Mechanics, Heat, Oscillations and Waves
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Lecture - 07
Plane Polar Coordinates

The next thing we are going to do is to talk about how you form vectors from scalars, in particular I would like to talk about the cross product of two vectors. But, we need a few preliminaries before that and one of them is this issue of a coordinate frame or systems other than Cartesian coordinates.

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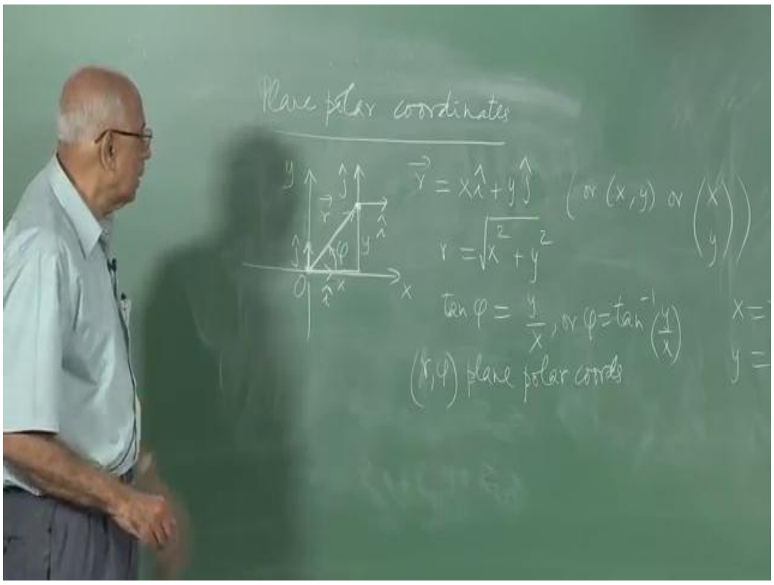

So, let us again look at the simplest example which is a two dimensional plane and look at what we already know from elementary geometry which is the following. On the x y plane, the position of any point here is given by a position vector r which comprises in terms of unit vectors along the x axis which I believe is denoted by i in standard notation and along the y axis by j. The idea is to write this as x times i plus y times j meaning that you go x units multiplied by i the unit vector in x direction here and y units here and the resultant of this vector plus that vector is indeed this position vector.

So, that is a standard way of writing this, but I will often write this also as just x comma y or sometimes depending on what I want to do I write it like this. You are all equivalent ways of writing the position vector of a point in two dimensions in a plane. Now, this is not the only way in which you can describe the position vector of a point, you could also say well why don't you write it in terms of this distance from the origin to that point and the angle made with respect to some reference directions conventionally taken to be the positive x axis.

So, if I call this angle phi, for a reason it should become clear when we go to three dimensions I do not want to call it theta at the moment, because I will reserve theta for another coordinate in three dimensional space. So, if I call this distance r in fact, I should use another symbol for r itself, but it does not a matter. So, position vector in a plane and a phi here I could as well tell you what this distance is and what this angle is measured from here in the positive sense and then the position vector, the point here is uniquely specified.

So, the coordinates of any point in a plane can either be written down by telling you what it is Cartesian components x and y are or what it is plane polar components r and phi r and it is immediately clear that r squared is related to x and y by r square is x squared plus y squared and similarly phi or rather tan phi is equal to y divided by x. In other words, the tangent of this angle is the perpendicular divided by the base in this fashion.

So, if you like r is defined as the positive square root of x square plus y square, because it is a distance and it cannot be negative and phi is given by this quantity here. Then, the question arise us to what are the ranges of these quantities, just to remind you minus infinity less than x or y less than infinity. So, that is the range of the original Cartesian components on a plane running all the way unbounded both directions.

On the other hand, 0 is less than equal to r less than infinity, because this distance cannot best be 0, the least value is 0 and on the other side it is unbounded and phi runs 0 less than equal to phi, phi 0 means you are on the x axis the point is on the x axis and goes all the way around up to 2 pi, but 2 pi is identified with 0, because you are back at the same point. So, I should really write this as less than 2 pi, so that the coordinate single value.

So, these are the transformation rules from the Cartesian to plane polar coordinates. So, theta r and phi are called plane polar coordinates and these are the forward transformations rules, you give me x and y and this is how I find r and phi. So, this is tan inverse of y over x in this range, so you choose that branch of tan inverse y over x which is in the range 0 to 2 pi.

Going in the other direction given r and phi what are x and y, well we remember this is x and this is y, x divided by this magnitude r is of course, cos phi. So, it immediately follows that the reciprocal relations to these two are x equal to r cos phi and y equal to r sin phi. So, you have either this set of relations going from Cartesian to plane polar or from plane polar to Cartesian, where this transformations here.

Now, whenever you have such a mapping from one set of coordinates to another, it is a same plane let we talking about, but two different descriptions there are some things called mapping singularities. For example, the origin the point 0 comma 0 the value of the azimuthal angle phi is indeterminate, r is 0 that is the point at the origin. The origin is specified by simply saying r is 0 and there is no meaningful value of phi, it is indeterminate, here when x and y both go to 0.

So, this is just a harmless singularity at this level, but it is there one should remember that it is there, for all other points. We have a unique phi and a unique r subject to this modular 2 pi restrictions here. Now, what about the unit vectors? Well, this is the interesting point, because if you have a point here I can still define a unit vector in the x directions, it is i and a unit vector in the y directions, that is j and they do not change as I move from one point to another. This one parallel to the x direction positive x directions in that parallel to the positive y direction and that is it.

And we know that i and j are perpendicular to each other in the remains so everywhere and they are independent of where you are. So, if I go to ask what is the basis, the Cartesian basis vectors in at this point it would be this and this or at this point it would be this and this. But, I can have another set of unit vectors and here is where we start leading a proper notation I would like to use another symbol for the unit vector which is e, little e with the subscript to say in which direction it is or which coordinates it refers to.

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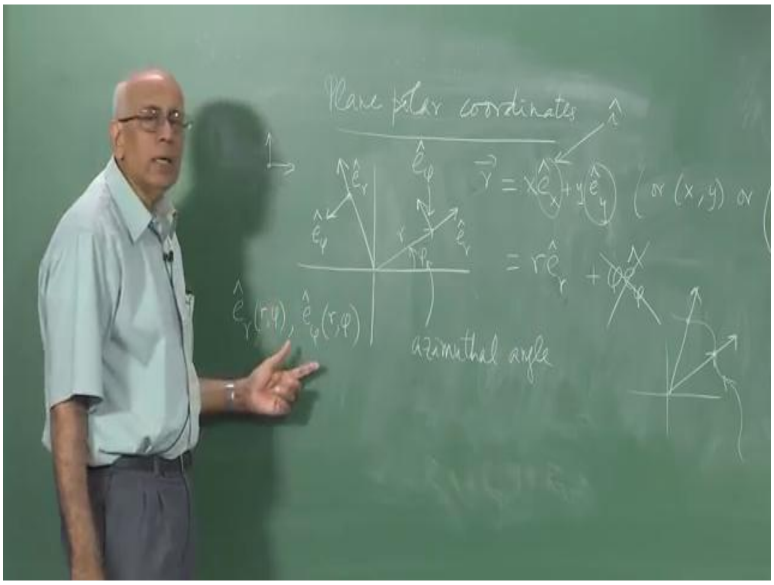

So, from now on I am not going to use i and j, but instead I am going to write r is x times e sub x plus y times e sub y. To remain myself of the fact that this quantity here is what we called i and this is what we called j and this is to remain myself, that these are unit vectors in the x and y directions respectively. What would you write for r in terms of a unit vector in the radial direction? Well, the magnitude here a little r without a vector symbol and the direction of this position vector is this. So, there is a unit vector here and let me draw a neat figure.

So, for an arbitrary point at distance r there is a unit vector in this direction which I should call e subscript r unit vector in this fashion, but to define the position of a point on a plane I need two pieces of information. So, I need a linear combination, I need the coefficients x and y for two mutually perpendicular vectors which form a basis in this plane. Well, actually you can use oblique axis also, but right now we are talking only about basis vectors which are perpendicular to each other is the simplest case.

So, I need a vector at right angles to this and since this is the direction of increasing phi, this vector here is the unit vector in the azimuthal direction, increasing azimuthal angle. So, remember that this phi the polar angle in plane polar coordinates it is also called azimuthal angle and this unit vector here is e subscript phi. Now, I can write down what this r is in terms of these unit vectors, remember that the physical dimensions of r is length. So, you have to on the right hand side have a length, these things have no dimensions at all the unit vectors, this quantity here and this quantity here each of them has a dimensions of length.

And of course, the vector r is the vector that starts here and ends here points in this direction. So, it is equal to the magnitude little r multiplied by unit vector in that direction, so it is equal to e sub r and nothing else. It would be incorrect to write something like e phi, it would be incorrect, it is dimensionally wrong in any case and you do not need it, it is not there, this already tells you what the position vector r is.

So, therefore, this is the crucial point, if you write this in plane polar coordinates, then the radius vector to any point is just the magnitude of the radius vector which is the distance from the origin times the unit vector in the radial direction, that is this quantity and that is it. But, here comes a catch, here comes the point, if you go to some other point and ask, what is the unit vector? It is equal to e sub r here and the azimuthal vector is going to be e sub phi here.

Unlike, Cartesian coordinates where the unit vector here was the same as the unit vectors here, they just transported rigidly from there to there. And therefore, our coordinate independent that is not true any longer both e r and e phi are dependent on where you are. So, you should really write this as e r of r comma phi e phi r comma phi, this is the characteristics of using coordinate systems other than Cartesian coordinate systems.

Except for the Cartesian coordinate systems in any dimensional Euclidean space, all the other coordinate system will have unit vectors which have position dependent, which has of course, advantages as well as disadvantages. The advantage here is I can write it down very simply in the fashion, but the disadvantage immediately is that this is position dependent.

Therefore, for instance if you are going to describe the position of a particle which is moving in this two dimensional space in this fashion, at this instant of time it is position is out there and this is the radius vector with the radial direction like that, at a later instant of time it is here with the position vector here, the unit vector in that direction. So, the unit vector also changes with time as the coordinate change with time and therefore, if I am going to compute velocities and so on, I must make sure that I also take into account the variation of this with time the unit vector. This is going to be important and it is going to prove to be an advantage as we will see later on. But, I hope this is clear that the unit vectors in the Cartesian system are in some sense rigid, they are fixed one some for all, independent of the coordinate of the position of the particle.

But, the unit vector in any other non Cartesian system immediately is position dependent and we have to take that into account at all times. Having said that and talked about plane polar coordinates, we will go back now to two dimensional vectors or vectors in a plane and discover what kind of scalars we can form from it, from vectors and what kind of vectors we can form from it.

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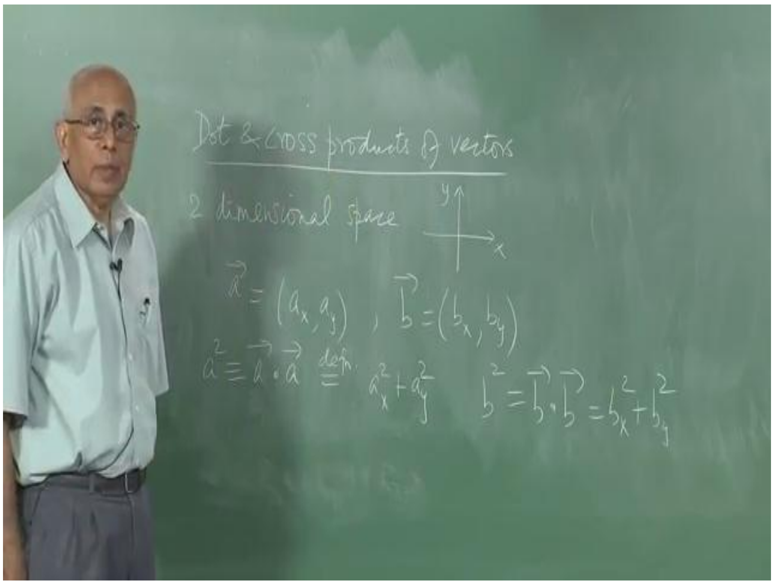

So, once again let us go back to this dot and cross products of vectors. The simplest of course, is the original dot product with which you are already familiar and we talking for the moment in two dimensions. So, two dimensional space the x y plane, if you like it is x y always. In this space, if you have a vector a and it is got components a sub x and a sub y and another vector b which has got components b sub x and b sub y. Then, the question that we can ask is can we form scalars from these vectors.

We have already seen how to do that in the simplest case, I have already pointed out that a x and a y transform exactly like x and y under a arbitrary rotation keeping the origin fixed and that a x squared plus a y square is unchanged, it is a scalar. So, the rule of the game is that a dot a is defined as a x squared plus a y squared. So, you take this fellows square it, you take this fellows square it and this is if you like the square of the magnitude of this vector a. And that is the scalar we have seen that already that is a scalar, explicitly from the fact that x and y had those trigonometric transformations.

Similarly, for b we have b squared equal to b dot b and it is conventional to write this as a square equal to this without a vector symbol and this guy here is b x square plus b y square, this is the scalar that is the scalar. But, now again using those trigonometric transformation laws, it is not hard to see that a x b x plus a y b y is also a scalar in exactly the same way.

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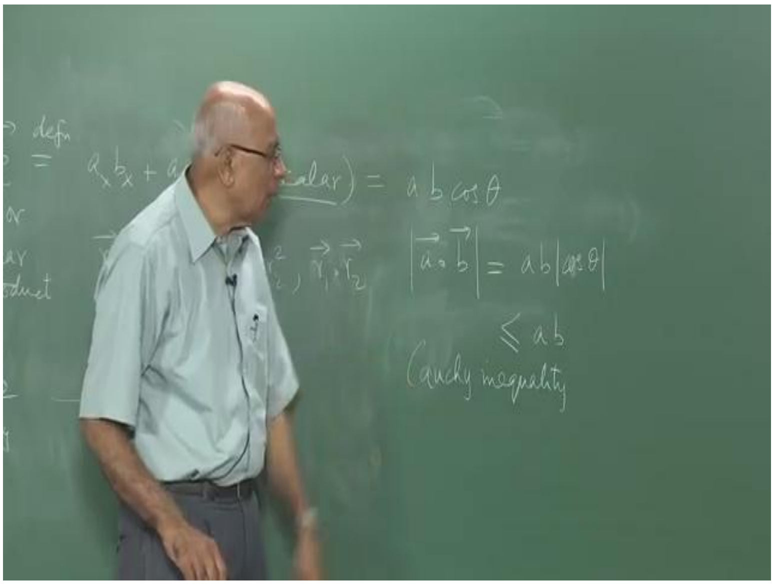

And this is written as a dot product b and it is defined as a x b x plus a y b y and this is scalar. So, that dot product which puts a dot between two vectors is also called a scalar product, dot or scalar product. The dot product of a vector with itself which is a square of the magnitude of the vector is a special case of the more general definition that you give me any two vectors and then I form a dot product between them by multiplying similar components like components. So, a x with b x a y with b y are some over this sum the two of them and I get a scalar quantity here. So, that is one way a finding a scalar from a vector.

If therefore, you had two points in space and one of them had coordinate r 1 and the other one had coordinate r 2. I could ask what are all the possible scalars you can form from r 1 and r 2 independent scalars functionally independent scalars of course, be r 1 squared, r 2 squared and r 1 dot r 2.

Now, what is the geometric meaning of this quantity here, well here is my coordinate system, here is my vector a, there is my vector b in this fashion and the angle between them let say it is theta then resolving a has a x cos whatever this angle is plus sin whatever this angle a y times sin at similarly for b it is a trivial matter to show that this quantity is also equal to a times b times cos theta.

And you can do this for easily by writing down the transformation rules or some elementary trigonometric to show that a dot b is the same as a b cos theta. So, here is a quick way instead of doing this which is a little messy here, the quick way of finding the dot products of two vectors is to take the magnitudes of the two and multiply them and multiply the result with the cosine of the angle between them.

Now, at least to an important and very, very interesting identity to like it is called an inequality, it is quite clear that by the way this is magnitude of a, this is magnitude of b and it also says that cos theta here is an angle which are ((Refer Time: 18:54)) 0 to whatever pi over 2 pi and the magnitude of cos theta is always less than equal to 1. So, it immediately tells you that the magnitude of a dot b which is a real number which could have both sin's, because when theta is in the second quadrant or third quadrant and so on it is a negative.

So, this quantity could actually be negative there is nothing it says that they should be positive quantity. Because, these individual components could have either sign positive or negative. So, it also says that if you took the magnitude of a dot b this is equal to a times b times the magnitude of cos theta, because a and b themselves cannot be negative the magnitudes of vectors and this is less than or equal to a times b. Because, the magnitude of this is less than or equal to 1, this is the very important in quality it is got many names at this level it is called the Cauchy inequality. Because, and it simply at this level expresses is the fact that the magnitude of the cosine of the angle cannot exceed unity, it is caught to be between 0 and 1 including of course, the end points. Now, the simple question is when does this become equal to a, b; obviously, when the magnitude of cos theta is equal to 1.

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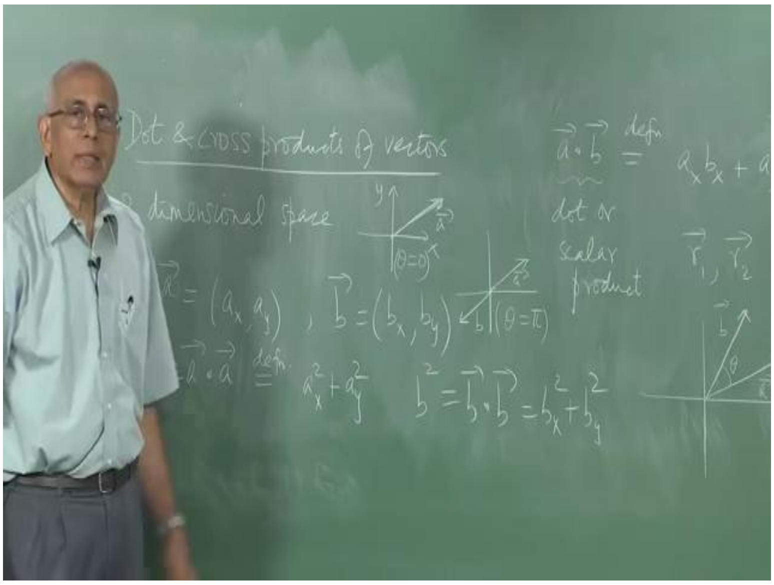

And when does that happen, it happens either when a is out here and b is out there the same direction. So, theta 0 or when a is in this direction and b is in this direction, this would correspond to theta equal to pi and this would correspond to theta equal to 0. So, what is that mean that implies that b is either parallel or anti parallel to the vector a, in other words b is some number times a, because that number could be negative, if it is anti parallel and positive if it is in the same direction if it is parallel.

So, it says b is not in a different direction in the plane, but a and b are co linear as soon as two vectors are co linear, then the magnitude of the dot product become equal to the product of their magnitudes in this side. So, this is a very important observation and it generalizes to other dimensions, it essentially says that when two vectors a linearly dependent of each other, namely one cannot be written as a product of number times the other vector plus one positive or negative number.

When this inequality strictly applies and the equality of implies if and if the two vectors are co linear with each other that is the very important observation which just got many, many generalizations including the uncertainty principle in quantum mechanics. So, it finally arrives from a generalization of this simple observation here. So, now, what we have this elementary thing done or let us look at what else we can do with these two vectors. The other thing we could do and instantly I need a little bit of a more elegant notation, because I am going to talk about this in handle higher dimensions and we will come to that in a minute. I have two quantities here and two there these two vectors I could form the following combination.

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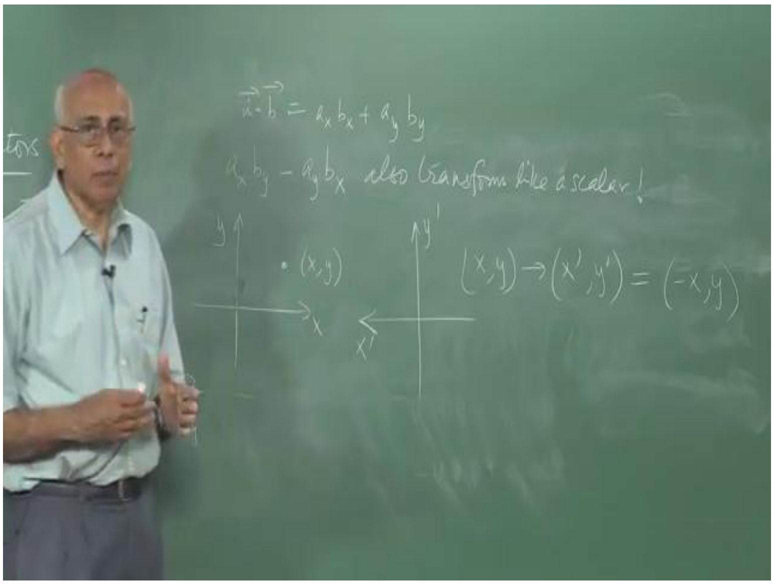

I could form a x by minus a y b x and ask what does it transform like, what does this quantity transform like I know the transformation for a x and a y and b x and b y. Now, what is this combination by linear combination look like? Well, incredibly enough this quantity also looks like a scalar, this also transforms appears to transform like a scalar, but it is not the same scalar as the dot product, this guy here dot here a dot b was equal to a x b x plus a y b y.

But, this also transforms like a scalar, but it is not the scalar product and it looks like something part of a more general structure of some kind. And because it is two dimensions, it is a little deceptive it appears in this very, very simple form, but this is caught a meaning here which you will become hereafter we talk about what happens in three dimensions.

It turns out that this quantity is the analog of the so called vector product of cross product in three dimensions which produces a vector out of two vectors by certain prescription vectors in the case of two dimensions, you do not have enough components you have only two components. And therefore, the same prescription leads you to a single quantity of this kind and it turns out to be a scalar in this case.

But, it is not an ordinary scalar in this case and let me explain what is happening here. If for instance, you said instead of a coordinate system which behaves like this, instead of rotating this coordinate system, suppose I said I reflect this coordinate system about one of the axis. So, my new axis x prime and y prime, if I reflect about the y axis then this is what my new axis would be x prime, but y prime would be the same as y. Because, I am just reflect in my put in a mirror here, so this axis just reflected in this side.

Now, this coordinate system cannot be obtained from this coordinate system by any rotation, because a moment you rotate x to bring it to this side y also rotates downwards with here. So, this transformation is not a rotation, the transformation here is to say that if I had a point out here, this point in the new coordinate system this x and y would go to x prime, y prime which would be equal to minus x and y. So, the x coordinate gets reflected, but the y coordinate does not change at all.

Now, this two is not going to effect to definition of a scalar, because the distance from the origin does not change x square plus y square is the same as minus x the whole square plus y square here. So, what is happening in this case is that one components changing sign in two dimensional space is not a rotation, is not the real number obtainable by rotation of the axis and then a transformation of this kind is called an in proper transformation as suppose to a proper rotation here.

And now what we will see next is that y this quantity has specific transformation properties under that kind of transformation does not change, this kind of quantity will change it will in fact, become minus itself and therefore, this is not quite scalar which is sort of pseudo scalar and the word in fact, is pseudo scale. So, even within scalars we have a little bit of sub classification depending on what these quantities do and a rotations and in proper transformations like reflections which to are valid transformations.

Because, when we say the loss of physics should not depend on how you oriented coordinate system, they should also not depend on whether you choose a coordinate system which is right handed or left handed should it matter. So, as long as that is true you need to be able to deal with quantities which also have specific transformation properties under these in proper transformations.

And here is one such example will come back to this, because this is will much clearer after we talk about what happens in three dimensional space. So, the next task is to go to three dimension vectors and ask what are the possible kinds of multiplications of vectors that you could have and how do they transform among each other. So, just remind you once again and when we talked about three dimensions, we are going to get something called a vector product or a cross product of two vectors.

But, it will turn out that the analog of the cross product in three dimensions, in the case of two dimensions is something which produces really a scalar and not a vector. So, took at a long story short, the idea that the cross product of two vectors produces another vector is specific to three dimensional space, it is not to in two dimensions, it is not true and any dimensionality higher than 3 like 4 or 5 or 6 or anything like that.

So, this is the peculiarity of three dimensions a very important peculiarity in will see what roll it place and whatever we are going to do. So, we will talk next about three dimensional vectors and how to find quantity which transforms like a vector when you take two vectors and combined them to what is called a cross product that is the next task.