Mechanics, Heat, Oscillations and Waves

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Lecture – 36
Circulation Vorticity

The last topic, I want to introduce in the context of fluid dynamics has to do with the concept of vorticity, which is a technical word for what we already know in terms of eddies or whirlpools or something like that. But, you do not necessarily have to have whirlpools or eddies in order to have vorticity in a fluid as we will see. I need to introduce a little bit of vector calculus to do this, but we will do this in slow steps.

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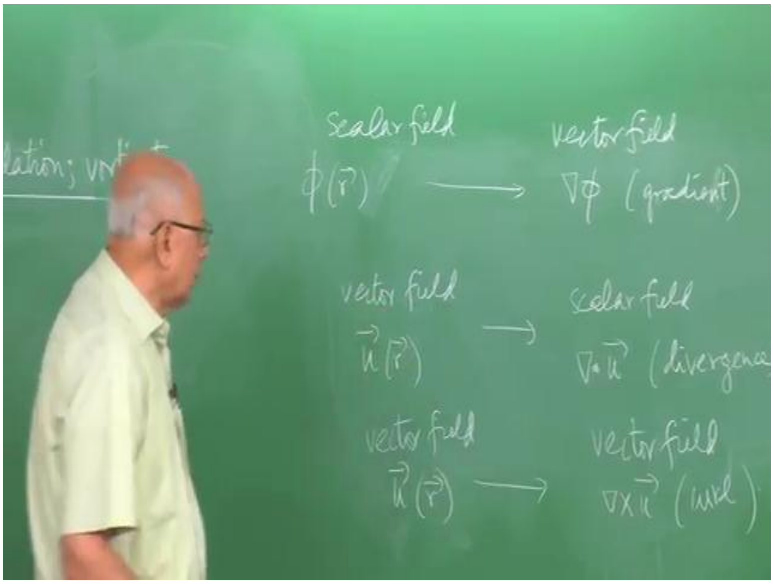

You already have seen that if you give me a scalar field phi of r and t; that is a scalar field; well let us forget about the time dependence for the moment. If you give me a scalar function of the position coordinates, then from here we can form a vector field, namely the gradient of phi or the directional derivative of the scalar field. And that is called the gradient of phi, written as grad phi or del phi in this form.

We have also seen that if give me a vector field, let say some u of r for instance. From here, we can form a scalar field, which is the divergence del dot u and that is the divergence of this vector field here. It gives you physically, it is equal to the flux per unit volume in a limit in which this volume goes to 0 at any given point r and we have a formula for it, which is written by writing these out for instance in Cartesian or any other coordinate system.

We can write down, what sort of derivatives with respect to the position coordinates, this involves here. For instance in Cartesian coordinates it is delta u x over delta x plus delta u y over delta y plus delta u z over delta z, but in other coordinate systems you gets a little more intricate, we would not go into that right now. The question you can ask now is, from a vector field can you form another vector field and the answer is yes.

So, here is a vector field u of r and from here, you can form another vector field, which is called the curl of this field and formally, operationally although that is not the basic definition. Operationally, just as this is del dot u, this other field is given by del cross u and I will write down this explicitly in a moment here.

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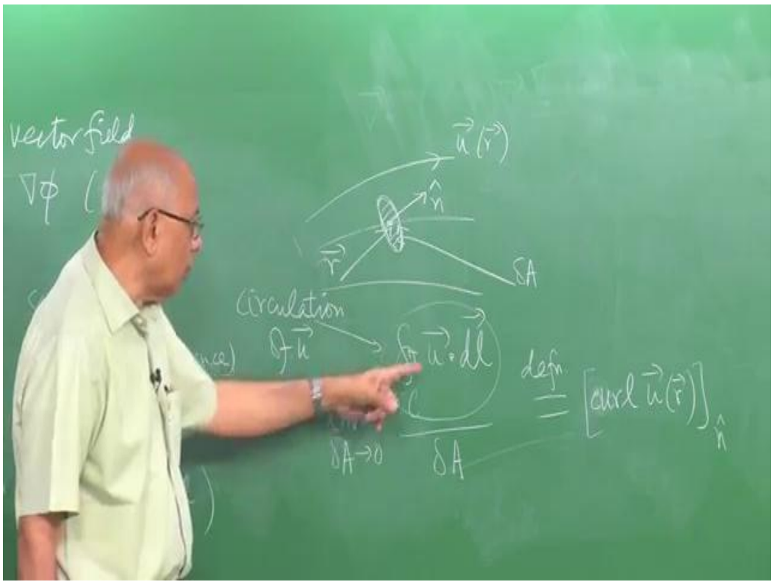

Physically, what it does is the following, if these are the lines of the vector field u of r, then at any point say here, you take an infinite decimal area element delta A pointing along some unit vector n. While in area itself, if it is a curve surface for example, you cannot associate a unit vector with it and area element is essentially a planner object and you can associate a unit normal with it.

So, if this is the direction of this normal n to this area element delta A, then, what you construct is the so called circulation of this vector field by taking u dot d l along a closed path around the periphery of this area in this fashion. This quantity is called the circulation of the vector field and d l is a line element, we changes the direction as you move along here, each d l is infinite decimal, a small tiny vector.

And you move along here, it changes directions, you got it with the u at that point and this is called the circulation of the vector field u around this circle here. This contour here C, this is the circulation, you find the circulation per unit area and you take the limit as delta a goes to 0, the limit is finite, just as we took the flux per unit volume, one infinite decimal volume element. And the surface L area delta S of that volume element went to 0.

In the same way, this periphery; this circumference also goes to 0. The area goes to 0 in such a way that this quantity, if it has a finite limit, it is called the curl of this vector field. So, this is the definition, it is called the curl of u at the point r. So, this is the point r in some coordinate system and the curl of u is defined as the circulation, this is the circulation of the vector field.

So, the circulation per unit area is the normal component, it is the component of this vector along the direction n or if you dot this quantity, so let us leave it like this. So, to find the curl, full curl, you going to have to take different area elements pointing in different directions and each case you find the normal component by computing this circulation per unit area. Now, this is a cumbersome thing to do, so the obvious thing to do it should take the convenient contour and find out, what it is in Cartesian coordinates.

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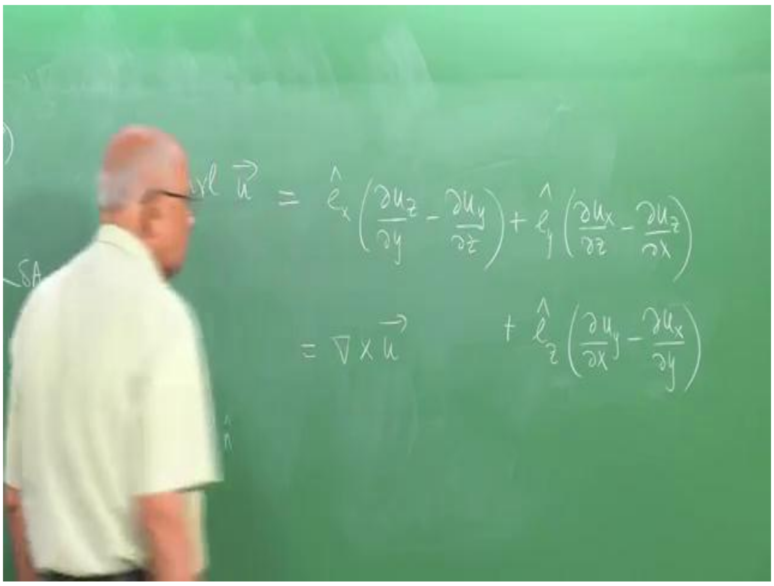

It terms out that curl u in Cartesian components is equal to e x times delta u z minus delta u y over delta z plus e y delta u over delta z x minus delta u z delta x plus is a delta y delta x u y minus delta u x over delta y. So, that is the formula for the curl and it is clear that this quantity, can also symbolically, it is exactly the same as writing del cross u. So, I emphasize again, although it is del cross u and delta u for the curl and divergence respectively.

These are not the definitions of the divergence in curl of a vector field. The definitions are in this case, the flux per unit volume and the limit of the flux per unit volume as a volume element goings to 0. And this is the circulation per unit area gives you depending on the normal to the element of area, it gives you the corresponding component along that direction of the curl of this vector field. And this is we like an algorithm for computing for the curl is one should tell me this vector in Cartesian coordinates.

Now, the question is, what is the physical meaning of the curl of the vector field, but before that, let me see, where this, let me write down, where this comes in, in our problem of the fluid flow.

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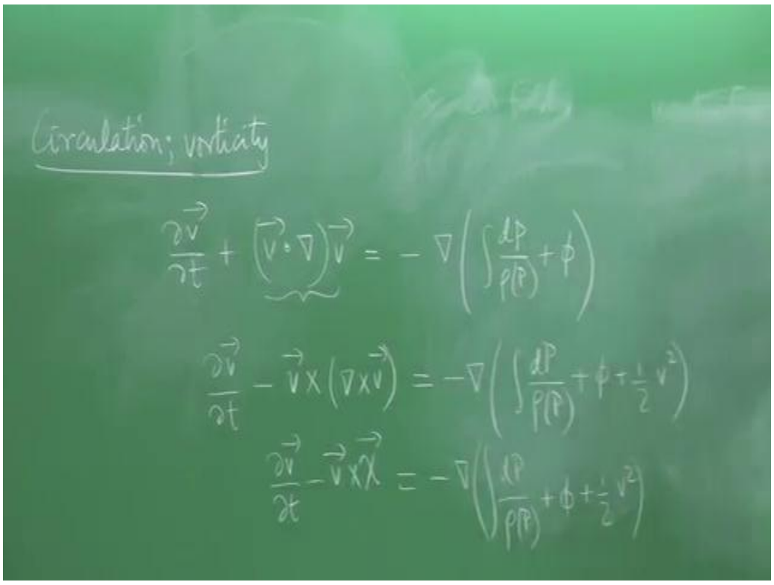

Recall that, our equation of motion of a fluid element was this plus v dot del v, this thing here was equal to on the right hand side, we had minus the gradient. And let us look at barrow tropic flow as usual d p over rho of p plus phi plus F, viscous which I am dropping all the time. This quantity, when we took the dot product with v, we discover with essentially the gradient of half v squared.

But, now without taking the dot product, we can simplify this alone and it terms out that this can also we written has there is minus v cross, the curl of v. So, that is del cross v, this form that is equal to minus the gradient of integral d p over rho of p plus pi plus

1half v square. So, the statement is v dot del on v is the gradient of half v squared, which I bring it to this side minus v cross the curl of v itself.

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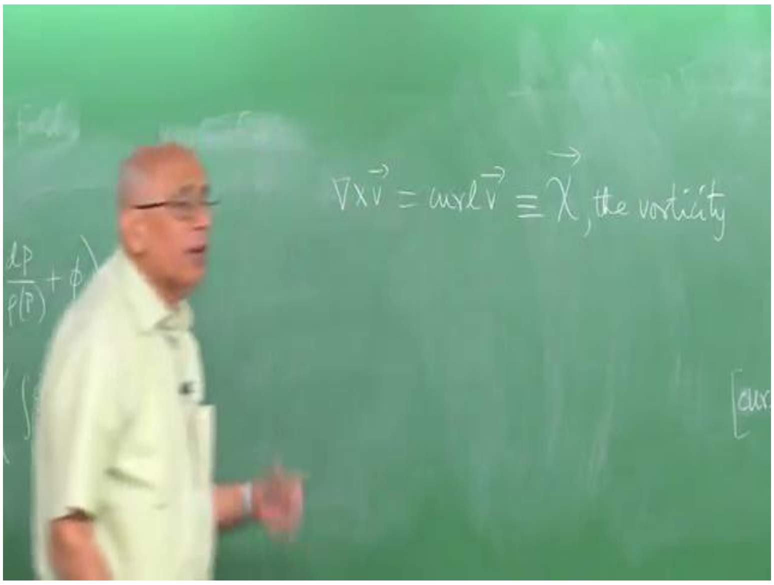

And this quantity del cross v equal to the curl of v equal to sum chi, which is of course, function of r and t in general for a fluid. This quantity here this chi is called the vorticity, you can define a vorticity at every point in space at every given time for a velocity field. If this is a function of r and t, so is this a function of r and t in general and terms of this vorticity vector, ((Refer Time: 10:08)) this quantity, this thing here becomes delta v over delta t minus v cross chi, equal to this quantity on the right hand side. That is the general equation to a fluid element.

So, let us write down on also ((Refer Time: 10:17)), now before I say, what happens to this quantity in general, when you have a fluid flow, which is got vorticity of this kind. Before I do that, let me also point out another remarkable statement, which is the following. We need to geometrically interpret this vorticity here and the simplest, we are doing this is as follows.

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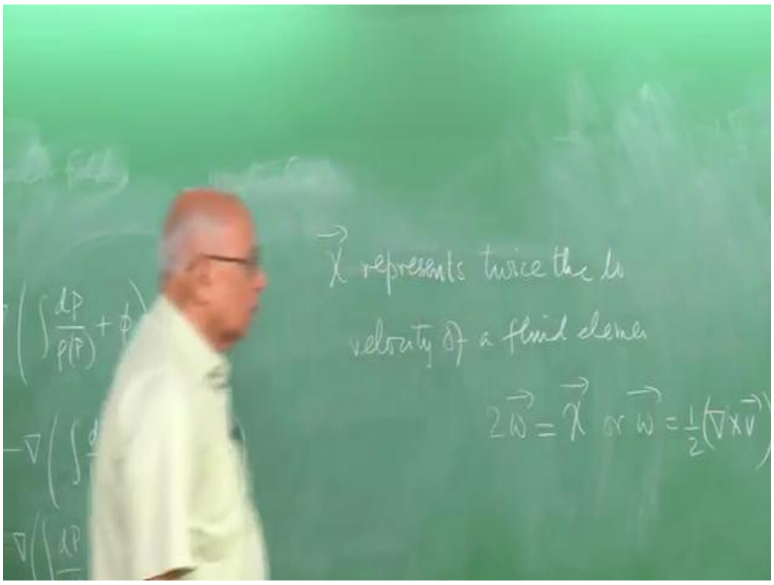

It represents, chi represents twice the local angular velocity of a fluid element, if you regard, it has a rigid body. And what is this mean, it means that if you have a flow and you focus on some fluid element, you are on that fluid element, you ask never mind how the fluid is going around it. You ask, does this fluid element rotate around itself or not, if it does, then there is a chi or there is a vorticity and if it does not, there is no vorticity and let us immediately given example to show you how this works.

So, let us look at a flow along a horizontal canal and for simplicity, I will look at the surface flow, we will not look at the depth of it. The depth is along the z direction say and the flow is the surface of the canal in the x, y plane.

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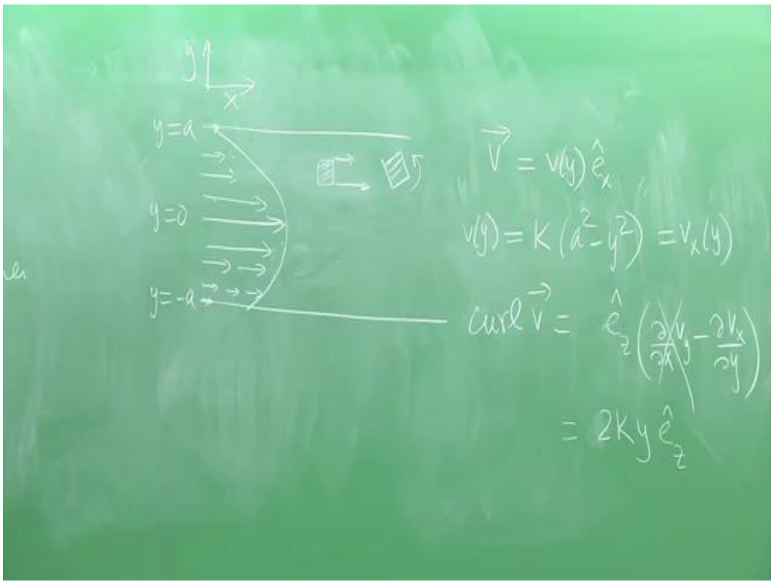

So, it looks like this, that is the x direction, this to y direction and let us put our coordinate axis, so that down the middle of this canal is the x axis and the flow stream line, all the flow lines are parallel to the banks of the canal on both sides. And let suppose that is viscosity in the problem, we have already seen this when we first did dimensional analysis for ((Refer Time: 12:54)) flow in a pipe.

I pointed out that if you have a horizontal pipe, the flow is maximum, the velocity is maximum at the center or axis of the pipe and it tappers to 0 at the two ends, because of this viscosity. So, if I have to draw the velocity vector symbolically, you can see that the vector is going to be very large here. And then, it is going to be little smaller here is going to be even smaller here, even smaller here and so on, till it is essentially 0 at this point and this point.

And let us give the some coordinates, let us this is the line y equal to 0, this y equal to a and y equal to minus a. So, the width of the canal is 2 a and this is what the flow in the velocity vector looks like at these points and of course, at this point is exactly the same in this form, etcetera. So, these are small vectors, these are medium vector, these are small vectors, this is even time in vectors and so on.

So, to simplify the pictures, so that, it does not complicated, let series all these arrows and you have a parabolic profile of this kind in which the maximum speed occurs in the center and it tappers of in this direction on both sides. Now, what is a velocity field look like in this problem, it is immediately clear that v at any point, we are only looking at the surface. We not worried about z coordinate at all, v at any point is the speed multiplied by e x, because it is along the x direction at every point.

And the speed is dependent on how for you are away from the center along the x axis, the further up you go, the further up or down you go in the y direction, the speed decreases. This v is a function of y alone and it is gotten x component of the velocity, which is dependent on y and therefore, you can see that, the curl is not going to be 0 in this problem.

And what is this say and how to this manufacture itself, well I have a proof this, but we can show that v of y equal to some constant times a squared minus y squared in this problem. It is symmetric about y to minus y; that is clear, it must be 0 at y equal to a and y equal to minus a, which is the banks of the canal and in between it is largest. So, the function is quadratic and one can show this straight forwardly and it looks like this v of y looks in this fashion here.

And this is of course, v x of y and that is the only component of v; that is not 0, because a flow is a along the parallel to the x axis everywhere on the surface. Once, you have this, the curl of the v only component of the curl of v; that is non-zero. If you recall the formula for the curl that we wrote down earlier is e z delta over delta x v y minus delta v x over delta y and this component is 0, because the velocity has no component in the transfers in the y direction.

So, you have e z minus delta v x over delta y, but the x of y is given by this. So, this is equal to minus k is a minus sign here and the minus sign here. So, this becomes k times 2, so I differentiate this with respect to y and going to get a minus 2 y and I put kill minus sign and you get a 2 k y times e z, this is the curl of the velocity. So, even though the stream lines are parallel lines, they are straight lines, you still have a curl and the moment you have a velocity with the curl, the flow is set to be rotational.

If the curl is not identically 0, the curl f of velocity field is identically 0, the flow is set to be irrotational, just a piece of terminology. But, what it implies is the following, it implies that fluid elements here have a local angular velocity which is given by half is curl here. In other words, proportional to k y equal to k y time e z, what it means is the following, if I put a small raft here in this fashion and it floats along, then because the velocity is larger here and smaller here.

As time goes along this raft will rotate by it is about it center in this fashion and the angular velocity of that rotation is what is measured by this curl here. According to this rule, ((Refer Time: 18:09)) which says 2 omega equal to chi. So, this example illustrates clearly, what is meant by the curl of the velocity field, it gives you the local angular velocity apart from this factor of 2. And not surprisingly, you can see that as you go down in this as you go up here or go down here the rotation rate changes and that is what this is want to tell.

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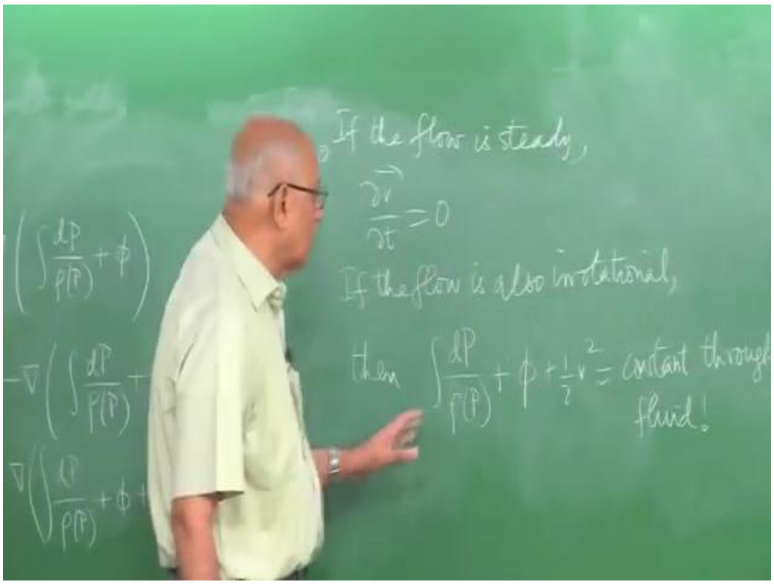

Now, where is that, take us here, well now you can see that, if the flow is study, this equation delta d over delta t is 0. And more over, if the flow is irrotational, when well this term is 0 and this term is 0. And now, even without dot in with v, this whole thing has a gradient, which is identically 0. That means, along any direction what so ever, this scalar function cannot change. So, it says that integral d p over rho of p plus phi plus half v square equal to constant, the same constant throughout the fluid.

So, we have a powerful statement, it says that this combination here, this scalar combination is represents some kind of energy as you can see per unit volume of the fluid, this combination does not change throughout the fluid. So, it is the same and all the stream lines. We already saw that rotational or irrotational flow, we do not care as long as to study barrow tropic etcetera, etcetera.

We had Bernoulli's principle, which told as that this combination was constant along each streamline. Now, if in addition, there is no vorticity anywhere in the fluid, then this constant is the same on all the streamline. So, it is constant throughout the fluid, this combination here. If the flow is rotational, then we have already seen that this quantity here is constant are given streamline.

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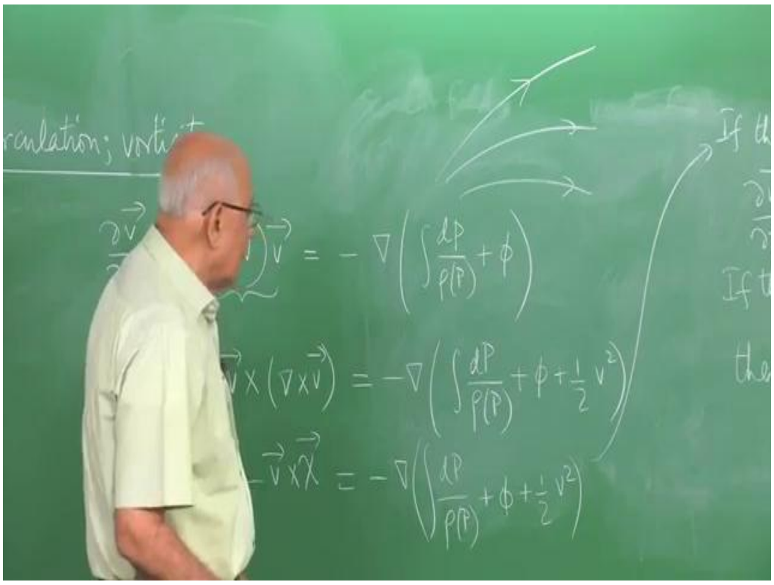

So, here is a given streamline and constant along that, might have a different value on this or this or different streamlines. But, again if the flow is irrotational all this constants are exactly the same. Even otherwise, if I took this equation and it was study flows of that goes away and I dotted this not with v, but with chi.

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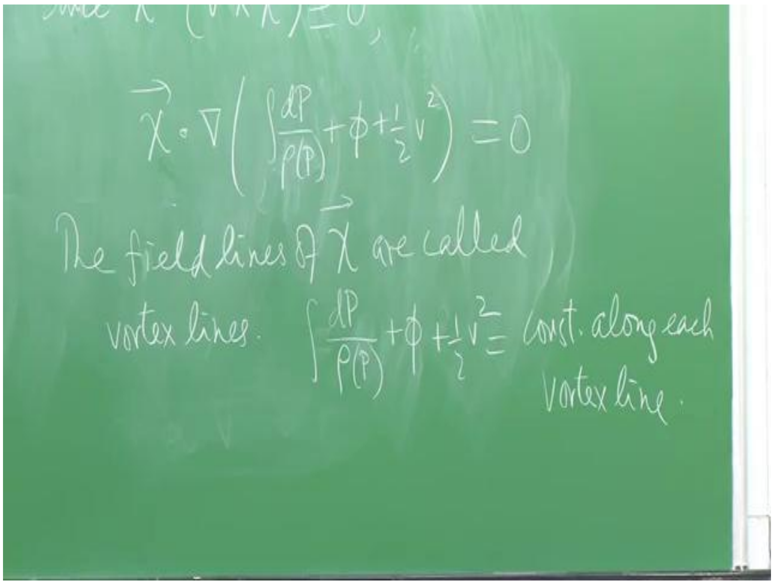

Now, if I dotted with chi, then since chi dot v cross chi is identically 0. This scalar triple dot product identically 0, because v cross cui locally perpendicular to chi and this is component of along chi with this dot product is 0. Since, this is so chi dot the gradient of this quantity in the brackets integral d p over rho of p plus phi plus half v squared is equal to 0.

And just as in the previous case, where it was v dot this quantity, which was 0 and I said therefore, this quantity is a same along a streamline of flow line of the velocity field v, a field line of the velocity field. Similarly, now it says, this quantity does not change, because no gradient component along a line on which chi flows. In other words, the field lines of this vector chi this circulation vector chi, the vorticity vector chi are such that, this combination does not change along these vortex lines.

So, the field lines of chi are called vortex lines and integral v p over rho of p plus phi plus half v squared equal to constant along each vortex line. So, this quantity is constant not only along each streamline, but it some given value along streamline with also has some given value along each vortex line. And then of course, you could ask, what happens if the vortex lines and the streamlines exhaust all possible directions in some specific sense, well then the whole thing is constant along entire fluid.

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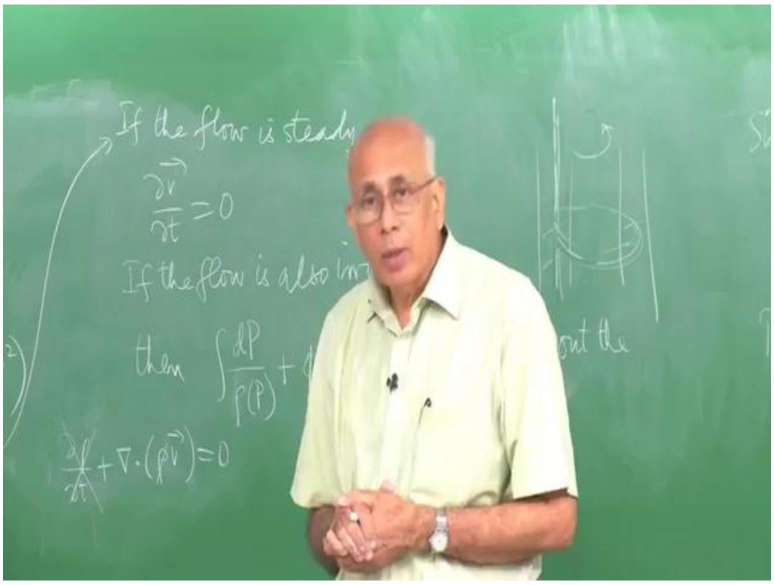

And this is what happens, if you took two cylinders, concentric cylinders and you rotated it, the inner cylinder keep in this one fixed and if put fluid here. Then, the streamlines are lines which are concentric circles with this center as axis. There is no flow in the z directional at all and then, each vortex line, each streamline is a circle of some kind like this and each vortex line is a line along the cylinder form by these little rings here.

And now, you are told that this combination is constant along this line, it is also constant along this line, therefore it is constant along that line and so on in the fluid have exact the value everywhere. So, where the special kinds of flows, where you can make very general statements based on these principles here of what happens to the vorticity are or what happens to this quantity here this conserve quantity along a stream line or along the vortex line.

So, the fact that remains constant both along stream lines and along vortex lines will be very powerful way of analyzing fluid flow. We have a dot at all about the other equation, the equation of continuity, but along with this statement, you also have the statement that the delta rho over delta t plus del dot over times v equal to 0. And if the fluid is incompressible, then this term goes out and that is a constant that goes out and you have del dot v 0.

So, you have left with smaller problem of analyzing the velocity field, such that, it divergence identically 0 and it is curl way certain simple equations, which we can explicitly write down. And then, when large number of formal theorems can be proof which a very useful in practice in fluid flow. So, to summarize in a nut shell, we have seen the most basic elements of what happens, when you took Newton's law and applied to the motion of a fluid.

The fluid is a describe by density field and a velocity field at every point, you also have to specify the pressure field on it, namely we what the pressure is at every point to take the time. Then, in cases like barotropic flow, where the density is a function of the pressure alone, we can go ahead of distance in analyzing this complicated partial differentially equation, even though it is non-linear. If we neglect the viscosity using these general principles on the way, we saw that concept such as divergence, gradient or curl and so on of a vector field at a very naturally.

The next step of course, would be too potent the viscosity term and ask, what happens when you have viscosity, when you had viscosity, I am not going to do that, you end up with an equation, which improves upon the Euler equation that we written down here. It is called the Stokes equation and in principle it contains all the information go on along the continuity equation to the solve in principle to solve the problem of fluid flow.

But, you must add to this boundary conditions initial conditions and so on, the mathematical problem of fluid flow is formidable especially the problem of turbulence. But, I hope this is given you some introduction to the basic ideas involve in analyzing this fluids and how the fundamental equation of fluid flow already follows from in assumption of new consequence of motion, I think I will stop here.