Mechanics, Heat, Oscillations and Waves
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Lecture - 35
Fluid Flow
Bernoullis Principle

The equation of motion we have derived for the fluid element leads us to a very important principle, which is very useful in practice called Bernoulli's principle. And today I am going to discuss that along with various other related matters.

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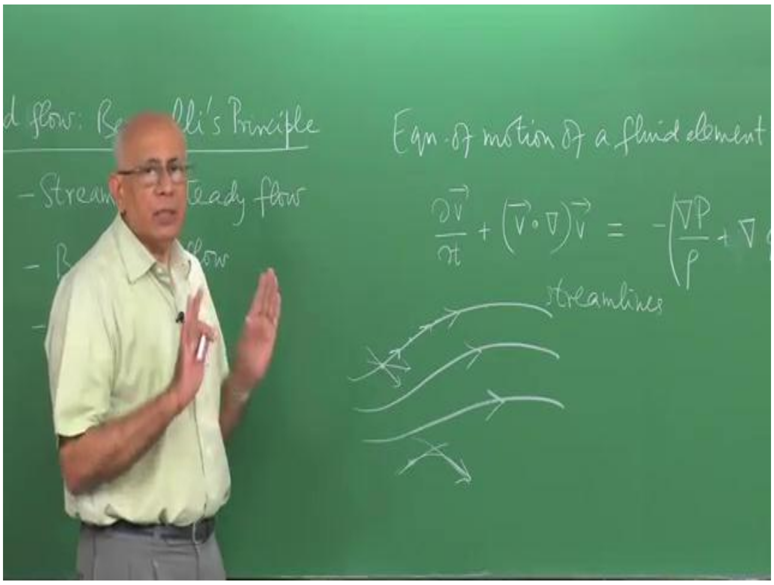

If you recall the equation of motion for a fluid element, so equation of motion of a fluid element was of the following kind, it made a statement about the rate of the change of the velocity field of the fluid element. And it was of the form delta v over delta t plus v dot del on v was equal to on the right hand side, we had a combination of terms. We had the terms, which went like minus the gradient of the pressure divided by rho plus. So, let us put everything inside minus gradient of pressure over rho plus the potential out here, the potential energy per unit mass of the fluid was sitting here plus where plus is a viscous term.

And if you drop the viscosity of the fluid, assume that the viscosity is low enough that it can be dropped, this in this it flow then this term it throughout and we had an equation of this kind. So, this is not a trivial equation to solve, it is a complicated equation as I repeatedly emphasize. The fact that this v appears here and here makes it non-linear and very, very non trivial to solve, even if you tell me, what the pressure is and what the density is and so on at every point in space and time.

You still have a difficult problem mathematically to solve this equation that was the basic equation of a fluid element here. Now, the way to derive this principle is as follows, is to first of all notice that this term here it should be very gradient of phi. It would be useful, if we could write this term here as the gradient of something or the other. But, the gradient of p divided by rho and if the fluid is not, if it is incompressible of course rho is a constant it comes out.

But, otherwise you have a rho here and it is a non trivial function possibly of the pressure, of the density and so on, you know of the velocity etcetera, it is coupled to the velocity. But, we can always write this term as a gradient term as follows in a very special instance and that has to do with bar Barotropic flow. But, before I do that let me come back and say a few things which are implicit in, what I said earlier, but I did not say it as explicitly.

What is a stream line? Well, a stream line is just the feel line of the velocity field in other words, if at some instant of time the velocity vector at this point is in this direction, at this point is in this direction and so on. You join all these up and you get the portrait of the velocity field throughout the fluid at any given instant of time, those are the stream lines. We have statically assumed that our flow is stream line flow, although I did not say it explicitly, this flow is assumed to be stream line flow.

In the sense that at every point it is assume that there is a unique velocity field, the unique direction of the velocity and magnitude of the velocity at very point in the fluid at any given instant of time, now these are the so called stream lines. The line of flow may actually be different in the sense, that if you give me a certain point here, look at a certain point, if this whole flow changes as a function of time, in other words if v has an explicit dependence on time. So, that this derivative is not 0, then if at time t 1 the velocity vector has it is along that direction, at a later time it could be along this or along this and so on.

So, the entire set of stream line could actually change as the function of time. So, this flow lines, then serves could be different, so if you took a given parcel of a fluid it could be on one stream line at a given instant of time and at a later instant it is on a different stream line, it moves this way and so on. So, this could be the path of the parcel of fluid and it is not on any one given stream line. So, the point is that if the flow is not steady, the stream lines and the lines of flow are different from each other. But, if it is a study flow, this pattern of the velocity field lines does not change with time.

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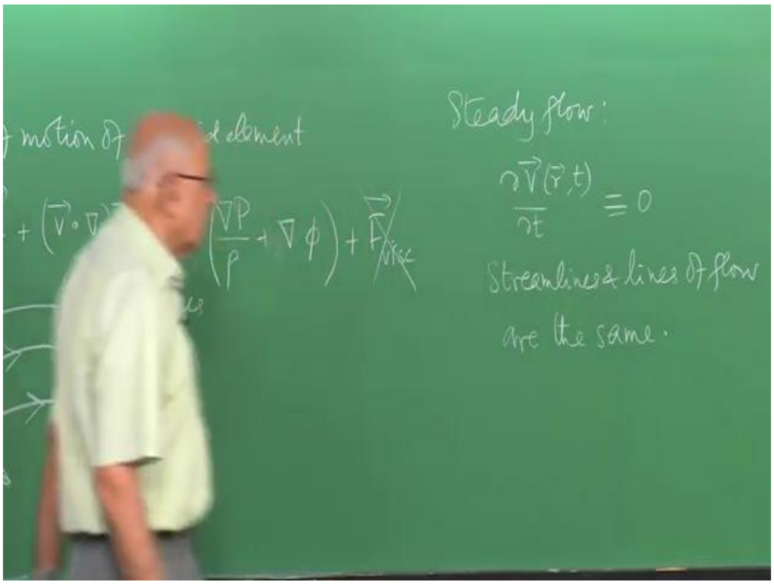

So, a steady flow corresponds to implies that delta v of r t over delta t is identically 0 at all points, no explicit dependence on time and moreover stream lines and lines of flow are the same. They coincide with each other and that is intuitively obvious, geometrically obvious here, that if the pattern does not change as the function of time, then we say the flow is steady. So, in the case of steady flow we can drop this term here first, so first assumption, let us look at steady flow.

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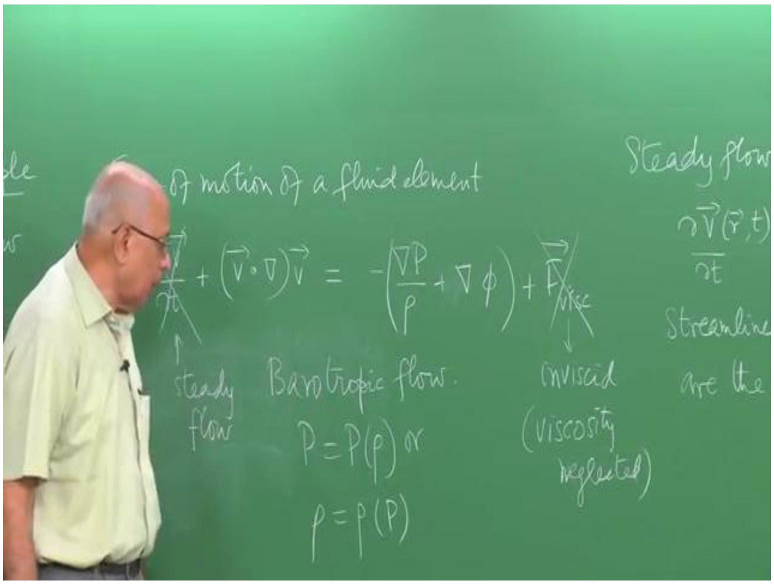

So, this goes away, because steady flow; this goes away, because I assume that the fluid is inviscid, viscosity neglected, so that gives you a little bit of simplification. Then, we would like to know what this term looks like, in general this term is quit complicated. But, there is one kind of flow called Barotropic flow where it terms out that the pressure is some given function of the density or reciprocally the density is some specified function of the pressure itself. Now, when this is happen? Consider for example, the ideal gas, which we are going to study in some detail a little later.

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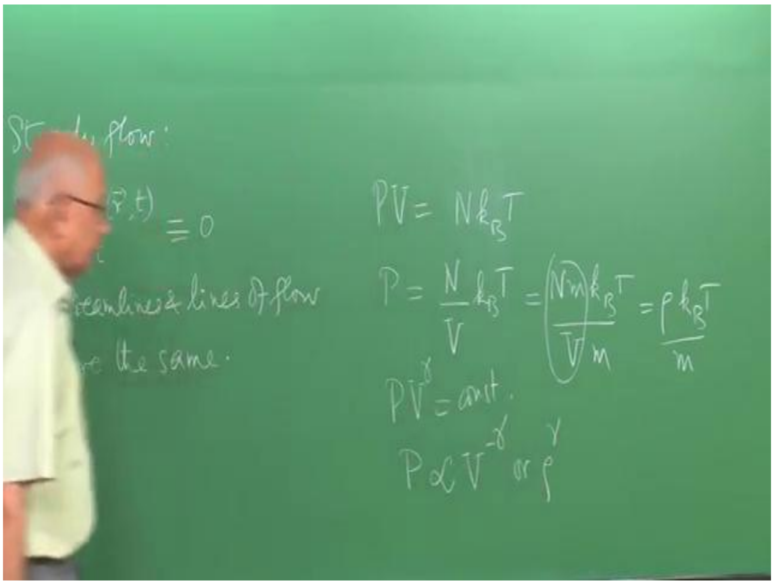

In that case you know that P V equal to the number of moles times the gas constant times the absolute temperature here. We could rewrite this in many, many ways we could write this in terms of the total number of particles times Boltzmann constant times T. And therefore, you can have, even write P is N over V k Boltzmann T, where this is the number of particles, it is the volume of the gas and k Boltzmann is Boltzmann constant.

If I multiply top and bottom by the mass of the molecule, then this becomes N m k Boltzmann T over V times m. But, N m over V is the total mass of the gas divided by the volume of the gas in therefore is the mass density of the gas. So, here is an example, at a fixed temperature for an ideal gas the pressure is directly proportional to the density and vice versa. So, this quantity here, we can actually evaluate in this case, because rho becomes the function of p.

So, for general Barotropic flow implies that P equal to P of rho or rho equal to rho of P. In other words, there is no other dependence, the density depends only on the pressure;

all other variables are irrelevant. In that case, the temperature would have to be kept constant and then, of course you have exactly that situation. In fact, over relation which says P is proportional to rho directly, but here in general it some non-linear function complicated function of flow.

Another example, if the gas undergoes an adiabatic process then P V to the power gamma equal to constant as you know, aware. And since, V is proportional to 1 over the density for a given amount of gas, this says P is proportional to V to the power minus gamma or density to the power gamma. So, that is another situation where you have a flow of this kind, pressure or density is given some function of the pressure. In that case this term can actually be written down as a gradient of something than the other.

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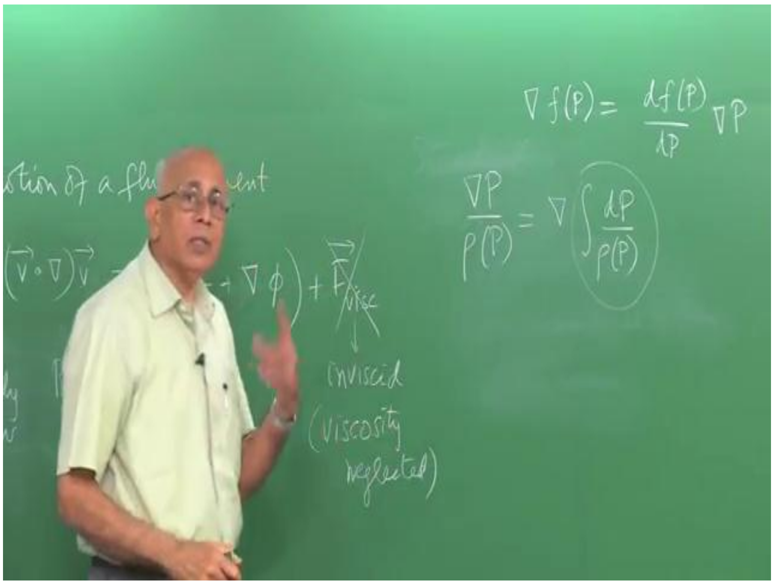

So, let us see how that happens, gradient of pressure divided by rho if it is a function of pressure alone can be written in the following way. If you have the gradient of some function of the pressure, this is equal to d f p over d p times the gradient of the pressure by gradient time in derivatives with respect to x y z with those unit vectors put in here. So, this is of course equal to the gradient of an integral in definite integral d p over rho of p.

Because, this quantity is an indefinite interval, which is the function of p and it is derivative is precisely 1 over rho. So, it is clear that by this rule the gradient of p over rho of p is a gradient of the indefinite integral d p over rho of p and if you tell me p rho as a function of p, I can compute what this function is here in principle.

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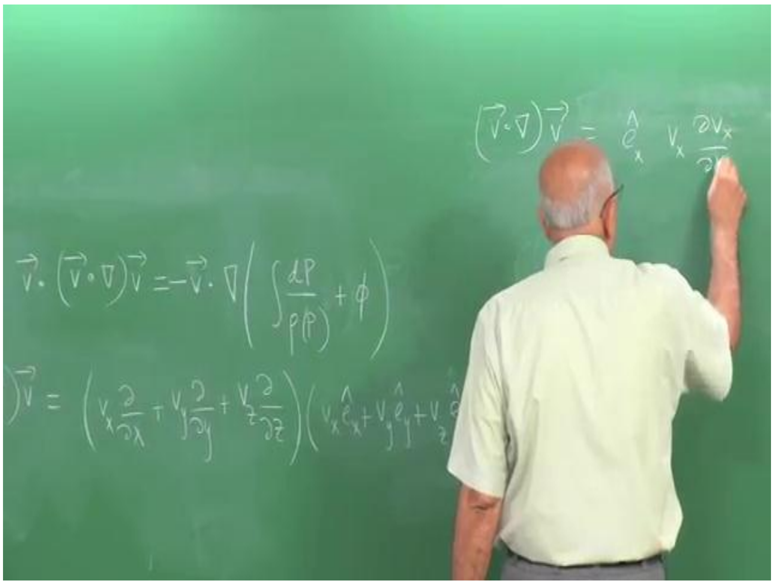

Therefore, under these assumptions of steady flow, inviscid flow, Barotropic flow, we finally have statement that v dot del on v on the right hand side becomes equal to minus the gradient of integral d p over rho of p plus phi and that is it. Next, let us take the dot product of both sides with respect to v, in therefore you have v dot this quantity equal to minus v dot the gradient of this scalier quantity. But, look at, what the left hand side is and lets write it out explosively I do not want to use any vector identity we can simplify this directly.

Just a little bit of algebra, but it is instructive to see, what happens this quantity is a scalar operator, because is a dot product here, it involves derivatives and it axe on a vector to produce a vector and you take the scalar product of that vector with this v. So, this thing here should be simplified as follows, lets write out v dot del on v is equal to v x delta over delta x plus v y delta over delta y plus v z delta over delta z. Acts on ex on velocity, which is v x e x plus v y e y plus v z e z in this form and there are nine terms in this expansion, but there finally, easy to write down.

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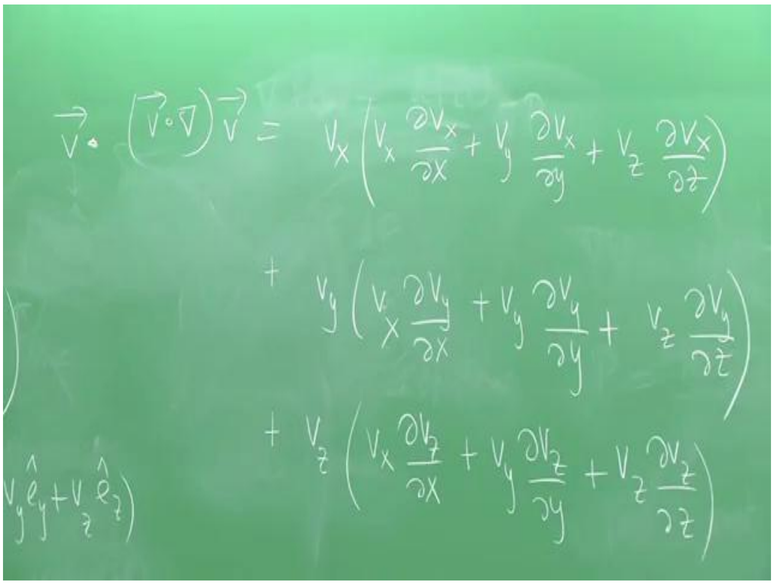

So, let us write the first one down this is going to be v dot dell on v equal to. So, pull the ex out all the way to the left and you have v x delta v x over delta x plus v y delta v x over delta y plus v z delta v x over delta z that takes care of differentiating this term here this three derivatives acting on this term, the next term will be a v y outside and in e y. So, we have a v y plus e y times this term a when this e for done this write, so you have a v x v y v z and then, I am going to take to this offer and act on that.

So, this is going to be a v x delta v y over delta x delta v y over delta we got do this properly. So, I have v x, what have by done? I differentiate it v x each time plus v y the term is quit correct. So, I have a v x delta v y over delta x plus v y delta v y over delta y plus v z delta v y over delta z, so that is perfectly correct plus z times, so we got this term here and it is a v x acting on this fellow here, so its v x delta v z over delta x plus v y delta v z over delta y plus v z delta v z over delta z, so those are the nine terms here.

And now, add them of in this form here this plus this plus this etc, what do we get, does not the best way to add it up. Now, let us dot this with v dot, so that is it this is going to be v x up course v x ex plus v y e y plus v z e z and a use the fact that ex dot e x is 1 e x dot e y is 0 ex dot e z is 0 etc. So, doting this is going to be give me ex dot e x is one and its multiplied by a v x out here. Then, the e x dot e y is 0, e x dot e z is 0 the v y part of it is just going to produce a v y here and the v z part of it is just going to produce a v z here that is it and we are in good shape.

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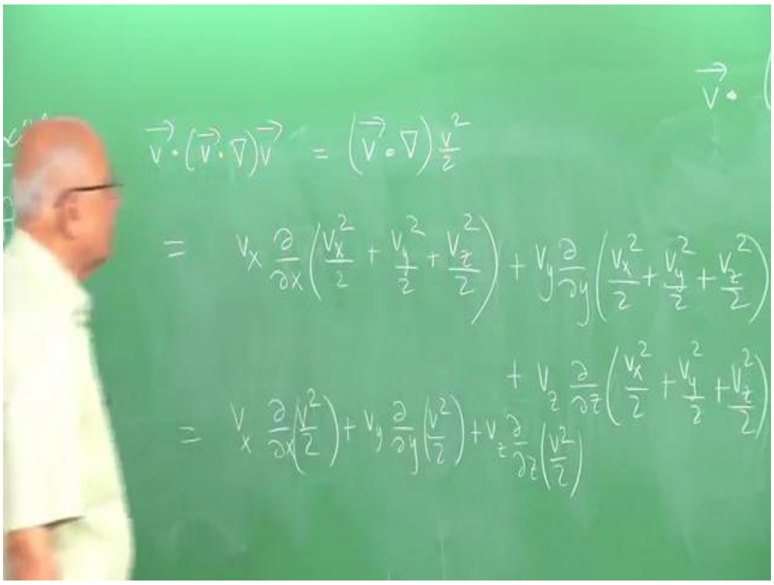

So, we now have a formula with says v dot v dot del on v is equal to lets collect terms and a collect this term this term and this term first and note is that v x is a common factor. So, this is v x times delta over delta x v x square over 2 plus v y square over 2 plus v z square over 2. Because, v y delta v y over delta x is half the derivative of v y square and similarly half the derivative above v z square.

So, its v x times this plus v y times delta over delta y exactly the same quantity v x square over 2 plus v y square over 2 plus v z square over 2 plus v z delta over delta z v x square over 2 v y square over 2 v z square over 2, which is up course equal to v x delta over delta x v square over 2, because that is the definition the square was the magnitude of the velocity plus v y delta over delta y v square over 2 plus v z delta over delta z v square over 2. But, this looks like v dot the gradient of half v square. So, we finally, have an identity, which says this quantity is equal to v dot del acting on v square over 2 this is the scalar quantity scalar operator acts on 2, which is, what this quantity is here.

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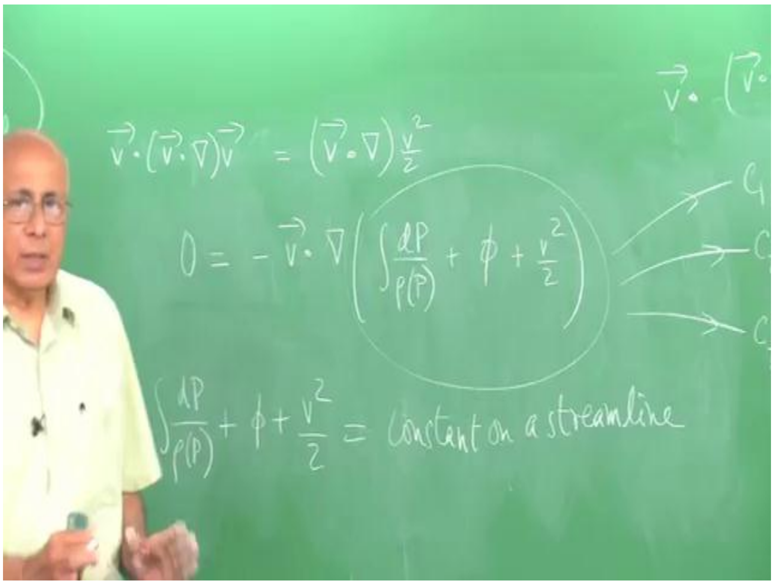

Putting this back in are equation we, now have a statement with says that 0 equal to minus v dot the gradient of integral d p over rho of p plus phi, whatever be the potential per unit mass of this fluid plus v square over 2. I move the thing this side from the left hand side to the right hand side and took the v dot outside, so its v dot gradient of this. Now, what is the meaning of this statement here, it says that this scalar quantity its gradient has no component along the vector v at any point.

But, the vector v at any point its direction is actually the stream lines direction. So, it says, that if you have a stream line like this at this point this is the direction of the velocity v and this statement says that this scalar function has no component its gradient has no component along this direction at all. Therefore, it cannot change along this direction this scalar function cannot change along this direction if you recall the gradient of a scalar gives me the rate of change of this scalar function along any direction, what, so ever one side dot it with a unit vector of the direction.

In this case the velocity v is tangent to the stream lines and it says that at every point this scalar function does not have any change assume move along this tangent. In other words along a stream line as you go bit by bit increment by increment this scalar quantity does not change. So, this is Bernoulli’s principle it says that integral d p over rho of p plus phi plus v square over 2 equal to constant on extreme line this is Bernoulli’s principle in a general form it does in say it is the same constant on different stream lines.

So, if a fluid flows like this and this is of the other stream line this are the other stream lines and that is the portrait of all the stream lines this quantity is some constant c 1 on this some other constant c 2 on this some other constant c 3 on this and so on. In general this is, what happens, but otherwise on each given stream line we as said that this quantity is a constant. What are the assumption, that went in to it we set that the flows in this it this no viscosity no dissipative forces be set that the force external force on it is conservative comes from a potential is a gradient of the potential the flow is barotropic.

So, write you have density as the function of the pressure of the fluid and the flow is steady, so delta v over delta t is 0. Under those conditions this quantity is constant on a stream line, now we can immediately apply this to the case of an incompressible fluid there of course the rho comes out.

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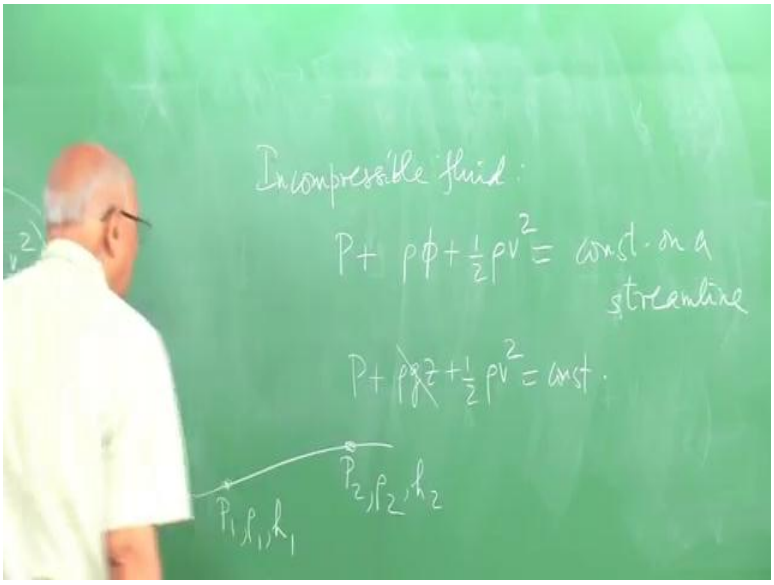

And, if the fluid is incompressible this will immediately in phi p plus rho times phi plus 1 half rho is square equal to constant on a stream line rho just a constant and in take role of d p gives you p and that is it this is the form in which you usually familiar with this. In fact if this comes from gravity and you have a flow, which is under gravity for instance, then this is written as rho g z in general, so p plus rho g z plus half rho v square equal to constant this the form, in which you usually learn an elementary treatments, what the Bernoulli's principles.

If it is a gas for instance, then the density is extremely small and in general it terms out that the effect of this pressure difference this gravity a potential difference is very, very negligible. And therefore, if you neglect this, then it says p plus half rho v square is constant, which is the form in, which we see, then its most elementary form here. Incidentally, if this is a stream line and you have a pressure p 1 and density rho 1 here and a pressure p 2 and a density rho 2 here and a height h 2 and this is a some height h 1 how of the ground level.

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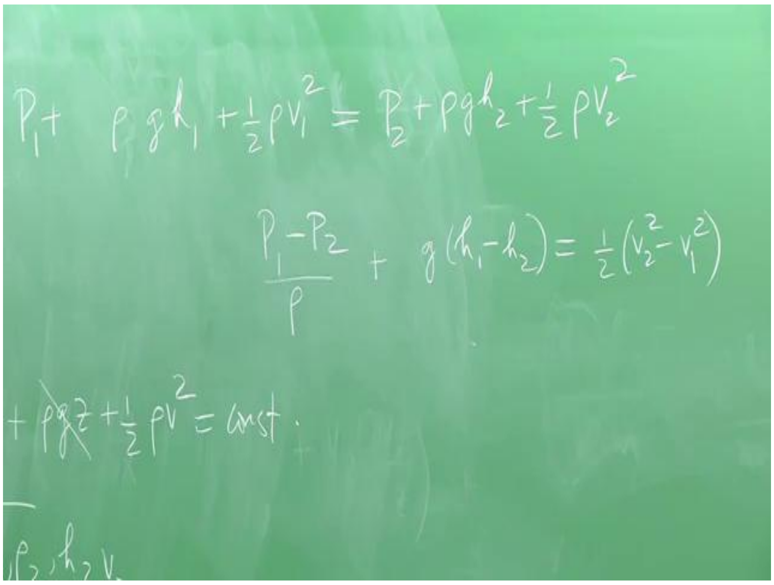

Then, this equation here tells as that p 1 plus rho 1 a rho g h 1 plus 1 half rho v 1 square equal to p 2 plus rho g h 2 plus 1 half rho v 2 square. The velocity is v 1 here there this velocity speed is v 1 here in v 2 there tells you this. So, it actually says the p 1 minus p 2 over rho let us take rho plus g times h 1 minus h 2 equal to 1 half v 2 square minus v 1 square. So, the change in the kinetic energy if you like is equal to the work done on this fluid by this pressure had on the gravitational potential.

So, the force of gravity and the pressure it together to do work and change the kinetic energy of this fluid and its written in this form. So, this is the form, in which Bernoulli's principle is used very, very half. But, we have a much more general relationship, which actually says that when you have barotropic flow of the steady flow of the bar tropic fluid under a conservative external force in the absence of viscosity it is this quantity that is constant along any given stream line here.

Next are like to introduce a slightly more shuttle concept in fluid flow and that has do with circulation this is not generally talk at in elementary level, but I want to do this, because a game we want it seen that fluid flow as offer does a very natural way of understanding set an operations like the gradient of a scalar this the flux of a vector field, which means the divergences of a vector field the flux per unit volume. There is a one more concept called the circulation of a vector field.

And since, it appears, so half in various applications of vector analysis such as a Maxwell equations in electro magnetism its worth doing it here and I intent do that. Next we will discuss in very elementary terms the concept of the circulation of a vector of a fluid and, what is meant by the curl of the vector should of the velocity, because it has physical implications.