Oscillations and Waves
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Lecture – 08
Vectors in a Plane, Scalars & Pseudoscalers

Let us continue today with our discussion of Vectors, specifically we start with Vectors in a Plane.

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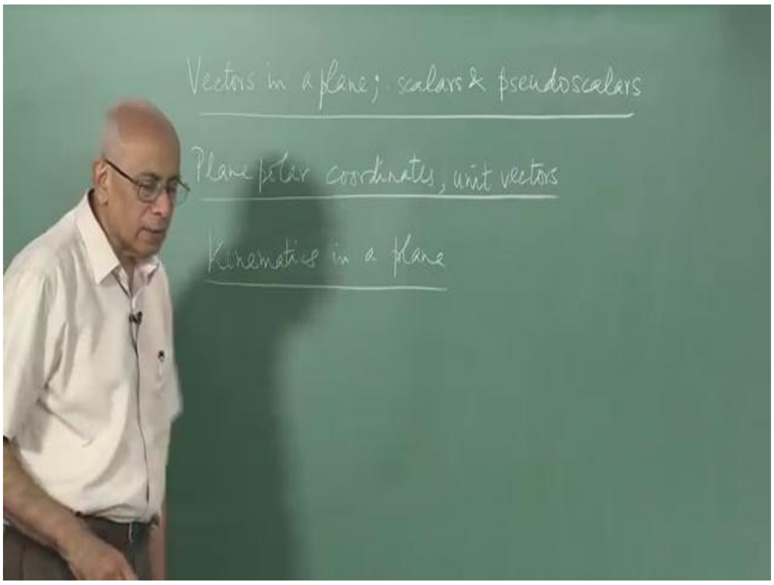

And I would like to talk about a few subjects today at few topics which would include for instance, the ideas of Scalars and Pseudoscalers and elaborate on that. And then we will look again at plane polar coordinates, discuss unit vectors, and finally the target is to try and understand kinematics in a plane in terms of plane polar coordinates, because it is of plane particles importance. So, let we start by pointing out that we define the vector as a set of quantities, the same number of quantities as the dimensions of the space that you have been interested in, which transform under rotations of the coordinate axis in exactly the same way as the coordinates themselves transform.

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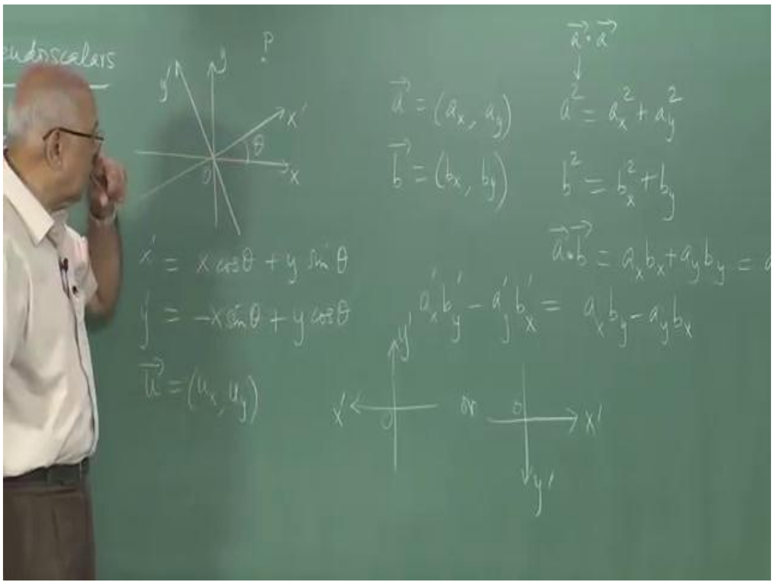

So, in a plane we have already seen that if you took a frame of reference, the x y axis like so and then you took a till that frame rotated at some angle theta with respect to the first coordinate system y prime. Then, we had a set of transformation rules which said, that the coordinates of an arbitrary point p, f their x and y in the original frame of reference, then x prime is x cos theta plus y sin theta and y prime equal to minus x sin theta plus y cos theta.

And the next statement just to repeat what I said already is that, if you have a vector any vector u which is got components u sub x and u sub y. Then, u is a vector if and only if, u x prime and u y prime are related to u x and u y in exactly the same way as x prime and y prime are related to x and y, that is the definition of a vector. Now, once we define a vector in this fashion we introduce the idea of a dot product.

So, if you have a vector a, which is got components a x and a y and another vector b, which is got components b x and b y, then from these two quantities from these two vectors a and b you can actually form three different scalars. One of them is a square which is a x square plus a y square and similarly, b x square plus b y square and the third scalar that you can form quantity which does not change under rotations.

The third scalar is a dot v which is equal to a x b x plus a y b y, that too is a scalar and that is why we call this the scalar product, the dot product or the scalar product more correctly the scalar product. Now, what this implies is that this particular combination of a and b, this rule of multiplication tells you how to take two vectors and get a scalar out of them, create a scalar out of them. And it is clear that this is just the same as a dotted with itself and this is the same as b dotted with itself, so this is a dot a by definition and similarly for b.

So, the dot product or the scalar product is a way of creating a scalar from a vector and so far, we restricted ourselves to a plane to two dimensional vectors. I also pointed out something else, I also pointed out that interestingly if you consider the combination a x b y minus a y b x, that two appears to be a scalar and this needs a little bit of working out, it is not a very hard thing to do. So, let me do that and show you how this happens.

So, fill it at a x b y prime minus a y prime b x prime, substitute for a x and b x and a y and b y formulas similar to this. So, you have a little bit of algebra to do, you would have to write a x prime is a x cos theta plus a y sin theta, and similarly for b, x prime and b y prime. Put that all inside this and do the simplification and it is a simple exercise to show that this will turn out to be equal to a x b y minus a y b x. So, just as, this quantity was equal to a x prime b x prime plus a y prime b y prime.

In exactly the same way this combination, this antisymmetric combination also remains unchanged under rotations, and therefore we would call it a scalar. But, it is really slightly different kind of scalar and the scalar we will have been taking about. Because, so far we restricted ourselves to pure rotations, but you can also ask what happens if for instance I go from a right hand coordinate system to a left hand coordinate system.

For instance, suppose I reflected about the y axis and the new coordinates where in the following fashion, they look like this. You had y prime and x prime here in this direction or for that matter, it reflected about the x axis and you had y prime and x prime over here. These coordinate systems cannot be obtained from the original x y coordinate system by rotating about the origin. No rotation about the origin, about any angle will ever produce this coordinate system from this.

Because, if we start rotating this, like so to bring the x axis point to this direction, the y axis will point downwards, but that is not what is happening here, not is it happening there. So, these transformations I mentioned briefly last time were in proper transformations. So, we have to worry about what happens to vectors and the in proper transformations as well. So, let us ask what happens if I have a transformation which say takes you from this coordinate system to that coordinate system.

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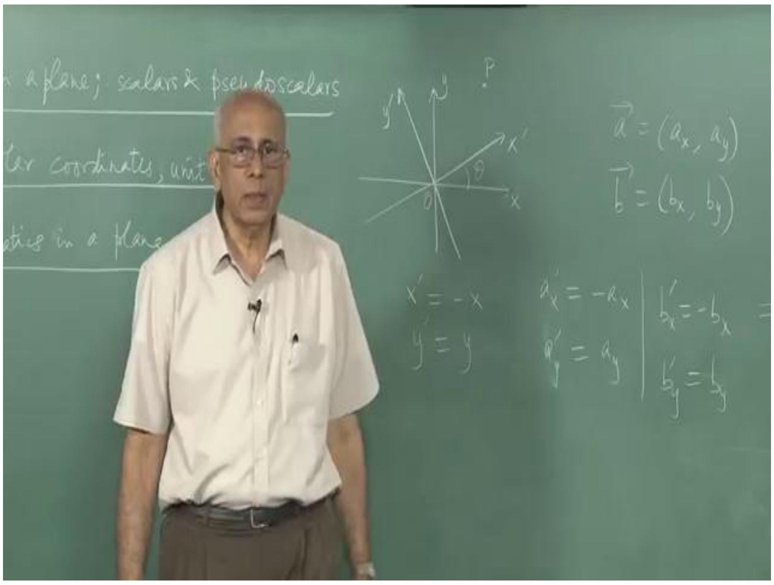

What is that imply? That says x prime is minus x, y prime equal to y, because the y prime axis is exactly the same as the y axis, but the x prime axis is the negative of the x axis, and therefore x prime is minus x. What happens under this transformation? Well, the coordinate transform in this fashion, and therefore any vector would transform in exactly the same manner, which means that this combination that you have here is going to be look like this.

You going to have a x prime equal to minus a x, b y a y prime equal to a y and similarly, b x prime equal to minus b x and b y prime equal to b y. What happens to this combination here? It is clear that if this thing ((Refer Time: 08:04)) changes sign a x prime goes to minus a x and b x prime goes to minus b x.

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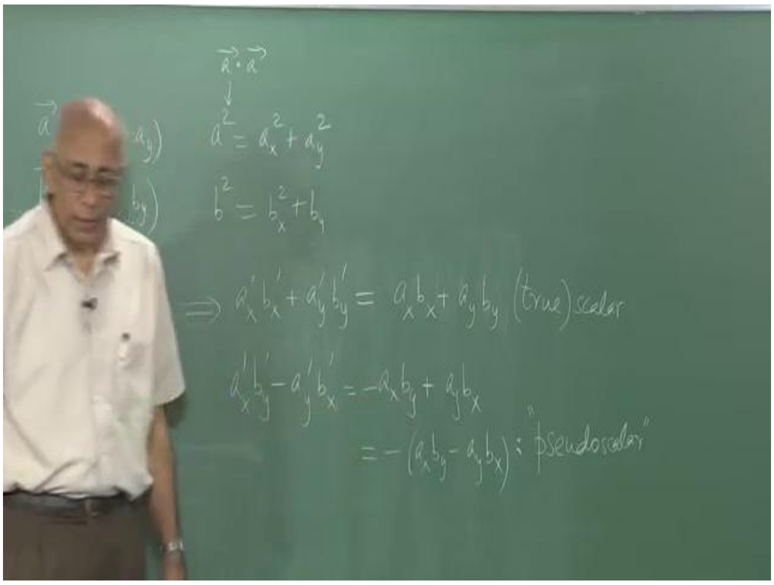

Under such a transformation, this implies that a x prime b x prime plus a y prime b y prime is equal to this changes sign, this changes sign, so the product does not change sign. It remains once again a x b x, and of course the y's would not change sign at all, so a y b y. Therefore, this quantity is a scalar not only under rotations, but also under reflections.

Of course, if you have a reflection about the x axis, it is a simple matter to see that what will happen then is that the x components do not change sign, the y is too and exactly the same thing goes through. So, this is a true scalar, not only under rotations, but also under reflections under in proper transformations. But, look at what happens to the other combination, look at the combination a x prime b y prime minus a y prime b x prime.

This is equal to well, a x prime changes sign b y prime does not change sign, so you have a minus sign. So, this is minus a x b y plus a y b x, because this does not change sign, that changes sign and cancels the minus sign give you plus there, but this is equal to minus a x b y minus a y b x. So, you see this combination which was the scalar under rotations has actually change sign under reflection.

So, under a proper transformation, the rotation is called a proper transformation, because you can continuously start from the original coordinate system and go to the new coordinate system, smoothly whereas, reflection is a sudden transformation if you like and this continues transformation. Under the proper rotation, this remains like a scalar, but under a reflection it changes sign, such an object is called a pseudoscaler.

It is still a scalar under rotations, but it is transformation and the reflection is different from that of a true scalar. And now, this tells you that there are different kinds of scalars that you can possibly have and in this case we see two different kinds of scalars. This will also generalizes as we go to higher dimensions to vectors, you have quantities which look like pseudo vectors and at that stage, I will give you physical examples of these quantities.

Just to cut a long story short, in three dimensions the position vector is a true vector, the momentum vector is a true vector, the angular momentum vector for instance will turn out to be a pseudo vector. It will have a different transformation property and a reflections or parity than a true vector has and we will see how. So, it makes, it make sense to deal with these to discuss these matters, because physical quantities have specific transformation properties and that is important to understand, how and why this happens.

Now, that we have done this, let us go back a little bit to what we have discussing namely plane polar coordinates. And I should now like to take about the unit vectors and what happens to them, as the position changes.

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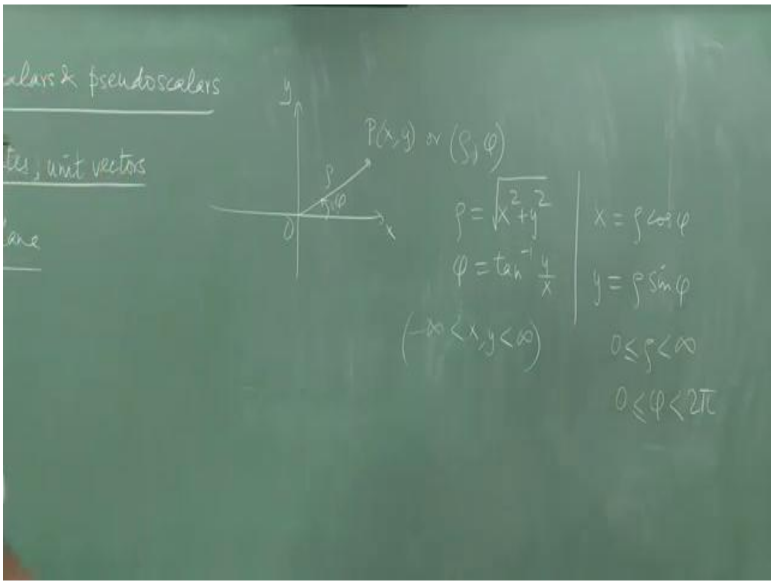

So, let us go on to this topic now that is over, let us go back to this and look at what we said about plane polar coordinates. Here is the x axis, here is the y axis, that is the origin and my point was that any point p which has coordinates x and y in Cartesian coordinates could also be regarded as having coordinates or plane polar coordinates. And I use the symbol r for this distance and an angle phi, the angle from the x axis to the line joining the point to the origin.

And this quantity I called it r, I should now like to change that notation a little bit, because I would like to reserve r for the distance from the origin to a point in even three dimensions and I do not want to confuse that with this r here. So, let me call this rho, this distance rho. So, it was rho and phi and this rho is just x square plus y square, square root and this phi is time inverse y over x and the corresponding inverse transformations were x equal to rho cos phi and y equal to rho sin phi.

I am rewriting the same formulas that you wrote down on the last time, except on I replace the symbol r with the symbol rho, because I do not want to use the symbol r in two dimensions. I would like to use rho, because when I go to three dimensions, then we will turn out that the projection on to the plane on to the x y plane that distance I will continue to call it rho for consistency. Now, what are the ranges of these variables?

Well, minus infinity to infinity in the case of x and y, that translates here to 0 less than equal to rho less than infinity and 0 less than equal to phi less than 2 pi to make it single value. So, those were the ranges, these are the transformation formulas from Cartesian to polar coordinates and these are the transformation formulas from polar coordinates to the Cartesians on this side.

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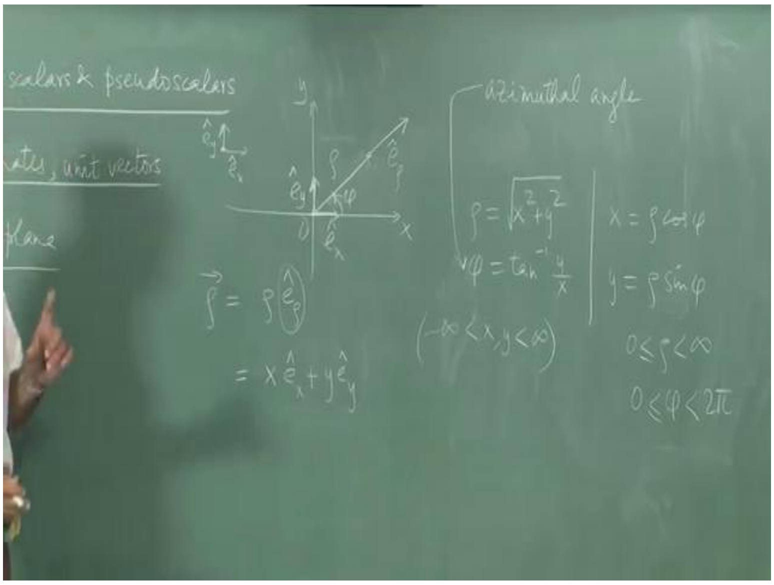

Now, any arbitrary point p such as the one I have written here could always be written, this point the radius vector to this point is the vector rho and it is magnitude is rho without the arrow. So, we immediately have rho equal to rho times e sub rho, where this is the unit vector in the radial direction, in the direction pointing away from the point along the line joining the origin to that point. So, this is the unit vector e sub rho, and then this vector rho this is, this distance is rho and it is pointing the direction.

So, it is clear that the radius vector from the origin to any point is the magnitude rho multiplied by the radially outward unit vector, I call this the radial direction. This angle that we have here is phi is the azimuthal angle, I could of course, also write this rho as x times e sub x plus y times e sub y, where e x and e y are unit vectors. So, this is the unit vector in the x direction, e x that is the unit vector in the y direction e sub y.

And I pointed out that wherever you are in the plane, the unit vectors in the x and y directions remain unchanged. Unlike the radial unit vector we changes, by go to this point it points outward like that, if I go here it points that, this fashion and so on and we would like to know what is the dependence of this e rho on the coordinate themselves and that is easy to write down.

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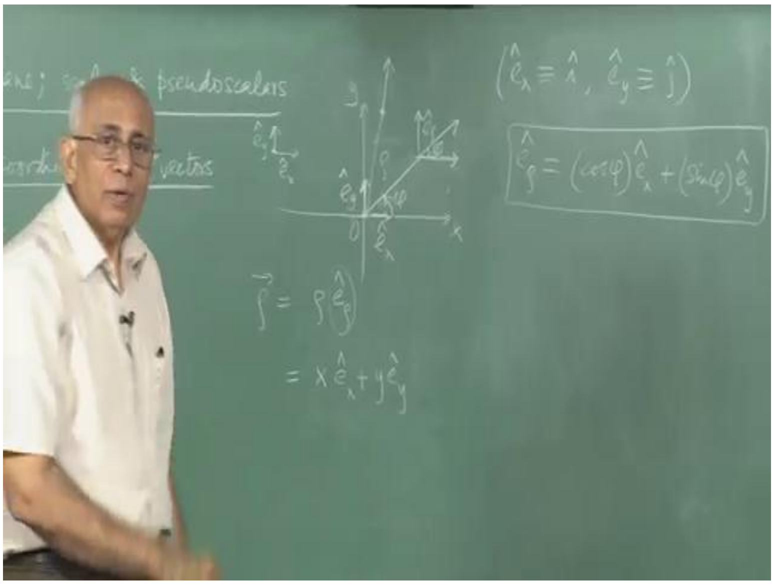

Because all you have to do is to take this vector e rho and resolve it along e x and along e y, in this fashion. Of course, by the law of corresponding angles or whatever, this angle is also phi. This direction is the direction of e x and that direction is the direction of e y, remember e x is the same thing as the unit vector i and e y is the same thing as the unit vector j. I prefer to use e x, e sub x, e sub y etcetera, because it immediately tells me along which direction it is and I am going to extend this to higher dimensions as well.

So, you have e rho pointing radially outwards and this is e x and that is e y and the question is, how do I write e rho in terms of e x and e y. And it is obvious from this figure that e rho is composed of a vector along this direction and a vector along that direction. The magnitude of this vector here e sub rho is unity and this angle is phi, therefore the x component is just cos phi. So, this is cos phi times e x plus and the vertical component is the sin phi, because this distance divided by 1 is equal to sin phi.

So, this is sin phi times e y that is a basic relationship.

It is a basic relationship and it immediately tells you exactly what have been saying all along namely, that the unit vector in plane polar coordinates is going to be dependent on the position, on the coordinates themselves. In this case, it turns out that e rho is a function of phi. If, therefore you go to some other phi in this direction, e rho points like that and it depends on this angle, it changes. Now, you could ask, why does it not depend on rho itself, it does not depend on the magnitude rho, that is immediately cleared.

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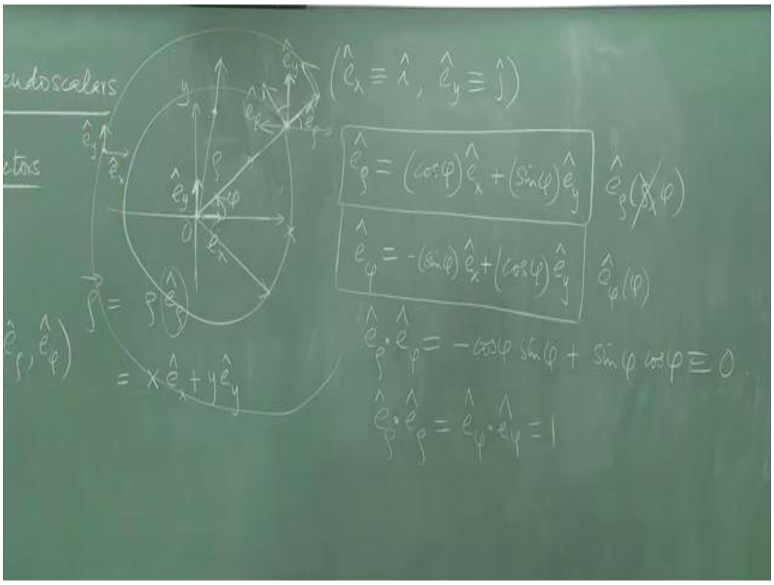

Because, if you are at this point that is the radial vector unit vector, but if you are at this point with a larger rho, you are still in the same direction, it does not a matter. Therefore, it is not surprising that e rho is a function only of phi and matter of rho. So, if I go to in general write this as a function of this distance in phi, there is no dependence on this quantity. It is a function only of the azimuthal angle and that is geometrically, immediately obvious from this figure.

What we need to do now is to ask, after all if I have an arbitrary point here, this vector I can resolve this vector along the x direction and along the y direction. Can I do so in polar coordinates? Can I resolve any point any vector here at this point along the radial direction and normal to the radial direction? What normal shall we choose? Well, we should choose the normal, just as originally I choose the direction of e x to be that of increasing x and increasing y.

Here I choose it in the direction of increasing rho and I should choose it the other vector in the direction of increasing phi, the azimuthal angle. But, this azimuthal angle is measured from this line in the positive sense, and therefore when you increase it, it should, therefore move in this direction here, perpendicular to e sub rho. So, if this is e sub rho the unit vector that is perpendicular to it is e sub phi. Again, it is easy to see what it look like in Cartesian coordinates in terms of e x and e y, because the same figure e x along here, e y along here, this angle is phi, therefore this angle is phi, immediately and e phi has a component along e y and this angle is phi.

So, it is clear that e phi equal to cos phi times e y, because this angle is phi and the projection of this on that direction has a cosine and it is along e y. But, the projection along the x direction is in the negative e x direction and it is this high which is sin phi. So, this rest of this formula is minus sin phi times e x plus e x cos phi. Again, as before it is immediately clear that e phi also is a function only of phi, does not have rho dependence on it at all, it depends on phi and that is clear too.

Because, if phi are at this distance e phi is along a direction which is tangential to this circle of constant rho, but if I move a little further it is again tangential to the circle and these two lines are parallel to each other with no dependence on the distance from the origin. So, immediately we will see that e phi also depends only on phi that does not have a dependence on rho at all.

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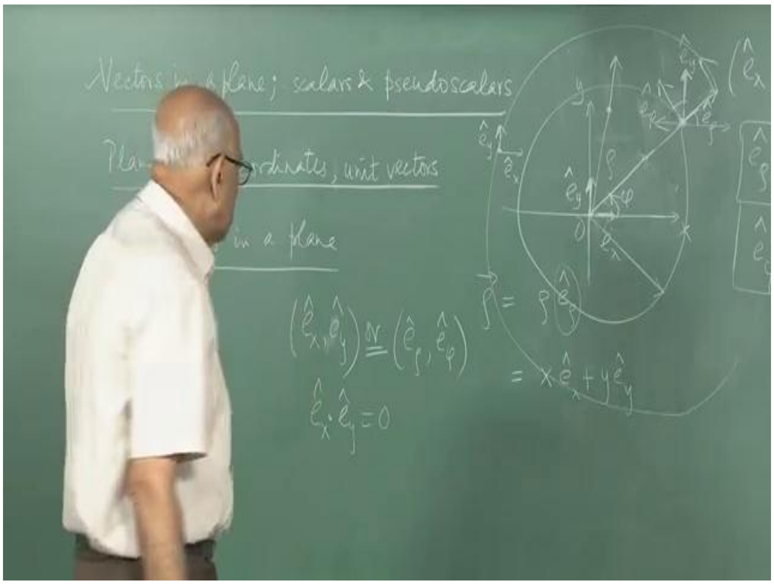

So, now, we can go to work, we have a unit vector system e x e y, you can use these formulas or e rho and e phi. When you work in plane polar coordinates, you should use these unit vectors and in Cartesian coordinates, you use these vectors. With the important difference that these two unit vectors will depend on the azimuthal angle, so therefore it will depend on the location of the point in the plane, but the unit vectors all the same and just as we know that e x dot product to e y equal to 0.

Because, the cosine of the angle between them is cos phi over 2, which is 0, there are right angles to each other. Similarly, this construction shows that e rho and e phi are perpendicular to each other, but it is a simple matter to verify that ((Refer Time: 23:48)) e rho dot e phi is the x component times the x component plus the y component times y components, and therefore this is equal to minus cos phi sin phi plus sin phi cos phi which is identically equal to 0.

So, there are indeed to right angles to each other and it is equally straight forward to see that e rho dot e rho equal to e phi dot e phi equal to 1, they have to be. It has to be so because the length of this unit vector is of course, 1 by definition and that follows in the fact that cos square phi plus sin square phi is 1, immediately. So, we have another orthogonal set of unit vectors to describe vectors in the plane either e x and e y or e rho and e phi depending on the application, depending on what you would like to do.

The next task now is to see whether we can use this to understand motion of a particle in a plane, whether we can understand kinematics in a plane and this requires the following kind of construction.

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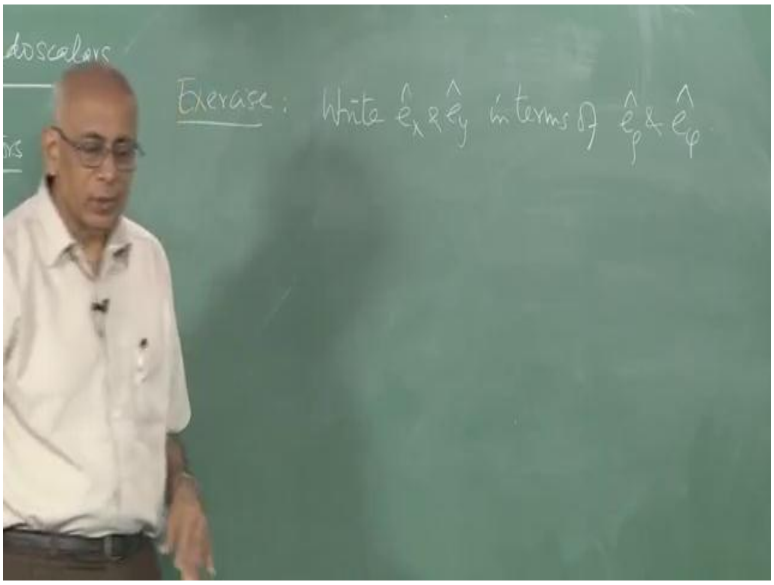

So, we have a unit vectors by the way an exercise, a small exercise. Write e x and e y in terms of e rho and e phi, I already wrote down formulas which expressed e rho and e phi as linear combinations of e x and e y. And now, what you should do is to write e x and e y as linear combinations of e rho and e phi and verify that e x and e y do not involve, it should not involve the coordinates themselves. Because, they are constant vectors, these are constants both in magnitude and in direction whereas, e rho and e phi are unit vectors magnitude is always 1, but the directions changes from point to point.

Now, what we like to do is to understand how a particle moves in a plane and the reason I call it kinematics is, because we are not going to specify at the moment what the forces on this particle are. So, we are not going to talk about it is actual motion on the some given force which would be a part of dynamics. On the other hand, kinematics is that, that part of dynamics that part of the study of motion which is independent of what the force on the particle is, which is independent of any specific given force or any given initial condition. This is just general formulas which arise essentially from geometry, just the definition of things like the velocity, acceleration and so on.

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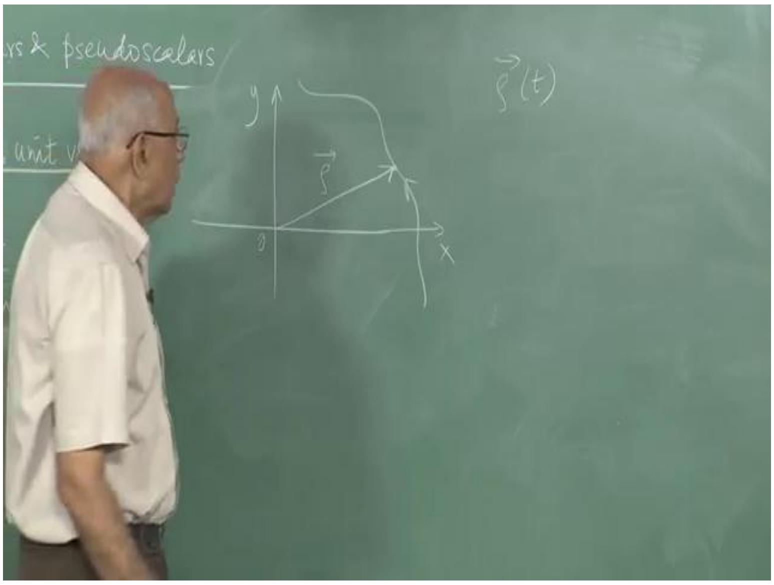

Now, first of all if you have a particle moving in the plane in some arbitrary trajectory in this fashion that is the part of a particle as a function of time say. Then, what you should do to understand the motion of this particle is to use the fact that it is position vector at any particular point is the radius vector which we call rho in this fashion and that is a function of time, it changes from time to time, so rho is a function of time.

The position vector of a particle in a plane moving in a plane at some arbitrary fashion is a function of the time. Then, I like to see what the velocity is, what the acceleration of this particle is and so on. Now, what is this mean?

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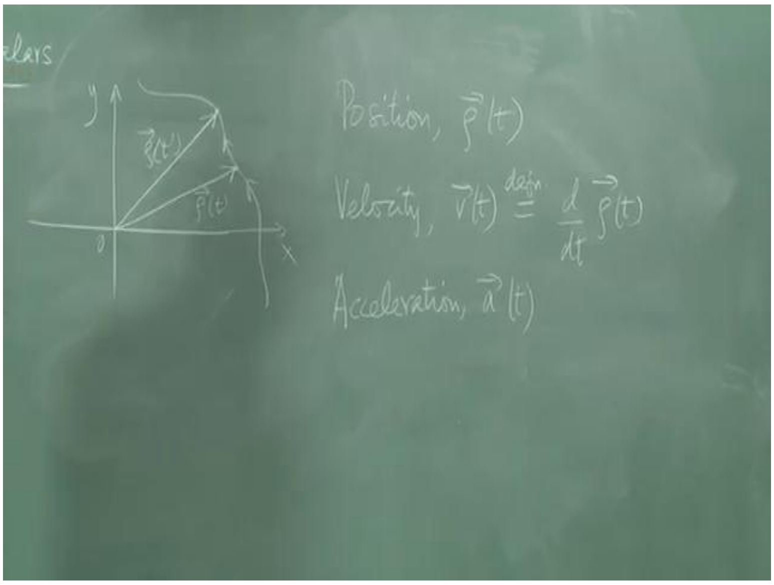

Well, at a little instant of time if the position vector is there or here for example, so that is the position vector at later instant of time. This is rho at time t, this is rho at some time t prime greater than t say, particle. Then, it differs from rho of t in two possible ways, one is it is direction is different in general and the second is, it is magnitude is different in general. So, therefore when I say the velocity which is the rate of change of this rho with respect to t, you have to take into account both factors.

The fact that it is direction may change as a function of time and that it is position may change as a function of time. So, the velocity of this particle and let us call this position rho of t, it is velocity v of t and we would also be interested in it is acceleration a of t. Because, dynamics is going to come in, when we specify the acceleration of the particle we are going to say. Under a given force, we going to use Newton's law and say the force is equal to the max times the acceleration, and then we going to have to solve for the position in general.

So, the idea is to give me the force on the particle at any instant of time, you are giving me it is acceleration and the problem in dynamics is to work backwards from that and discover, what it is velocity is, and then again backwards to discover what it is position is as a function of time, so that you can protect the trajectory of this particle, so that is the basic problem of particle dynamics. Now, what is this velocity? As you know from calculus, this is equal to the rate of change d over d t of rho of t and that is the definition of the velocity.

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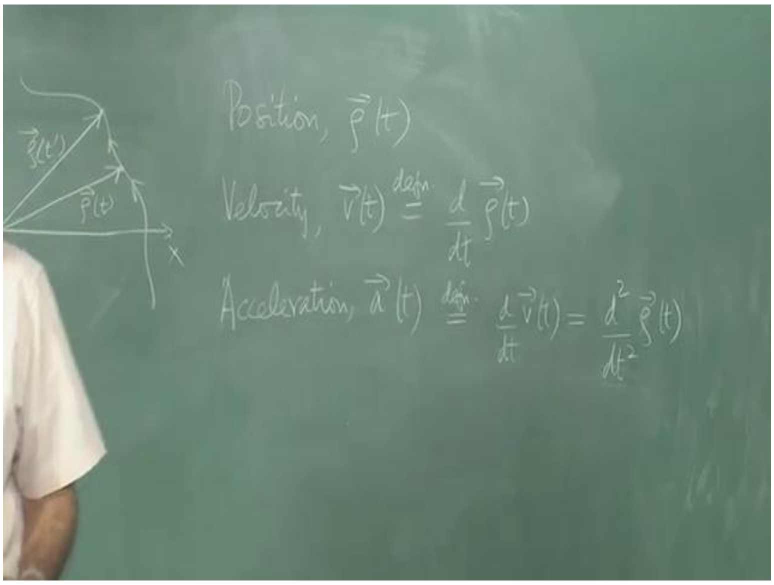

In the acceleration by definition it is d over d t of v of t and that of course, is equal to d 2 over d t 2 the second derivative of the position. So, if you like the problem of a dynamics, particle dynamics at this level is to say, if the second derivative with respect to time of the displacement or the position is given to you, can you work backwards and find the first derivative namely the velocity, and then can work backwards and find rho of t.

These are derivatives, differentials, therefore when the inverse problem will involve integration. You have essentially do two integrations in order to discover this point and that is called solving a differential equation for this quantity. Right now we are not going to do dynamics, we going to do kinematics. So, all I am going to do is to write down formulas for the acceleration and the velocity given the position and given the fact that this position will change both in distance from the origin as well as the angle, azimuthal angle, so this is the target.