Mechanics, Heat, Oscillations and Waves

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Lecture - 04
Sketching Elementary Functions

Today, we are going to take a little digression and speak about a topic that you might think properly belongs in Mathematics. But, it actually belongs in Physics in a very real sense and this has to do with the way functions look when you sketch them, when you plot them. We are not going to plot graphs to some fixed scale or anything like that, but you see in the study of Physics you often come across mathematical formulas, you come across them all the time and then, you would like to know how one quantity varies as another quantity is varied.

You would like to know the dependence of x and y or one variable on another and so on and this happen all the time. We have a mathematical formula, but then it does not speak to us unless we are able to visualize it in terms of a picture, which tells you what the function roughly looks like and I would like to call this the idea of sketching elementary functions. So, we are not going to try to plot this on some specific scale of course, you can program a computer to plot these functions very trivially.

But, the point about physics is that when you have a formula in front of you, it should be able to speak to you. It should give you information about how these variables change as other variables are changed and this is very important a little art. There is a little art to this and it is the art of sketching functions qualitatively rather than quantitatively.

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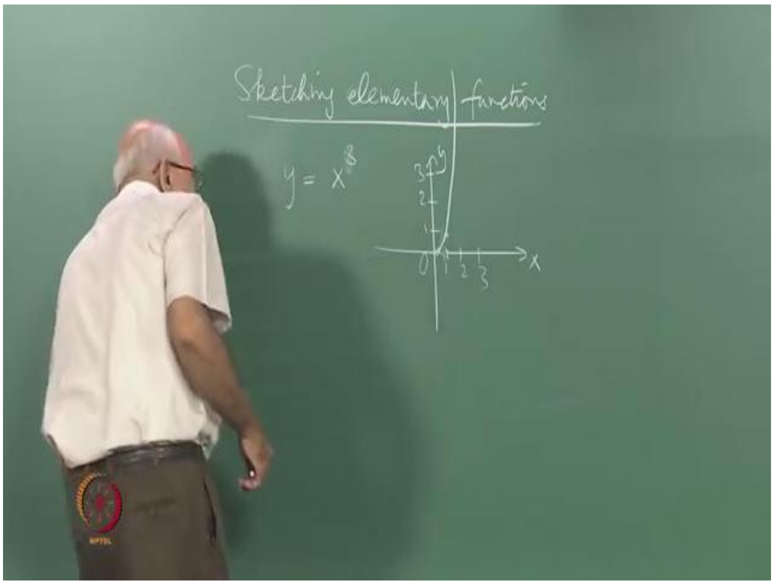

For instance, you take an extremely elementary example, if I plotted this function y equal to x cube for instance, that is already a non-linear function, it is a cubic function. If I want to start sketching this function by linear scales, if I did x here and y here and that is the origin and this is 1, 2, 3 and so on 1, 2, 3 etcetera. As long as you are between 0 and 1 there is no problem, because this point is a point on the graph, when both are 0 there is a point on the graph here.

But, the moment you come to 2 here y has gone to 8 and therefore, this graph increases extremely rapidly, there is something like this. So, linear scale is immediately useless as far as such a function is concerned. And since, most functions are non-linear it is immediately obvious that the, you should be able to sketch these functions qualitatively without specifying this scale which can always imported in subsequently. Even when you program on a machine or a computer to sketch these functions, you have to display little bit of cleverness in choosing different scales for the x and y axis; otherwise, you do not get any graph that you can recognize.

So, today I am going to spend a little bit of time telling you how these functions are sketched, what you should look for in these functions, how does the graph speak to you ((Refer Time: 03:07)) speak and that will make the sub sequence study of physics and physics formulas much more easy for you to understand, when I use terms like this is exponential in something and it dominates over a power law behavior or a such and such a quantity goes to 0 faster than any reciprocal power, inverse power of this variable and so on.

You should be able to make sense out of it right away and for that purpose, let me spend a little bit of time telling you how graphs are sketched qualitatively. What you look for any function? Let us start with simple examples.

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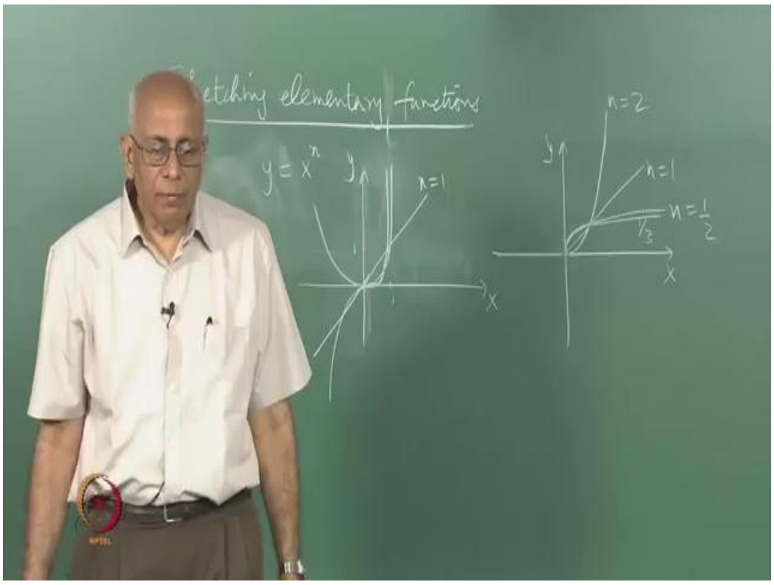

So, I am going to start with this example of y equal to x to the power n and we will start with an example of n integer first, positive integer to start with. So, here we go and the idea is to plot everything on the same graph y here and it is obvious that whenever n is any positive integer x to the power n and when x is 1, y is also equal to 1 whatever be the value of n. So, there is a point here which remains on the graph and there is a point here which is on the graph.

But, if n is equal to 1 it is just a linear curve, it is just a straight line goes off like that. So, that is n equal to 1, when n is 2 you have a parabola which passes through the point 1, 1 and the slope here is flat, so it goes like that. So, parabola when n is 3 it is an even flatter curve and rises even more rapidly and when n is 4 and so on it goes even faster and when n is an even number, the graph is simply reflected on this side. So, the parabola is reflected here and then n is an odd number, then the graph bends down under this, that is it.

And you can therefore, plot n equal to 4, 5, etcetera they going to rise even more steeply after 1, go towards infinity as x extends to infinity and minus infinity or plus infinity depending on whether n is negative or even. So, we got a graph, we understood the qualitative behavior of the function x to the power n, but it is interesting to ask what happens if n becomes a fraction between 0 and 1.

So, again on the same graph here is y, here is x I am not even going to mark the unique point anymore, this is what a positive power is going to look like. Depending on how large that power is, it gets flatter and flatter at the origin, x to the 4 is extremely flat, x to the 6 is even flatter and so on. And here is n equal to 1, y equal to x what happens if n is a fraction, like a half for instance. When 2, the graph is going to go through 1, but it is going to go like this say n equal to half, it is very steep at this point in fact the slope here is infinity at that point.

And this let say is n equal to 2, it is also going go to infinity as x goes to plus infinity, but it is going to go slower than linearly which in turn is going to goes slower than quadratic ally. So, now, we immediately have this concept of how the function becomes infinite, how fast is 10 toward become unbounded, if n equal to a positive integer greater than 1 it is certainly increases faster than linearly and less than 1, it will increases slower than linearly.

What happens if n is 1 3rd, just like x cube was flatter and steeper this fellow is sleeper here and also goes to infinity, but a little slower. So, we have this whole family of curves x to the n, when n is a positive number as n goes from 0, where the graph goes like essentially goes flat 1 and remains after that 1 as a extension as n goes to 0. As n increases, it goes to the linear graph and then it goes through graphs which can flatter and flatter of the origin and becomes steeper and steeper as x increases on the positive side and tends to infinity.

Now, the graphs for integer values of n, when n is even or odd or either reflections in this plane or in this quadrant or in this quadrant depending on whether n is even or odd. So, we have the possibility now of understanding that the function x to the power n, for general n is one member of a family of curves, for all sorts of values of n and that is the way to understand this simple function here. Now, what happens to a function like log x for instance?

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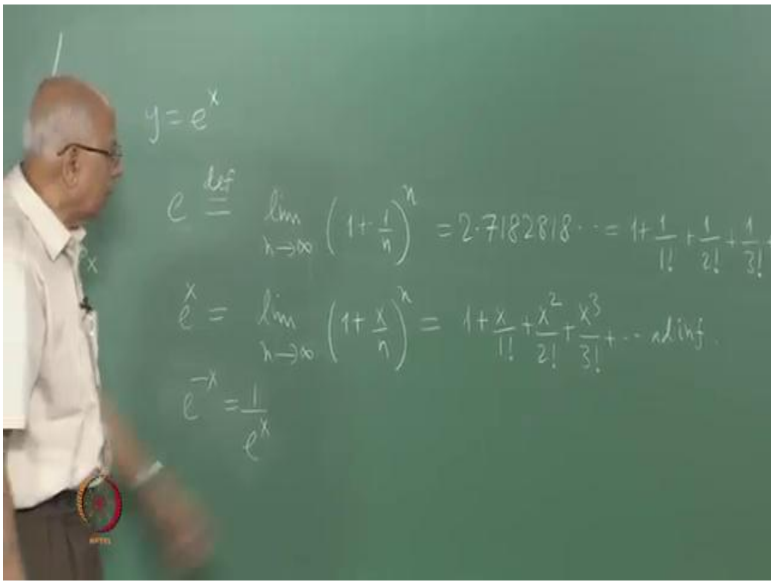

But, before that let us look at a function like the exponential of x, you are familiar with this graph y equal to e to the power x. I presume that you are familiar already with the idea of an exponential, but if not let me explain very quickly what it is. This number e the base of natural logarithm can be defined as the limit as n tends to infinity of a certain number is tends to 1 as n tends to infinity, but raise to a power which increases as n tends to infinity.

So, it is the limit of 1 plus 1 over n to the power n that is the definition of this number e. Now, it is a remarkable fact that this limit actually exist and this is quite finite. In fact, it is numerical value is 2.7182818 etcetera, it is a non repeating decimal all the way to infinity number of places, it is an irrational number it is called a transcendental number. It is even more than irrational, so it is a special kind of irrational number.

But, what is remarkable is that you see when n is a large positive value, 1 over n is the small positive value. So, you have a number 1 plus a small positive value, so this number is bigger than one and therefore, when you raise it to a power it increases all the time and now it is raise to a power n which is itself increasing. So, you therefore, saying if you had just 1 here no matter what you raising it to get 1, on the other hand you have 1 plus 1 over n here a little increment over 1 and that is raise to the power n.

So, you would wonder what is going to happen is it going to become unbounded, because you have a number bigger than 1 and that is raise to a positive value power and it is therefore, increasing all the time. But, the fact is whatever you raising your argument is decreasing here, because this is tending towards 1, had it be 1 like I said no matter what you raise it to continue to get in 1. So, there is a competition between the fact that this number extending towards 1.

And the fact that this power is tending towards the infinity and the result of this competition is that there is actually a finite limit and it happens to be this number here, it is the base of natural logarithms and it is a very fundamental number e. Now, extra polluting that a little bit the number e to the x it can be written as the limit as n tends to infinity as 1 plus x over n to the power for any value of the variable x.

You cannot in fact, define it not only for positive values of x, but also negative values of x fractional, irrational it does not matter you can in fact, define it even for complex values of this number x, but we would not get into that now. Can is there any other formula for e other than this? And the answer is yes, this is also equal to 1 plus 1 over 1 factorial plus 1 over 2 factorial plus 1 over 3 factorial plus etcetera and infinity, this is an infinite series and it converges in other words the sum of this infinite number of terms is finite and it is guaranteed to be equal to this number here.

You can see that these numbers are becoming smaller and smaller as n increases as you go further and further. And finally, in the limit you end up with this number which is convergent, a series which is convergent converges to this value. Similarly, this quantity also has an expansion which goes like 1 plus x over 1 factorial plus x squared over 2 factorial plus x cubed over 3 factorial plus etcetera and this too goes on infinity and the fact is that this make sense this infinite series converges. It is sum and produces to a finite sum as n go all the way up to infinity, infinite number of terms and the answer is e to the power x. So, one could ask what is this function look like, so let see how the function e to the power x is plotted and it goes like this.

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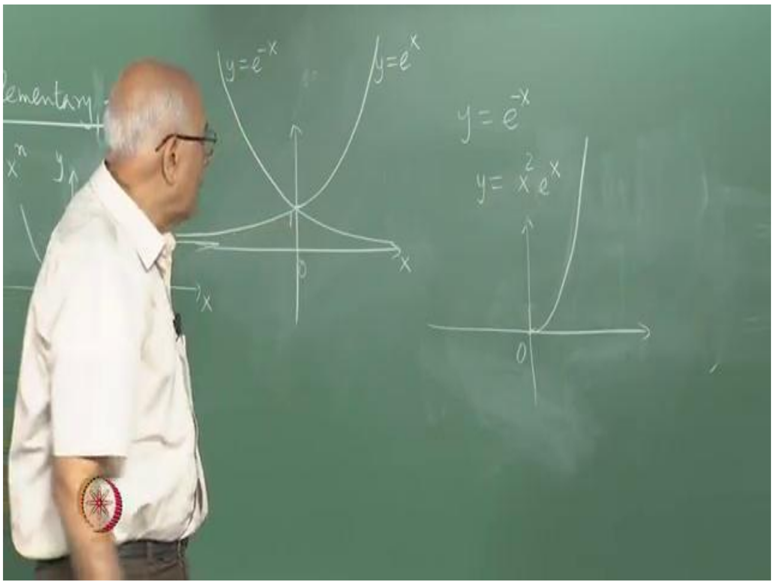

So, here is x, here is e to the power x out here, when x is equal to 0 e to the 0 is 1. So, it is starts at the value 1 and as extends towards larger and larger values it goes up in this fraction and tends to infinity more rapidly than any positive power of x. So, we says that e to the x tends to infinity exponentially rapidly; that means, it goes to 0, it goes to infinity as extends to infinity more rapidly than any x to the end for any positive n here.

But, as for is a graph is concern I just shown it in this fashion, this is not quantitative as you can see not only the same scale for anything like that. So, this is what e to the x does as long as x is positive, what happens when x is negative well that is easily found, because e to the power minus a number e to the minus x is equal to 1 over e to the power x. So, e to the minus x for negative x, e to the x for negative x remains positive, but it tends to 0, because it is the reciprocal of large positive number, so it goes to 0 in this fashion.

Once again it goes to 0 faster than any reciprocal power of x 1 over x to the 100 certainly goes to 0 when x tends to infinity much faster than 1 over x to the power 10. On the other hand e to the minus x goes to 0 even faster goes to 0 much faster than any power in the reciprocal power negative power of x. So, this is what the function e to the power x looks like, but you guaranteed that the numerical value of e to the x for any value of x and the real value of x is given by the sum in the series or preceding to this limit which is a compression way of doing it.

But, that series there converges very rapidly and within a sort number of terms depending on what the value of x is you get the numerical value of e to the x to whatever degree of accuracy that you desire. So, so much for e to the power x from you could ask what about the function e to the minus x, what is that look like is the reciprocal of either the x. So, we need to reciprocal of this function, so this is y equal to e to the x I would like to know plot y equal to e to the minus x and that is just the reciprocal of this function.

So, it goes to one once again, but it is the other curve namely, because down like this and equations. So, this is y equal to e to the minus x also remains positive is this, now you ask a little more complicated question, what does the function y equal to x e to the x look like, what is this look like. Once again, the argument is as follows let us plot it for positive values of x, so that you can see what is actually happening for positive values of x for very small x remember that e to the power x goes like 1 plus x over 1 factorial plus order x square and so on.

So, if a x very small compare to unity, but positive this number here is negligible and you have 1 plus x plus x etcetera, the leading term is 1, but there is already in the x here and therefore, this function looks like x, near x equal to 0 looks like x means it does this with the 45 degree slope and then the higher order terms kick in had it been just x to go on forever. But, because the higher order terms kick in this functions tool starts at any the aspect of an exponential.

Of course, it is going to go to infinity has x becomes large positive faster than need to the x, because the some other x multiplying it. But, the important thing is to look at the structure of the graph here ((Refer Time: 16:44)) in this function, I leave it you as an exercise to find out what this function does for negative values of x is got to have some continuity here and it does something on this side and ask to find out what is do when x is negative on this side.

Let us look at what x square e to the x looks like. So, let us plot x square e to the x as before by the same argument is before e to the x square form less near x equal to 0, it just 1 plus order x etcetera. And therefore, the leading term is going to come from x squared which has been known is like a parabola and then it takes of exponential this function. So, this is the way this graph looks like once you multiply with an x square and therefore, x cubed x 4 etcetera, etcetera.

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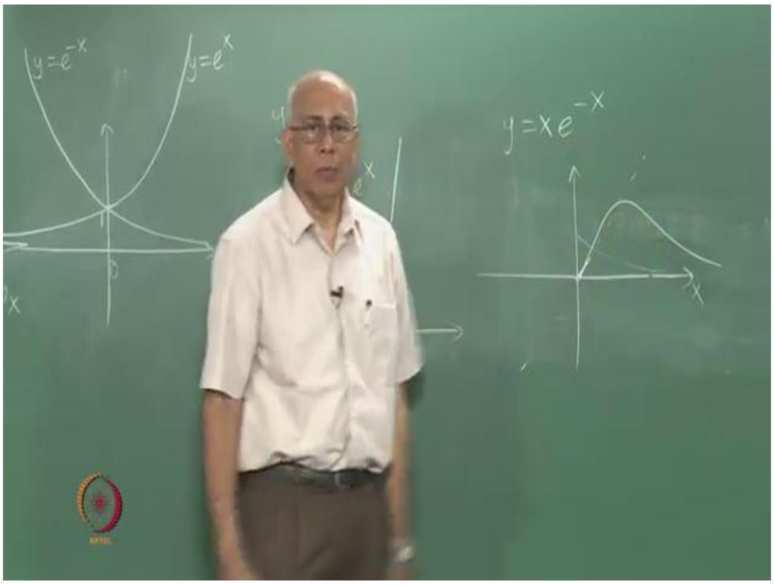

More interesting is a question of what is this look like y equal to x e to the minus x, what is this graph look like. So, here is x and this y what would this look like, again when x goes to 0 this function here goes to 0 that goes to 1. So, the limit is 0 passes through the origin right here, on the other hand for very, very small values of x this function is approximately x and then there are corrections of order x squared, in approximately x means it takes of linearly and then it could have continued for ever had this been just x, but it is now being multiplied by e to the minus x which if you recall starts at 1 and goes to 0 in this fashion.

So, we have the product of a function which starts at 1 and goes to 0 very rapidly and we have that multiplied by a functions it is starts at 0 and claims to infinity not so rapidly because it is a linear function whereas, the exponential drops to 0 much faster. So, as x goes to 0, this 0 is going to dominate because multiplied by finite number. So, the graph indeed takes of like that, but then once x becomes significant the functions drop back to 0, because e to the minus x is dropping to 0 very fast.

So, it is start at 0 it is guaranteed to go up and then it must drop to 0 eventually. So, the only thing it can do is to actually go like this and drop down to 0 in this fashion, it is not hard to the little bit of calculus to find out when the peek occurs. But, the qualitative structure of this function is clear just by looking at it by this little way of sketching this functions that this function here the product of an increasing function and the decreasing function, where this one dominates for large x is going to like this with a single peek.

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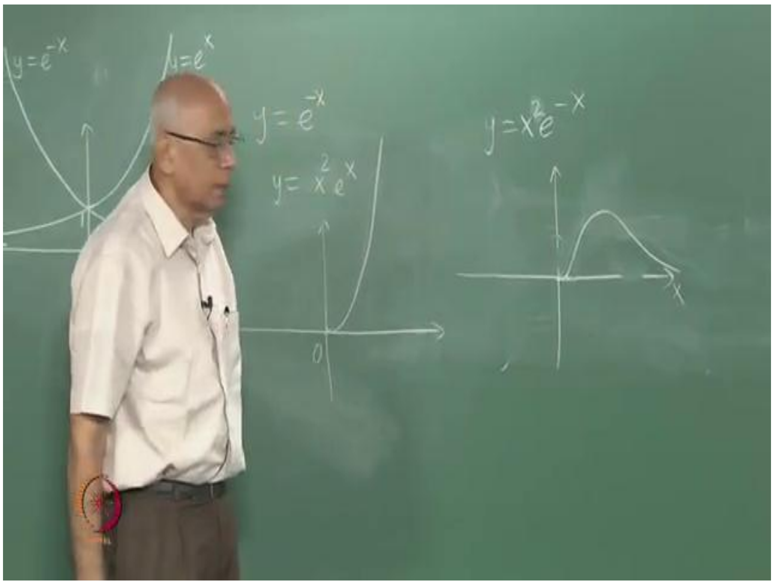

What about x square e 2 the minus x, it is going to behave an more or less the same way the only difference now is going to be exactly the same argument as before. But, the only difference now is that near the origin, the behavior is like x squared which is flat at the origin and therefore you have a graph it is look like this fashion, etcetera. So, one can now multiply by other function of x and so on.

So, the interesting thing is in this is what is going to happen in general very often you have a function, what is of interest in following facts, where does it have a peek, where does have a minimum, where does it have a maximum, what does it do for a small values of the argument near 0, what does it go for a large values of the argument near infinity does it become infinite for any become unbounded for any finite value of x and so on and so forth.

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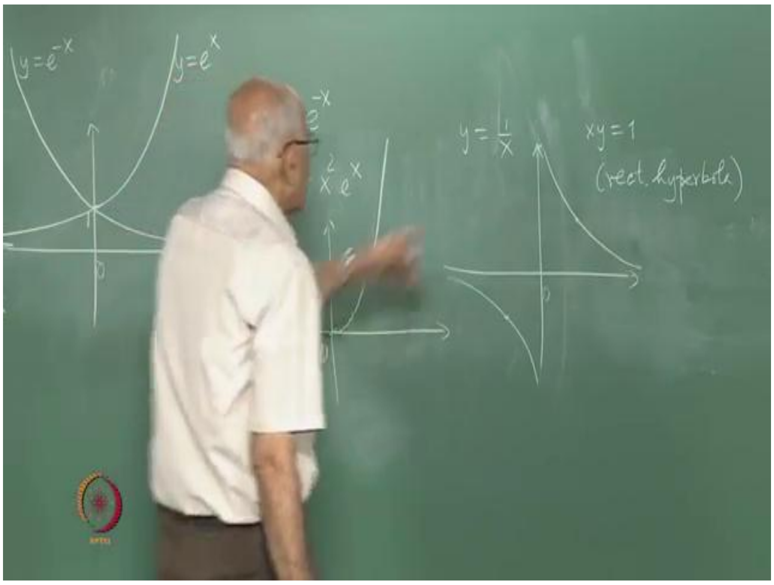

How about the function 1 over x, this function is an odd function of x, in other words when x is negative it just the negative or whatever the function was corresponding positive value of x. So, in this case for positive value of x, this is 0 the functions starts goes down it tends to infinity as extends to 0 plus infinity and I extends to plus infinity the function tends to 0. But, for negative values of x is just the reflection of this graph into the third quadrant and it looks like this.

And as you know this can also be written as x y equal to 1 which is called a rectangle are hyperbola, it is the very special kind of graph. But, this is what the function looks like, this quadrant is decreasing this way and this quadrant it goes up in this fashion, just reflected and of course, in this case at passes through the point 1, 1 or the point minus 1 minus 1 on this side. What happens if x y could 10 for instance it is again qualitatively just hyperbola except it will pass through the point here square root of 10 square root of 10 and minus those values here on this side qualitatively to look at exactly like this in this fashion ((Refer Time: 22:03)).

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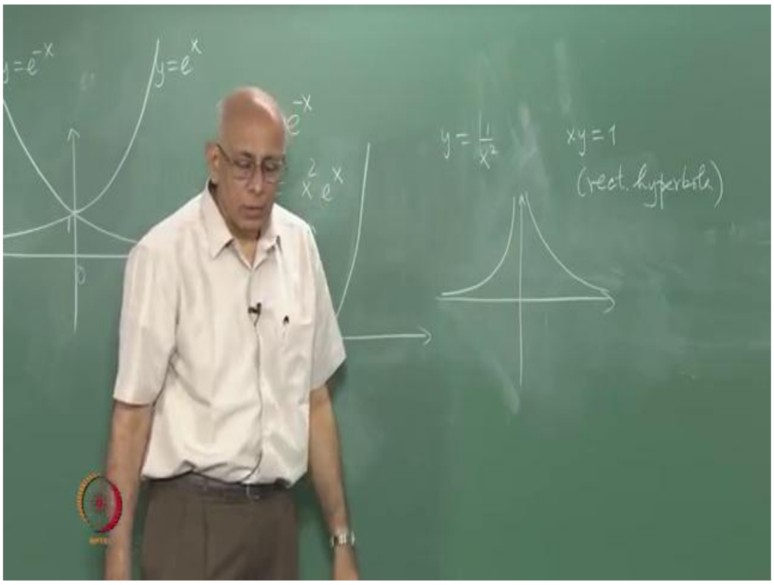

So, that takes care of 1 over x how about 1 over x squared and that is not a hyperbola, what would this function look like. Well, the first thing you going to ask in all these cases is the function in even function of x or an odd function of x. Because of the function is even you save yourself of trouble, because it simply reflected on the vertical axis, whatever happens in the first quadrant happens in the second and if it is an odd function whatever happens here happens in the third quadrant here.

So, the symmetric property is worth always being attention to because it will save your lot of labor. This function is an even function of x as x tends to 0 it becomes infinite and as an x tends to infinity goes to 0 both these are faster than the way 1 over x goes to infinity or 0 more rapidly. Because, it is the power is higher and known this side; however, it look like this and the reason is that this is an even function of x 1 over x cube it will again qualitatively look like that except this portion would be sitting on here on this side. So, we have y equal to x to the power n for negative values of n, as well we can plot what this function looks like, how about a function like the following.

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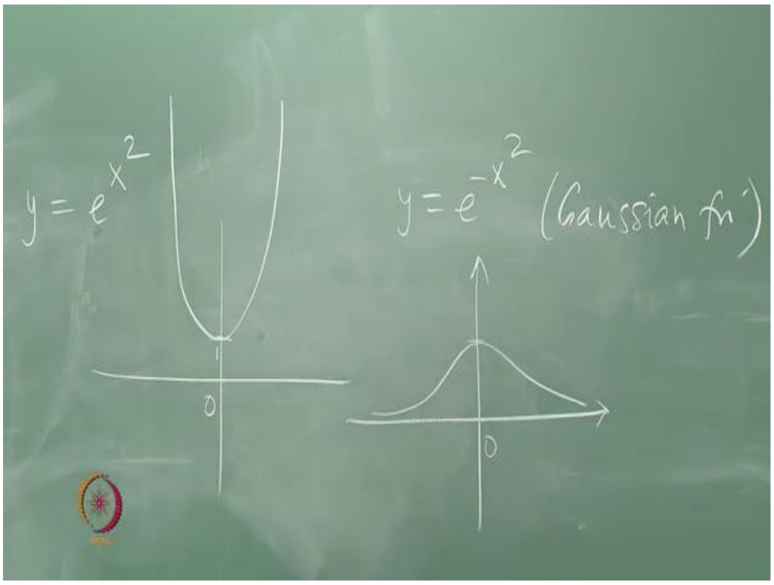

How about a function like y equal to e to the power x square, what is that look like well it is an even function of x. So, once again same argument is before whatever happens here is going to happen here, e to the x square starts at 0 at x equal to 0 starts at the value 1, e to the 0 is 1 and increases faster than exponentially, because x square goes to infinity faster than x itself does. So, e to the x square goes to infinity even faster it goes often in this fashion.

But, because it is an even function this function deep in this function, it is not a parabola it just a function which in looks like a bowl and this symmetric about x is going to minus x and it goes off to infinity on both sides this fashion. What about the reciprocal of this function, y equal to e to the minus x square that is a very, very important function it is called the Gaussian it appears all over the place in many, many physical applications and statics. So, it is worth remembering all it looks like, here is the x axis, here the 0 and what is the reciprocal of this function what is it look like.

Well, remember that the function is always bigger than one therefore, the reciprocal is always less than 1 and the function is positive and therefore, the reciprocal is also always positive and as extends to infinity. Now, e to the minus x square goes to 0 extremely faster on both sides and it is a bell shape curve which is therefore, looks like this. So, that is what this Gaussian function e to the minus x square looks like you will learn a lot more about it study of physics as you go along.

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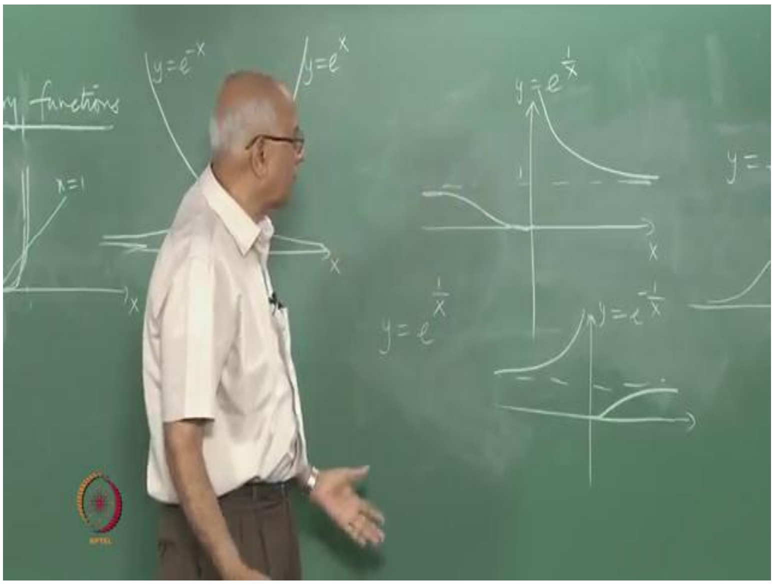

How about y equal to e to the power 1 over x, what would that look like? Again qualitatively, because it is fairly complicated function in that sense, but we can sketch what it looks like quite easily. Again, so y is e to the 1 over x as a function of x is it in even function or odd function of x can we said what it is, well since if we put x to the minus x when we write e to the minus 1 over x that is not the same as minus e to the 1 over x.

So, this is not in even function or an odd function of x it mixed, we can say what it does on this side without examining in this function more carefully. But, let us argue in the following the way as extends to plus infinity 1 over x tends to 0 e to the 0 tends to unity. So, it is clear that if this is 1, this function must go to 0 must tend to this value of a very large x. On the other hand, it is a positive number e to a positive number is bigger than one always.

So, the function remains on top of this graph and then what is it do as x tends to 0, 1 over x tends to infinity plus infinity. So, e to the infinity becomes infinity, so the function very rapidly increases and drops on in this fashion, what about negative values well if x is a large negative value 1 over x is nearly 0 e to the 0 is again 1. So, the function is going to approach the value 1 on this side.

But, because it is a negative e to the negative number it less than 1. So, it remains below this curve, but then a x tends to 0 from the negative side it is going to e to the minus infinity that is like 1 over e to the infinity which goes to 0 very rapidly. So, the function indeed does this goes to 0 extremely rapidly and very flat it looks like that. So, that is what e to the 1 over x look like for both positive x and negative x here.

And once I write that you should be able to say very easily what does y equal to e to the minus 1 over x look like and this case. Because, now all I have done is to change x to minus x, so it is immediately clear that the graph is going to look like this and this side and on this side it is going to do this in this plot portion that just by examining in this and saying instead of x and I call it minus x. So, it is like flipping this graph over twisting it 180 degree about this access rotating the graph and that is it this is what it looks like.

So, as we come across more and more complicated functions, you will see that I will use such arguments to plot these functions qualitatively schematically to try to understand what these thing look like. But, the take home lesson here now is that the exponential in general tends to dominate over any powers, but when you take limits and something becomes very large, very small and so on. In general rule of ((Refer Time: 28:32)) that the exponential will dominate this issue and it is behavior will dictate what happens when you have a product of such function, we will have a occasions to do many more of these examples as we go along.