Mechanics, Heat, Oscillations and Waves
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Lecture - 34
Fluid Dynamics
Equation of Continuity

Having looked at the condition for hydro static equilibrium of a fluid, let us move on now to fluid motion itself. Now, have in mind the motion of a fluid in stream lines, you already familiar with the idea of stream lines, namely if some kind of smooth study flow and I will explain these terms carefully as we go along. But, it is flow, which is not chaotic or turbulent as suppose to smooth study flow, which is not chaotic or turbulent or like a waterfall or anything like that.

But, a flow which is kind of regular as you can put a little piece of dies somewhere and you know, it moves in some kind of smooth curve. So, we have a flow of that kind in mind. Now, the first concept we have to understand is the idea of a current density.

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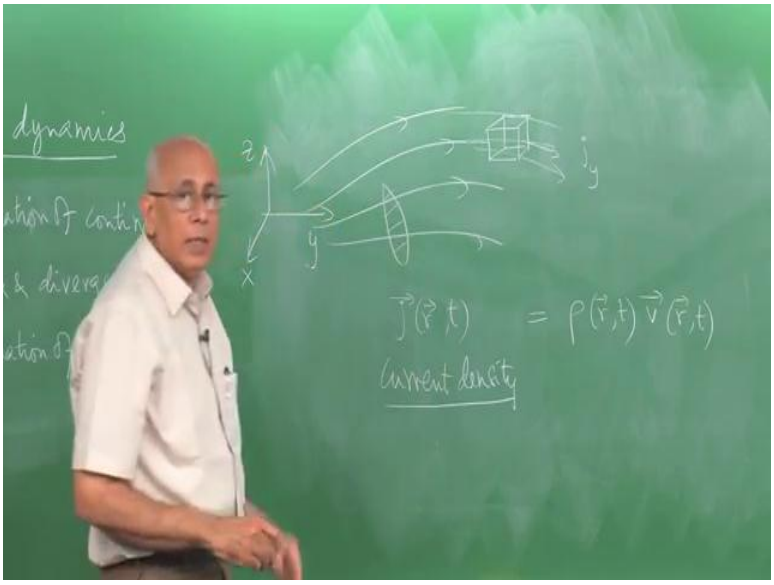

In other words, if these are the flow lines of a fluid, if you put a unit area here an area element, we could ask, what is the mass of fluid crossing this area per unit time, per unit area per unit time, this mass is called the current density j as a function of r and t. It is the mass of, at the point r at time t; it is the mass of fluid crossing unit area per unit time. And of course, it is caught a direction j and that is the direction of a local fluid velocity.

So, it is not hard to see, that this is in fact equal to rho, which could be a function of r and t general, the density multiplied by the fluid velocity r and t. So, by definition the current density j is the product of rho and v here. Now, we could ask, when this fluid is in motion assuming that, there are no points in the fluid, which are either sources of fluid coming out or sinks into which the fluid disappears.

So, without any sources or sinks, we could ask then, what is going to happen to a little volume element d v at some random point, at some point here. There is some fluid, which is going in and there are some fluid, which is going out through certain phases of this volume element. So, we could ask, what is known as the net flux of the fluid through this volume element, which would corresponds to adding up all that fluids, which is going in, ignore all the fluid which is going out and subtracting the formal from the latter gives you the outward flux of the fluid from that point here.

And let us find out, what this quantity is, now clearly if I use a volume element of this kind which is parallel, whose sides are parallel to the coordinate axis, then the contribution of the velocity vector pointing this direction on that phase would be the component of this velocity vector normal to this phase. So, that would be in this case, if I choose my coordinate system in this fashion as usual x, y, z, this phase is normal to the y axis. So, the current component j y is what is going to play a role here at that point.

Similarly, across the two other phases, it is going to be j x and j z. Now, what is coming in here is j y at an earlier point d y over 2 to the left of x, y, z and here it is d y over 2 to the right of x, y, z. So, exactly the same way as we found the flow of force on a fluid element was minus the gradient of the pressure.

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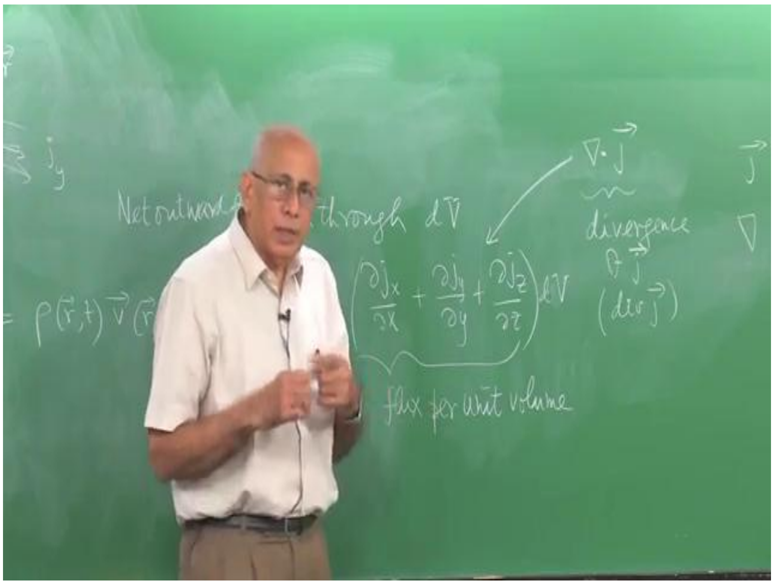

In this case, the same argument, exactly the same argument, but using this vector j will tell you that the net flux, the net amount of fluid that is flowing out through this volume element. Net fluid flux through the volume element d v is equal to delta j x over delta x; ((Refer Time: 04:38)) that is the portion contribution out phases normal to the x axis plus delta j y over delta y plus delta j z over delta z multiplied by d x, d y, d z which is d v.

So, net I should write this more carefully. So, let us write this as net outward fluid flux, it is equal to this quantity here. This is the mass of fluid that is flowing out through the l volume element d v through all the phases of it, put together. Whatever goes in with a negative sin, whatever goes out with a positive sin will automatically give you this quantity here.

By the same argument, exactly the same argument that we use in finding that the gradient of the pressure gave you minus the gradient of the pressure gave you the net force on the fluid on the volume element. In exactly the same way, you end up with this combination here d v. Therefore, the flux per unit volume is just this quantity in brackets, this is the flux per unit volume at the point x, y, z.

((Refer Time: 06:08)) So, at any point in the fluid at the point r, you have to find these derivatives and remember, these derivatives would occur, because each component of this is the function of r and t. They would involve derivatives of rho and v in general, but that is the quantity that appears in the flux per unit volume. This quantity has gotten name, this thing here.

Notice that, you could also write this as del dot j, the reason you write in this form is because j x, because j we call is e x, j x plus e y, j y plus e z, j z and the del was an operator which look like this. So, you can write this as a dot product of this vector operator del with the current density j. This quantity is called the divergence of j and it sometimes written as the divergence of j inwards.

I emphasis again that this idea of the flux per volume at a given point is applicable to any vectors field at all, it does not have to be the current, any vectors fields such as the electric feel or a magnetic feel and so on. The flux per unit volume by calculating the influx surround in infinite decimal volume element at any point is defined as the divergence of that vector field at the point. And then Cartesian coordinates, it is written in this form or in this form.

So, this is not the definition of the divergence, it is an algorithm, if you like for finding this quantity here is the algorithm for finding that the divergence. So, the divergence itself is defined as the flux per unit for volume of a vector field at any point here. And what we found here is the flux of the current density, which in the fluid case happens to be rho times b. But, what is this quantity equal to ((Refer Time: 08:42)) what is this going to be equal to and that is interesting to ask.

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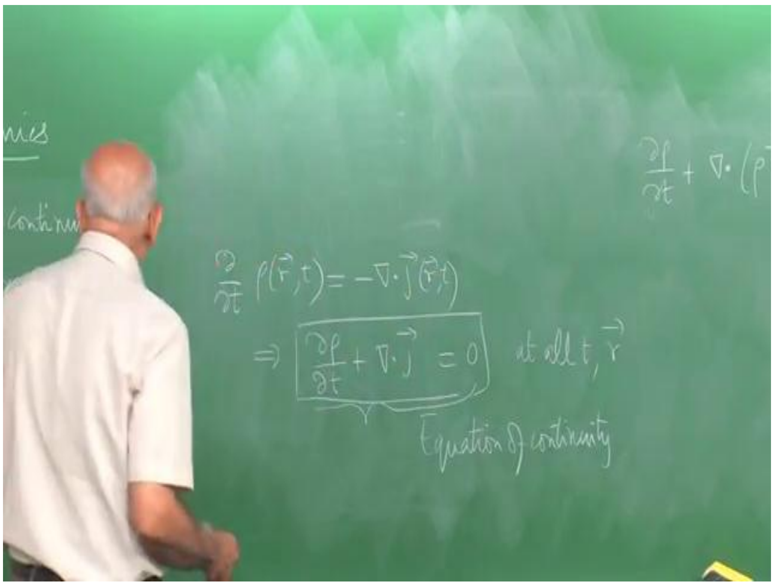

Because, it is obvious that in the absence of sources and sinks, you must have the rate of change d over d t delta over delta t, rho of r comma t, this is going to be the rate of change of unit of the mass of unit volume of the fluid at the point r at any given instance of time t, it could increase, it could decrease etcetera. And clearly, if the system has the flux going in and nothing coming out, the density is going to increase, this is going to be positive.

If on the other hand, the flux is mostly outwards, then more fluid going out, then going in, then this quantity is going to decrease at that point. In any case, this is what is going to balance minus del dot j of r comma t, because this with the minus sin is the invert flux per unit volume and this is the rate of change per unit volume, rate of change the mass from unit the volume at the point r at time t.

So, this is got to be, if there is no source are sinking here, this is got to be given by whatever flow the flux of the fluid here, the net invert flux of the flow, which is minus del dot j. So, together, this implies the delta rho over delta t plus del dot j equal to 0 at all times at all points in the absence of sources and sinks. This equation, which is essentially conservation of matter or conservation of mass here is called the equation of continuity.

And it is a crucial input, because notice, that what the equation of continuity telling us is that, there is the constraint on the way rho and j behave, remember rho j times v. So, delta over delta t plus del dot rho times v equal to 0 is connecting rho and v. Means, these two quantities, these two fields are not independent of each other. We call what I said earlier namely the task of fluid dynamics in some sense of hydro dynamics is to determine, what rho and v, r, this scalar field and the vector field r, given the pressure and the external force and so on and so forth.

So, this is the task of fluid dynamics, but you already have independent of what forces you apply in the fluid, you already have the constraint equation in the equation of continuity. So, we must keep that in mind here, before we go on to finding out, what the equation of motion actually is. The next step is the following, before I do that, let us point out the following simple case.

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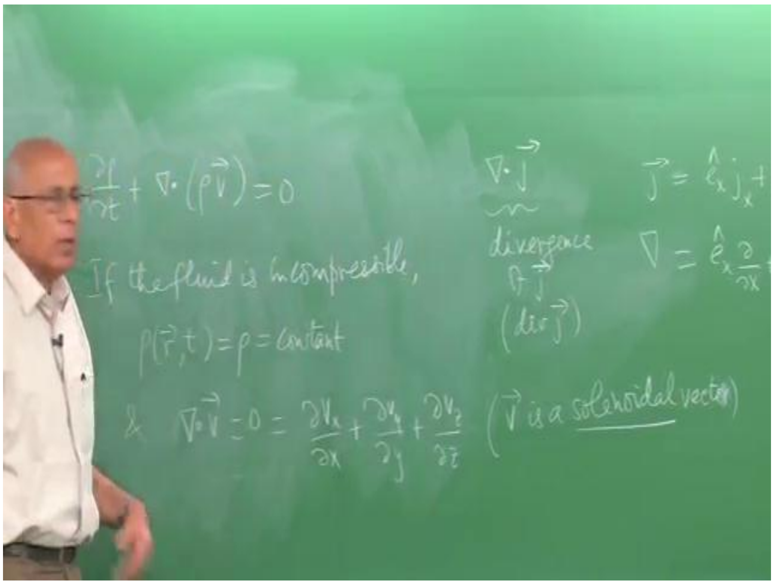

If the fluid is incompressible, which is not a bad assumption in many cases, even in the case of gases as long as you are dealing with very high temperatures. You are not dealing with very voile and fluctuations in the density due to different temperatures or going near the speed of sound in the medium in the fluids, so that you have shock waves and so on. As long as you are dealing with such velocities or such conditions, then for velocity is much lower than the speed of sound in the fluid essentially the fluid is incompressible, even a gas is essentially incompressible, when it is flow.

If that happens, if fluid is incompressible then rho of r t equal to rho equal to a constant in space and time, which immediately implies that this constant comes out of the derivative operators, this del involves basically differentiation. It comes out and this quantity is of course 0 and as a consequence del dot v 0. So, this says that, wherever you are in the fluid, this must be 0, such a flow is set to be solenoidal, these set to be solenoidal vector.

For reason, we would not going to that is the technical term, but this simplifies matters enormously, because it is says the equation of continuity, does not involved the density at all, it just says the del dot v, the divergence of v must be 0 at all points. So, it puts kinds of condition on the velocity v here in the very important case of incompressible flow.

Next, we now turn to real business which is to write the equation of motion f of fluid elements. So, we need to start by writing Newton's equation for a fluid element and that is now straight forwardly done.

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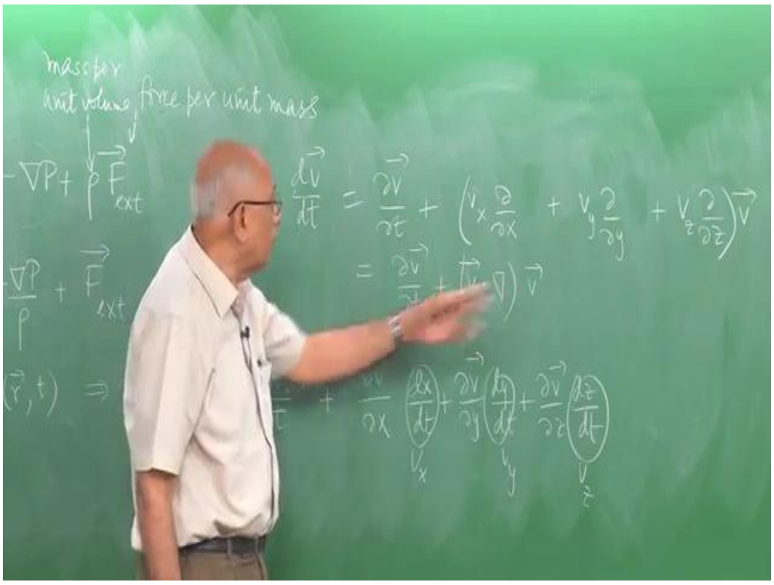

Because, we already know that the force on this fluid element per unit volume is minus gradient of p plus rho times F external, if such force is exist two times F external, this remember is the force per unit mass. And this is the density, which is the mass per unit volume. So, this term is the external force per unit mass and per unit volume. So, the force per unit mass multiplied by the mass per unit volume, this is the force per unit volume; that is already the force per unit volume.

This must be equal to the mass per unit volume, which is rho times the actual ratio d v over d t. So, that is the equation of motion of a fluid element at any point r at any time t, given this quantity and this quantity, you should able to determine this quantity. Assuming that we know already, but remember rho and v are connected by the equation of continuity.

So, if you like in some sense this equation gives you that dynamics, it is the analog of Newton second law, it is Newton second law and this gives you the conservation of mass. So, together these equations are supposed to determine rho and v, one can divide by rho and write this immediately as d v over d t minus radiant of p divided by rho plus F external, where remember F external is a force per unit mass on the fluid here.

So, this is the basic equation that we have to deal with in fluid motion, where knowledge of this should be sufficient and this should be sufficient determine this v. But, this is easier set that done, it is a very, very non-trivial task, you will see for complications that arise a way as you see in the moment. First of all, this is valid even rho, when rho is not constant, even when the fluid is mot, it is compressible; it is compressible say and changes with sometime, this equation still valid.

The next step is to ask what is this time derivative, here is acceleration here, remember that v is a function of space and time. So, if you followed a fluid element, which happens to occupy the point at time r at time t, then the rate of change of this v with t, the total time derivative will have two components. It implies that d over d t, v of t must be equal to 1 v of r and t, must be equal to one part of the time derivative time change.

Arises because, there is an explicit depends on time of the flow of the velocity vector at any given point. So, here is a point, this is the velocity vector one instant of time, little later it is here, little later it is here and so on, due to this inexplicit dependence on time here. There is a change with of this v with respect to time and that is of course, equal to delta v of r delta t.

But, that is not end of the story plus there is a contribution which arises, because this change is the fluid element moves as a function of time and therefore, there is a change of it is coordinate as a function of time and this of course, is x, y and z. So, there is portion which comes due to delta v over x times d x over d t plus delta v over delta y, d y over d t plus delta v over delta z e z.

So, the fact that remember now, we are tracking now the unit element, we tracking now the force per unit volume of a fluid element located at the point r at time t and the fluid is moving, perhaps in unsteady way. So, the velocity filed is self changes explicitly with time and it changes implicitly, because the fluid is moving, because this r as time dependents and the velocity at that point, the rate of change this r is precisely what we called v.

So, this quantity here can be rewritten in the following way d v over d t forget about the arguments here, let us forget about the r and the arguments here it is delta v over delta d t plus. Now, what is this, well as you can see delta v over delta x, delta v over delta y, delta v over delta z arises by taking that del operator, the gradient operator and acting on the vector v here. So, we let us write this out, let us write this out explicitly.

So, you have v x delta over delta x, v plus v y delta over delta y, v plus v z delta over delta z, this is v x, this is v y and this is v z. So, we have the total derivative of the velocity v is given by the partial derivative of it is explicit time dependents plus v x times this, v y times this plus v z times that. But, you see these quantities delta over delta x, delta over delta y, delta over delta z arise from the components of the gradient operator, the del the triangle operator.

So, this can be written as delta v over delta t plus v dot del, where this v dot del stands for the following quantity. It is stands for this combination of quantities acting on v, the partial derivatives act on v, so that is by definition v dot del here on v.

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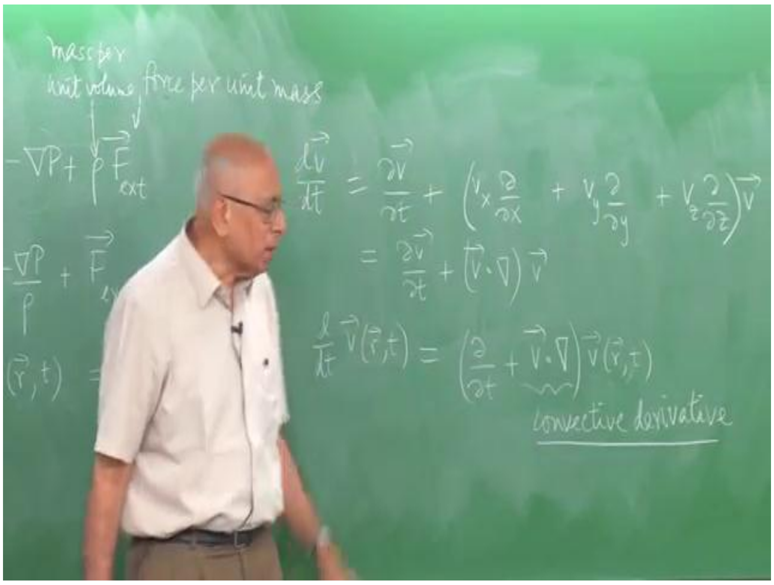

So, we have this beautiful formula which says that d over d t of this function of r and t is in fact equal to delta over delta t plus v dot del acting on v of r. So, the total derivative is the partial derivative with respect to time plus v dot del, this extra term here appears in the total derivative acting on whatever the velocity vector is out there. This quantity here is like a derivative due to the fact that fluid is being dragged along which be moved along and it is called the conductive derivative, it arises due to the motion of the fluid.

So, notice that v appears in two places, we started by writing and equation of motion for the fluid elements to this mass time acceleration here. So, we wrote down everything for unit mass of the fluid and we discover mass time acceleration gave you d v over d t, but it was a total derivative, but in the case of fluid, it becomes the partial derivative plus the conductive derivative.

This is crucial this extra thing here make the whole thing non-linear, because v appear is now in two different places in a product form. So, this immediately means that fluid dynamics is inherently non-linear. We started with Newton's equations, which look a very simple equation, but it inherently for the velocity field gave us this little piece of complication here is which it responsible ultimately for much of the complication in fluids not all of it, but much of it as we see.

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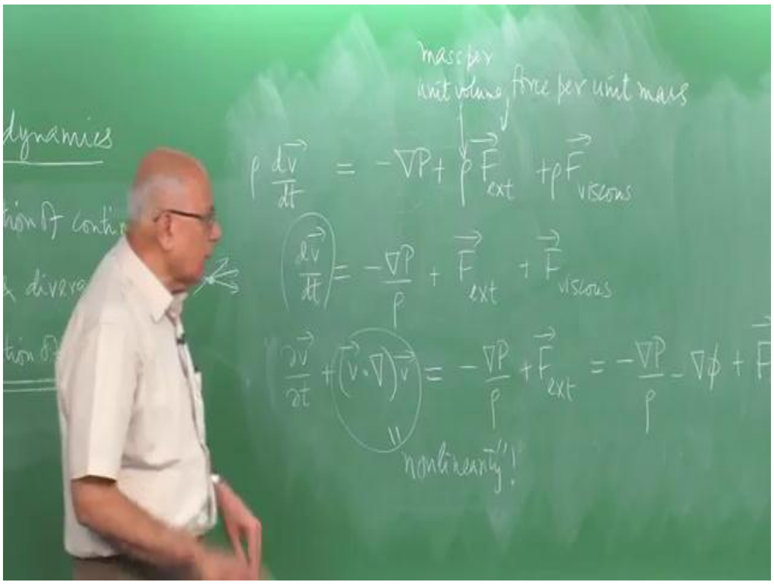

So, now, where are we, let us write that down, so the fluid elements the equation now has become delta v over delta d t plus v dot on v equal to minus delta p over rho plus F e x t. now if it should so turn out, this of external is a conservative force, such as gravity. Namely, it is obtained by from the derivative of a potential, then we already know that for a conservative force, let us write that here.

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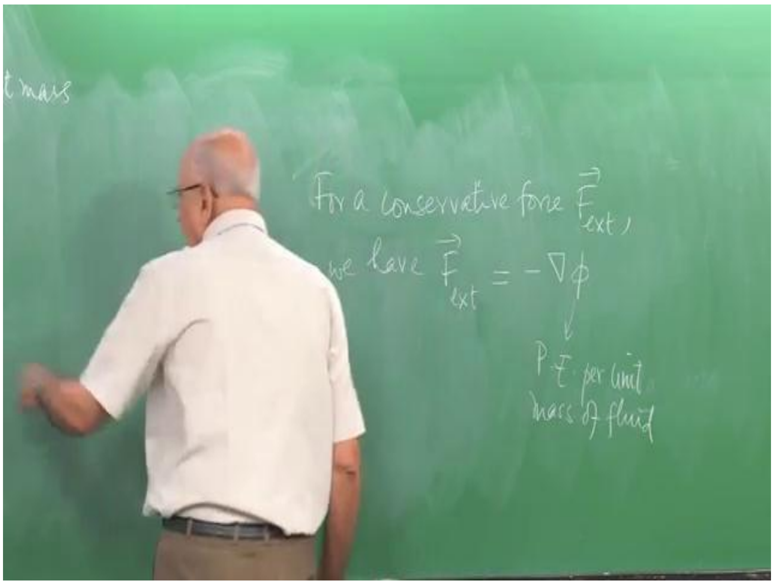

For a conservative force F external, we have F external equal to minus the gradient of some scalar potential, which has now the physical meaning of the potential energy per unit mass, because this F is the force per unit mass. So, this is must be a form gradient of some phi, such as gravitational potential rho g h, rho g z would be the gravitational potential. Then, would have a force in the z direction, which is minus g times e z.

So, we have the external force, which is derive from a potential and this is the potential energy per unit mass of fluid and if you put that in, this becomes equal to in that case minus gradient p over rho minus the gradient of phi. So, this is the equation of motion of a fluid element, but you might now say, I put a conservative force, but the fluid also has the viscosity and the innovative that to account.

And viscosity not a conservative force, it is a dissipative force, because ultimately it is represent some loss or dissipation a friction inside the medium yes indeed that term two exist. And in general, ((Refer Time: 26:22)) this should really the added. So, should add a rho times F viscous, which is the viscous drag per unit mass of the fluid. In which case you get here F viscous and you would continue to get here viscous.

So, it is getting fairly complicated, you can see that already there is a complication due to this quantity here, which I will call non-linearity. It is a quadratic non-linearity, there are two powers of v here, this is a v here and a v here. You would have expected linear equation, because we would set well mass times acceleration, acceleration is rate of change of time derivative velocity, but it is now appearing in a implicit form as well with the v dot del on v here. This complicates matters as we already said, but this is the equation, basic equation of motion here.

Solving this equation in any really complicated situation is really, really tough, but we will see various simple cases of it. But, I thought I should bring this show you how the very basic application of the basic principles, namely just the equation of motion will give you, automatically lead you. Using the fact that, we are dealing with continuous medium, automatically lead you to this fairly complicated equation. This is called the Euler equation.

Actually, I thing that, Euler's equation is the one that get without viscous term, but it differs on what the actual terminologies. But it is equation of motion of a fluid element and like to leave it like that. Our next task is the following, I am going to drop this as viscous, there are models for which I might mention it the end, but go to the drop this thing here and assume that, you have just the conservative force here plus this special gradient.

And then, we need to have some knowledge of how the presser changes function of the density and we will look at the special cases. In particular, if you had an incompressible fluid, then there is no problem, think simplify enormously, no viscosity, this is a constant and then, one can actually try to make head way and see, while this gets us. In particular way we like to integrate this equation once to see in some analog of something is looks like the energy in the case of particle motion. We found the most integrate motion was the energy can we do the same thing here is what we will look at next.