Mechanics, Heat, Oscillations and Waves
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Lecture - 06
Scalars, Vectors and All That

Today, I would like to talk to you about Scalars, Vectors and all that with a very specific aim in mind and that is to give you the proper definition of these quantities, after motivating the reason for using these quantities in the first place. Now, all the physical quantities that we deal with have certain specific mathematical characteristics. For instance, they are classified as vectors, tensors, scalars and so on and so forth, sometimes you classify them depending upon the action such as operators of various kind, matrices and so on there are lots of mathematics quantities.

But, the most common ones that you come across in elementary physics are scalars and vectors and we would like to understand what these quantities actually are and why we need them at all in the first place. This is crucial, because all physics is based on the use of such quantities. First, you would have heard of what a scalar is and what a vector is and let me recall the high school definition of these quantities, the naive definition of it.

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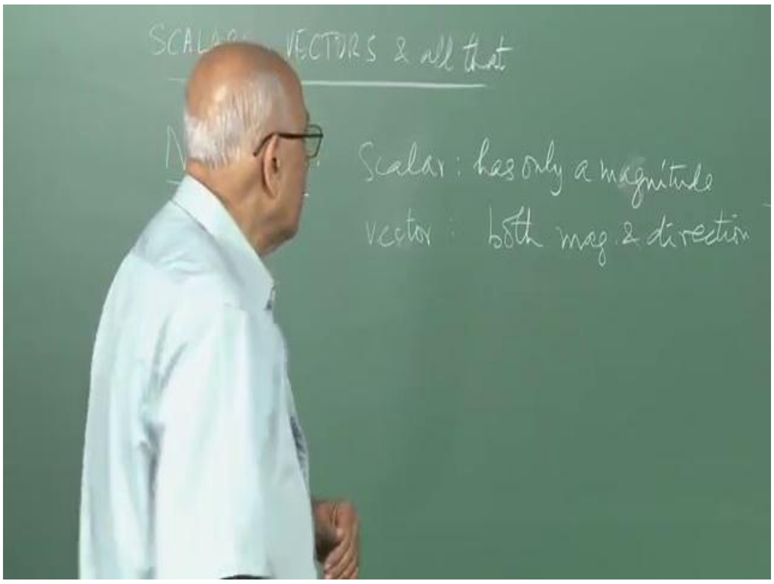

So, the naive definition says, naïve are simple minded definition says the following. A scalar has only a magnitude, on the other hand a vector has both magnitude and direction. I am going to right away puncher these definitions by saying them they are not adequate they are not enough. For instance, suppose you look at the coordinate of a point in a plane.

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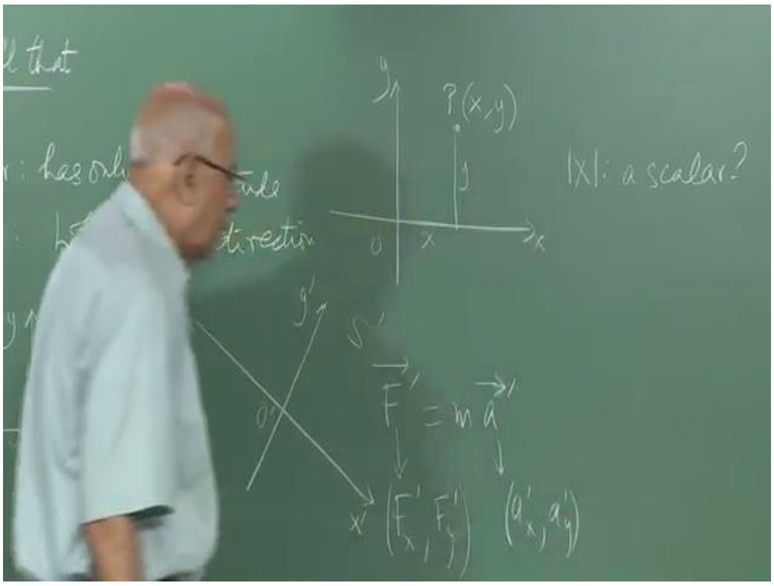

Now, as you all know if you take a plane, the x y plane for instance here is the x axis, the y axis, here is the origin. And let us look at some particular point p whose coordinates are x and y in Cartesian coordinates. What this means is that this distance here up to this point is x and this vertical distance is y, now this x can run all the way from minus infinity to infinity and so can y on this infinite plane.

And the question now asked is x is scalar, well you would say x has a sign, it is got both plus and minus possibilities. So, why not this, why not this quantity, is this a scalar, because it has only a magnitude and that magnitude runs anywhere from 0 arbitrarily high to arbitrarily high values. So, the question is it a scalar and the answer is no, it is not a scalar. Similarly, when you say you have a vector is a quantity with magnitude and direction, the immediate question that arises is direction with respect to what and the answer is direction with respect to some fixed coordinate system.

But, there is no special coordinate system and who gives you this coordinate system anyway. So, the point is this definition is inadequate, while they are convenient and once we know what we talking about we could use these definitions, the fact of the matrix is they are not rigorous definitions and they are highly incomplete. So, we need a proper definition of these quantities as well as an explanation as I have said earlier of why we use these quantities at all and here is what it looks like the answer is and let me do this in terms of an example.

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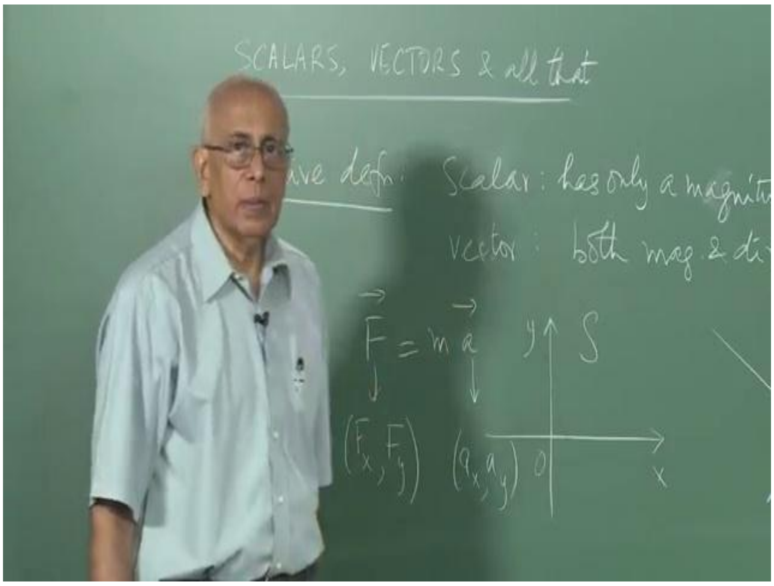

Here is the example, you are familiar with Newton's second law which says that the force is the mass times the acceleration. For a body of mass m in Newton in a mechanics, once you apply a force the result is an acceleration of the body and the acceleration is proportional to the force, this is Newton's second law of motion in a simplest form. Now, the question is these directions of f and a as specified; obviously, with respect to some fixed coordinate system.

So, here is my coordinate system and let us for the simplest case look at just two dimensions, here is x and y and my coordinate system let me call it s, my fixed coordinate system here and in this coordinate system this quantity f has two components an x and a y component. So, let me write it in this way f subscript x, f subscript y and this quantity a has two components a x and a y. On the other hand, the same physical force on the same physical body with the same mass could be examined in some other frame of reference.

So, there could be a coordinate system for an observer s prime which whose origin is sitting here o prime and this is y prime and this is x prime. Now, the question is what is he going to observe in this experiment on this body of applying this force. It is quite clear that since f sub x and f sub y are the components here with respect to this origin here and this frame of reference. These numbers are specific to this frame of reference, these numbers are specific to this frame of reference.

On the other hand that might lead to a totally new result all together and yet your text book says f is m a without bothering to specify where the coordinate system is. So, the logical conclusion is that it is probably independent of this coordinate system, in other words in this coordinate system the presumption is that the rule must be f prime which is the force observed in s prime or measured in s prime must be equal to m times a prime.

We made a little assumption here which I have pushed under the ((Refer Time: 06:08)) that m does not change and it is a scalar. So, in some sense I am already begging the question, but let us take it as m as you know the naive definition of m which is the inertial mass, it is a property of the substance has nothing to do with coordinate system or anything like that. Then, we except that in this frame of reference the law will be the force in this frame is the mass times the acceleration in this frame.

Once, that is true then it does not matter you do not have to specify which frame you are in, because it is true in all frames appropriate to each frame. These presence components would be f sub x prime f y prime and that acceleration would be a x prime a y prime on this side. Just one thing left to do and that is to say that since the law is the same in form between this frame and this frame of reference, it can only be expressed in terms of quantities whose transformation property is encoded in the quantities themselves.

In other words, this quantity f and this quantity a vectors must carry a dictionary in the very meaning which tells you that if you tell me how the coordinate system s prime is situated with respect to the coordinate system s and you give me this rule f is m a when I predict for you what the numbers f x prime, f y prime, a x prime and a y prime are.

In other words, if you want the physical law to be unchanged in form, to be a form invariant under a coordinate transformation, you must express them in terms of quantities whose transformation law is known is prescribe once and for all and such quantities are called covariant quantities. This is such an important principle and it is of the root of modern physics that I am going to write it down.

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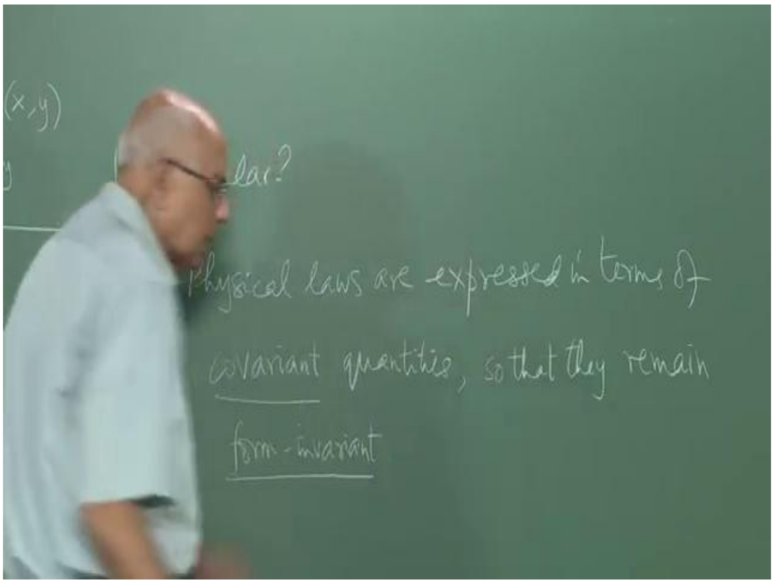

Physical laws are expressed in terms of covariant quantities, so that they remain form invariant. Now, that is a lot of big words for something which is a very straight forward matter, it simply says that our laws we believe more and more generally are independent of the observer. The whole of modern physics is an effort to discover laws of nature in such a manner that they are completely independent of the observer special status.

For that you need to express these laws in a form which does not change, when you go from one observer to another and to do that you need to express them in terms of quantities which carry their own transformation laws, their own dictionary and scalars and vectors and generalizations are such quantities under a specific kinds of transformation. So, this is what I would like to emphasize, this what I would like to show today, demonstrate today.

I repeat once again, the reason for us to use scalars, vectors and so on in particular vectors you can write in three dimensions, you can write three equations in one line and you write in terms of vector, it is not to save space. It is to make sure, it is necessary to make sure that you expressing form invariant physical laws in a manner where you can translate from one frame to another or from one observer to another.

In this case, if f and a are vectors in the definition which I am going to give shortly, then you are guaranteed that the law f equal to m a is form invariant under the transformations from one observer to another or one frame to another under which these quantities are vectors that set of transformations will be seen shortly and under that set of transformations you are then assure that this law will not change it is form, numerically yes it will change.

Because, any acceleration you measure in this frame, the numerical values that you have for these components at any instance of time would be different from the numerical values you have in this frame as would the force components. But, what you are assured is that if you give me f and a, the components of f and a in this frame and tell me how o prime is related to o, how far is it move, how much it rotated then I will predict for you that in this frame you will discover f prime is m a prime, where the f prime quantity the quantities which are prime can be found from the quantities which are given in the unprimed frame, so this is the idea behind the whole thing.

Now, what is the definition of a vector? What is the definition of a scalar? This is now going to be a matter detail and what kind of transformation we are talking about and the one that is used conventionally for vectors at this level are called rotations of the coordinate axis and let me explain what that does. So, as before we going to take a very simple example, which is rotations in a plane. So, for the moment let us pretend our world is two dimensional that we have just the black board as the world that we are talking about.

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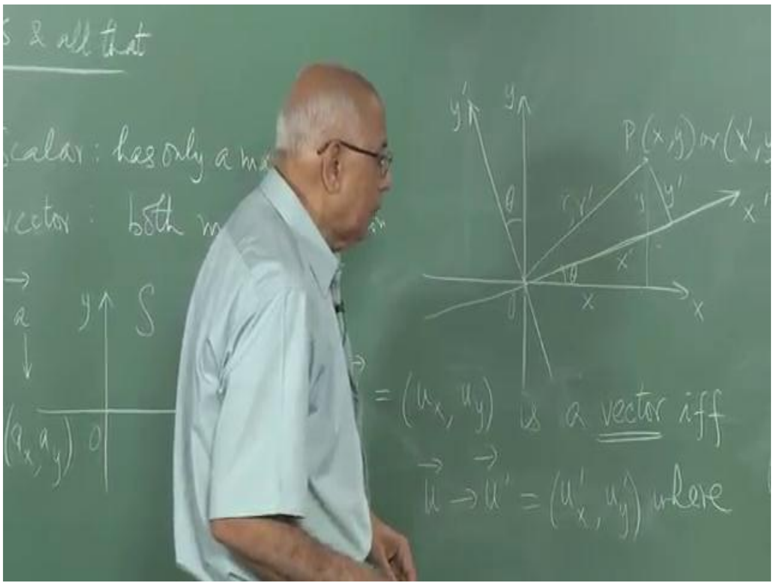

And there is a coordinate axis here x and y, here is the origin and you have an arbitrary point with coordinates x and y in the frame s, in the original frame s, so this quantity is y and this is x for an arbitrary point x comma y. Now, let us look at the same thing in a frame of reference which has the same origin, but it is rotated with respect to the x axis, the new x axis is at an angle theta, some arbitrary angle theta.

And here is the new y axis, this angle is theta that is the x prime axis, this is the y prime axis and this angle of course, also theta. Then, how are the components x prime and y prime of this object have this point prime? So, here is a point p which is either x y or x prime y prime in the new coordinate frame.

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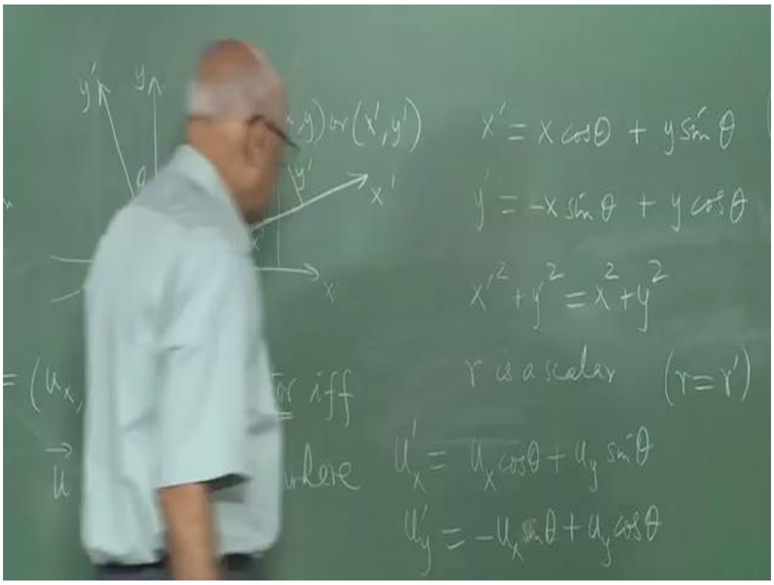

And then, it is a simple matter of trigonometry to see that the transformation rules are x prime is x cos theta plus y sin theta and y prime is minus x sin theta plus y cos theta, those are the transformation rules which follow from elementary trigonometry. And together of course, I call the point r this is the position vector of this point is written as r either in the original frame and r prime in the new frame.

So, r equal to x times a unit vector in the i direction plus the unit vector in the j direction, you can write it either way you like. But, I am just going to write it as x comma y in this form here and under this rotation r goes to r prime which is x prime, y prime and a rotation of the coordinate axis. So, each point although the point remains same the coordinates change, but I could regarded it as a transformation from x, y to a new set of pair of variables x prime and y prime.

And the rules by which you get x prime and y prime from x and y are written down here in this ((Refer Time: 14:45)). Now, it immediately tells you right away that x prime squared plus y prime squared is x squared plus y squared that is the simple matter of using the trigonometric identity cos square theta plus sin square theta is 1 and then you get this relation here for in arbitrary point x comma y. Now, what does that mean, the physical meaning of this x squared plus y squared is of course, this length of the hypotenuse.

So, this distance here is r, but it is also equal to r prime which is what this quantity is, because remember this number here was y prime and this number here was x prime. So, that x prime that is y prime and the square of this plus that is this squared which is the same as square of this plus this by the Pythagoras theorem. So, you know are first relation which is r and r prime are exactly the same, in other words the distance from the origin to any point is independent of the orientation of the coordinate access.

If you rotate the coordinate axis keep in the origin and change the distance to any point does not change at all. And physically it is immediately clear that the distance between any two points arbitrary points also does not change at all, because the distance between a point x 1, y 1 and x 2, y 2 is just x 1 minus x 2 squared plus y 1 minus y 2 squared in that two does not change at all.

So, now comes the definition, if you have a quantity which does not change and a rotation I call it a scalar, it does not change in magnitude, it does not change at all and a rotation it is then called a scalar. So, r is a scalar r equal to r prime, the prime quantity and the unprimed quantity are exactly the same, any quantity that does not change and the rotation is go from the frame of reference to another is a scalar under rotations I need to say that all the time because I might look at more kinds of general kinds of transformation.

But, for the purpose of this course you will restrict ourselves rotations, because that is what normally vectors, scalar, tensors, etcetera at defined in terms of. So, this quantities is the scalar, what then is a vector well as you know the naive definition does not magnitude and a direction and little more sophisticate we have saying it is got two components, once you have two dimensional world a vector as two components.

For instance, the position vector of a point is r x comma y, this is our primal vector, this thing here you have primitive basic vector this quantity here and now the definition is that any quantity with two components. So, anything lie let us call it u and put an arrow there and it is got two components u x and u y, any quantity with two components written on this form which transforms under the rotation in the same way that the coordinate r itself transforms is called a vector.

So, this is a vector f and only f under a rotation of the coordinate axis through the some arbitrary angle theta, only if u goes to u prime which is equal to u x prime and u y prime, where u x prime must transform exactly like the corresponding component of the coordinate itself. So, it must be equal to u x cos theta plus u y sin theta and u y prime must be equal to minus u x sin theta plus u y cos theta that is the definition of the vector in two dimensions a vector under rotation and so on and so forth but that is the definition.

Therefore, any two arbitrary quantities you put them next to each other and put them inside the bracket cannot form of vector, a vector occurs only when these two quantities a such that the pair is such that under a rotation, the pair transform in precisely the same way as the position vector itself us. So, it is with reference to the transformation law of position vector in point in space that we define a vector itself.

Once, you guaranteed that the quantity is a vector when we use the factor, it transforms in this fashion to predict what the prime quantities are from the unprimed quantities by a rule which is the same as the rule for coordinate transformation itself. So, this is the correct definition of a vector and it is extendable to any number of dimension, if you have a third dimension, the third axis coming out of the paper, they would be a u and they would be z also as well z prime coordinate and it should transform in exactly the same way the x, y, z transforms to x prime, y prime and z prime any set of three quantities which transform in the same way as a three Cartesian components of the position vector would then be called the three dimensional vector in three dimensional.

So, this is the notion of what a vector is and that is how the transformation law is encoded in the word vector itself. The moment I say something is a vector, I have also told you how it is transformation law is going to behave and that is going to make sure that any law I express in terms of vectors is transformable form one frame to another without a change in form, because I know how each side of the equation transform the transform like the vectors.

Now, it is a small matter to see that if you have an equation with the left scalar on the left hand side you should; obviously, have a scalar on the right hand side. Because, either side can change if you have a vector on the left you know have a vector on the right and so on and so forth, there are generalizations of vectors and scalar which we will talk about little later very straight forward generalizations, but this is the basic idea behind the vector here.

Now, we need a little bit of notation, so that we can generalize this a little more and that is as follow, it is very convenient to notices that I could write those two equations down x prime and y prime I could write them down in the following convenient way.

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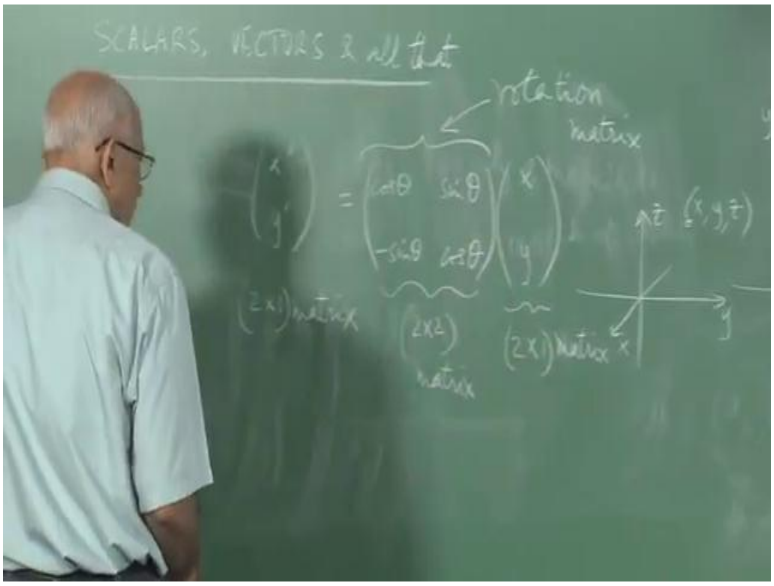

I put these two as a column vector one below the other, this thing here is a 2 by 1 matrix it has got 2 rows and 1 column. So, it is a 2 by 1 matrix. I presume there is some familiarity on the part with element matrices, but whatever is needed we will look at it as we go along will introduce that. This quantity here is equal to x comma y this 2 is a 2 by 1 matrix and it must be acted up on as you say by something which is a 2 by 2 matrix and that is the matrix of coefficients there, whatever is multiply x prime and y prime.

So, that thing is cos theta, sin theta minus sin theta cos theta and by the rule of matrix multiplication, this is the 2 by 2 matrix and as we know when we multiply 2 by 2 by 2 by 1 the 2's cancel out. So, the ((Refer Time: 23:13)) 2 by 1 once again. So, this gives you a column matrix here 2 by 1, this 2 by 1 column matrix one acted up on by this matrix here gives you 2 by 1 the new column matrix, will represents the position vector in the new coordinate system.

Now, what would you call this matrix, well it is a matrix which depends on the angle through which rotate could have been any angle I specify the angle I tell me I give one coordinate system. So, it is not a natural to say that this thing here is called a rotation matrix and it depends on only on the rotation, it does not depend on the coordinate on in of any particular point.

Because, the whole issue any point anywhere, any point here is transform in the whole coordinates it is got a certain components in the new coordinate it is called a different components which are related to the whole once by the action of this rotation matrix on the whole components this fashion, this matrix is 2 by 2. Because, we have two dimensions and what is happen under a rotation is that you have a linear transformation.

So, each component of the new vector is a linear combination of the components of the whole vector, all the components. So, x and y components of the whole vector have the mixed up to give the x component of the new vector and similarly for the y component as well, it is the linear transformation in the sense that the origin has been unchanged. So, it is clear that if you are at 0, 0 the origin in the whole frame you are still at the origin in the new frame, which means that if you set x and y equal to 0 you must get x prime and y prime equal to 0 as well which you do.

Because, these are homogeneous transformations, a proper rotation is a homogeneous transformation, in the sense that the origin is left unchanged under the transformation. Now, the generalization of this to three dimensions is easy in principle, in practice it is a little messy, because as soon as you have three components as soon as you have a three dimensional space and here is x, here is y and here is z, then you need for any point you need three coordinates x, y, z.

And now a rotation is not such a simple matter as keeping the origin unchanged and rotating the coordinate axis, because if you have a three dimensional space, you could rotate about any direction in that space. So, here is the direction in this space original coordinate frame and I can rotate about that axis in this fashion to produce the new set of axis here. So, we have to specify the direction about which I rotate and have to specify the amount by which I rotate to specify the direction in space you need to coordinates, we need to angles just as to specify by the position of a point on the surface of this earth, you need a latitude and longitude.

So, to specify any direction the unit vector in any direction, you need to know two angles essentially and then you have a third angle which is the angle of rotation. So, you have three parameters to specified rotation in three dimension space and therefore, the matter is a little more intuitive, but you are still guaranteed that under the rotation in three dimensional space you have the following.

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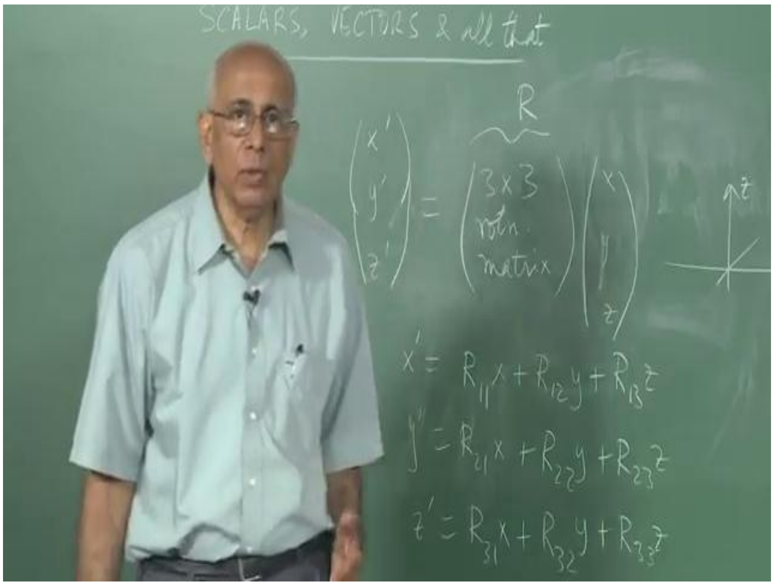

You still guarantee that under this rotation x prime, y prime, z prime now will be equal to some 3 by 3 rotation matrix multiplying acting from the left on x, y, z and this rotation matrix will not involve x, y and z themselves, because this is two for every x, y and z in space, this rotation matrix will involve the three angles which are the analog of the single angle here is specifies a rotation in a plane.

So, it is a complicated matrix which got very special properties, but all the same it is a rotation matrix and you are guaranteed that in general what is going to happen is that x prime if I call this matrix R, then this x prime is going to be R 1 1 the first element time x plus R 1 2 times y plus R 1 3 times z. And similarly y prime is R 2 1 times x 2 2 times y plus R 2 3 times z and z prime equal to R 3 1 x plus R 3 2 y plus R 3 3 z.

So, each component of the new vector is a linear combination of all the components of whole vector, some of these components may or may not exist depending on what kind of rotation may taking about the specifics of the rotation. But, that is the general structure here and you are guaranteed you are guaranteed that this matrix is such that the analog of the property that R squared equal to R prime squared is still going to be valid. So, it is still true that the distance to any point from the origin, clearly does not change under a rotation which keeps the origin and change.

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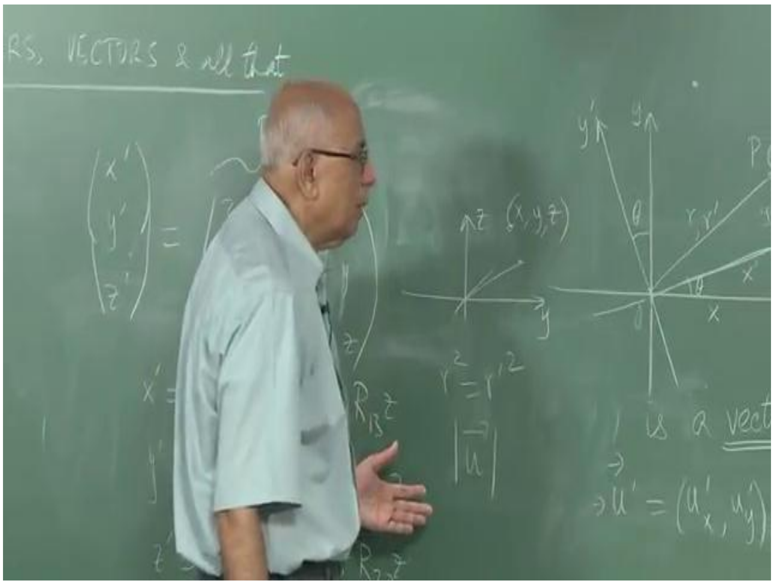

So, you still guaranteed that r squared is r prime squared and that r is a scalar in this quantity. Now, you can answer this question of whether in this example or in this example mode x is it a scalar, the answer is no, because x goes to x prime and x prime is this complicated combination here. So, clearly the modulus does you not in variant does not change.

So, x is a component of a vector and it is not a scalar by itself that is an important point, one should not imagine that when you have a vector comprising several components like this are in the case of u like this, one should not is a this is the scalar or that is the scalar not true, because these quantities do not remain and change under a rotation, the combine with each other the mix up components and they do not remain invariant.

On the other hand, you are certainly guarantee that u squared the magnitude of u x prime squared plus u y prime squared in quantity is equal to u x squared plus u y squared just as x prime squared plus y prime squared is x squared plus y squared. So, this quantity u this follow is in the unchanged and the rotations and it is a scalar, but the magnitude of any one component is not the scalar by itself.

So, this is the idea behind the scalars and vectors in the most elementary level and what we will do next is to see how we can forms scalars from vector and a little later will form vectors from scalars. So, that is because these are natural quantities that appear all the time and we need to be able to manipulate these quantities an understand, you also need to see how we can multiply to vectors, what is the geometrical meaning actual meaning of the dot product and the cross product in the case of vectors and there are few minus surprise on the way, so we will see that next.