Mechanics, Heat, Oscillations and Waves

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Lecture – 33
Fluid Dynamics
Hydrostatic Equilibrium

We looked at a little bit of the kinematics and dynamics of system of particles of individual particles in an external force and a few aspects of what happens when particles are interaction with each other and what the dynamic equations or equations of motion look like. We also went on and looked at a case where you had a continuous medium, such as a string and a tension and we saw what it is equation of motion would look like and in general it was the wave equation that we obtained for the displacement transfers vibrations of a string.

We also saw very briefly what happens, when you have a three dimensional medium such as air and you set up pressure waves in it which magnifies themselves as sound. And we saw a few aspects of what happens there and wrote down the corresponding wave equation. In the same logical progression, we now like to go on and study the dynamics of a continuous medium such as a fluid, when it is in motion. So, for the next few modules we will look at fluid dynamics, namely the most elementary aspect of a fluid in motion as a continuous medium.

Now, because it is a continuous medium we have certain very special features which arise in the motion of this system of such a system, which I will try to emphasize here. I am saying right away that we are not going to look at fluid dynamics at the atomic or molecular level. Because, we know ultimately the fluid is composed of individual molecules which move around in close interaction with each other. We are not going to get to that level of description, we are going to look at it at a more coarse grained or rougher level by we will talk about infinite decimal fluid elements as if they were all part of a continuous medium.

So, we will not go to the atomic level, but we looked at it effectively at a much more coarse level which is good enough for understanding fluid dynamics as it magnifies itself in normal life. Now, there are several aspects to this and the something that I want to say right in the beginning which is that fluid dynamics offers a very natural and physically intuitive way of understanding many concepts of vector calculus. And in fact, vector calculus to some extent was derived or invented if you like with the specific purpose of applying it to a fluid dynamics.

So, this gives us an opportunity to understand certain concepts in vector calculus such as the gradient and the divergence and the curl and so on which really are part of a formal mathematical structure called vector calculus. But, the physical interpretations of these objects of these various quantities is so simple and so straight forward, that is it actually worth understanding it in the context of fluid dynamics, which is what we are going to do.

As always, this is going to be pitched at a slightly higher level than is generally in the case at the high school level or even in the beginning undergraduate level. But, I hope that we will make things very clear in a physically intuitive way, so that there should be no difficult in understanding it as long as one understands the elementary operations of the calculus, so let start by asking how one would describe the continuous medium such as a fluid.

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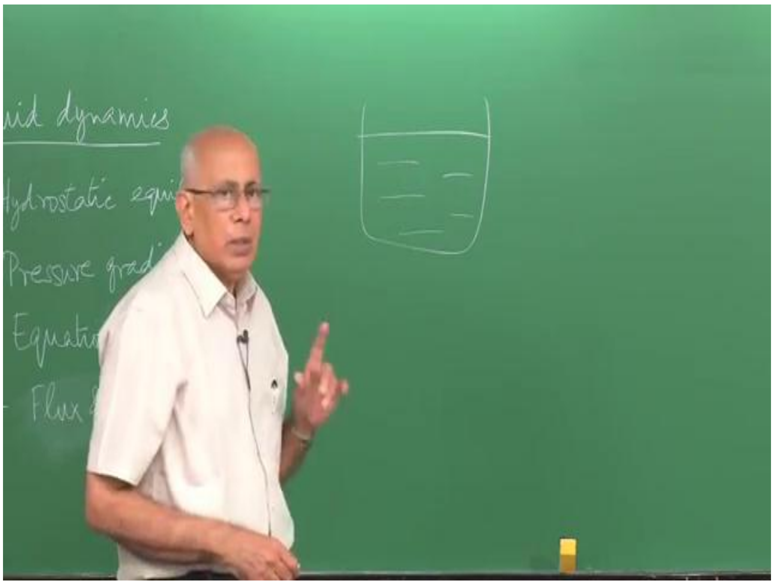

So, you have a fluid in some container perhaps and I would like to understand what happens to this fluid or maybe it is in motion in a pipe of some kind and we would like to understand what the flow is like, what the equations of motion are like etcetera. Now, there are two ways in which you can approach a fluid dynamical problem. One of them is the Lagrangian method, where imagine that you have a medium.

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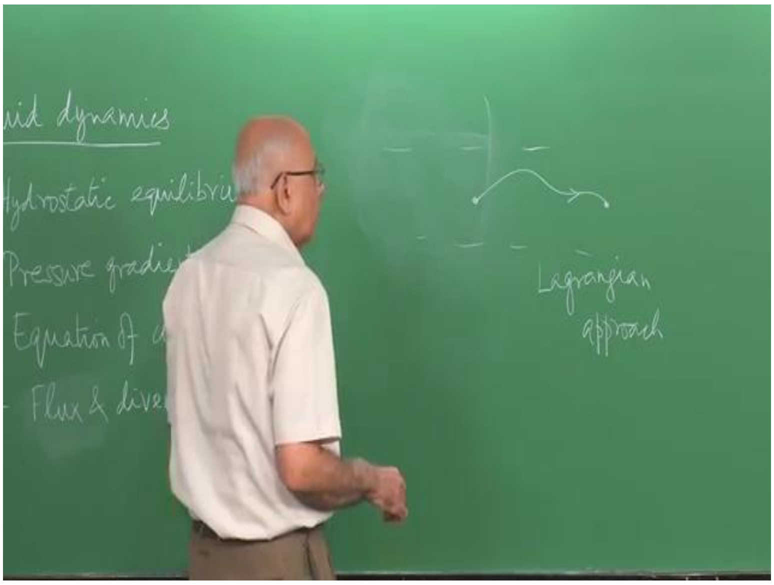

So, there is fluid everywhere and you focus on a particular fluid element at some instant of time and then, follow the path of this fluid element as a function of time. And you really have to do this for all the fluid elements in the fluid allowing for various constrains such as the fact that two fluid elements ((Refer Time: 04:32)) on the top of each other etcetera and this approach is called the Lagrangian approach, where you focus on any given individual fluid element and what it is motion as a function of time through the fluid. On the other hand, there is another approach to it which is the following.

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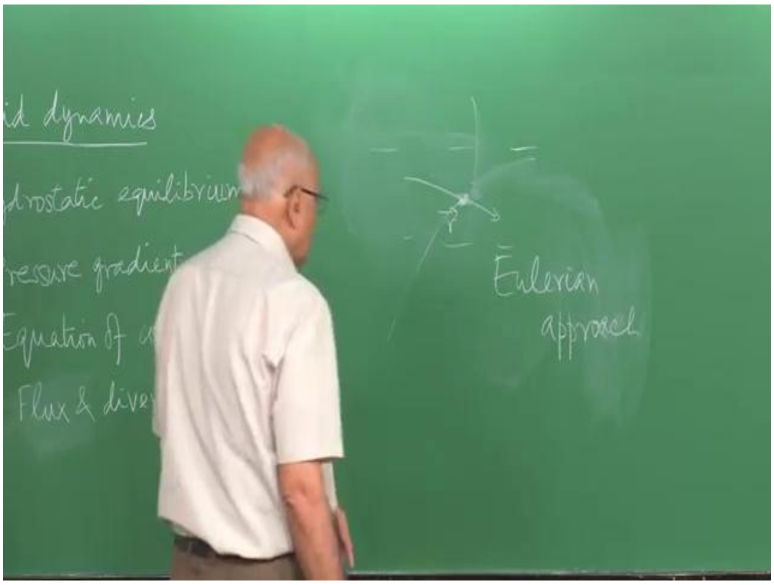

We focus on a particular location, so I focus on a location whose coordinate is r and look at what sort of flow you have at that point at any instant of time. So, it is possible that at some instant of time this, the element the velocity vector at this point points like this. And of course, it should be part of some continuous curve and at a later instant of time it is possible that the flow changes in such a way that this velocity vector points like this and so on.

So, the idea is to focus on a location rather than in individual fluid element and the two approaches are related to each other. The second approach is called the Eulerian approach and that is the one in which it is easiest to write things down specifically, especially at an elementary level and therefore, we will look at the Eulerian approach. Then, what are the variables that would describe and the element of fluid at this location at any instant of time?

Well, in the case of particles you talk about the position of the particle in all three Cartesian coordinates for instance, plus the momentum of the particle or the velocity of the particle at any given instant of time and you do this for each particle in the system.

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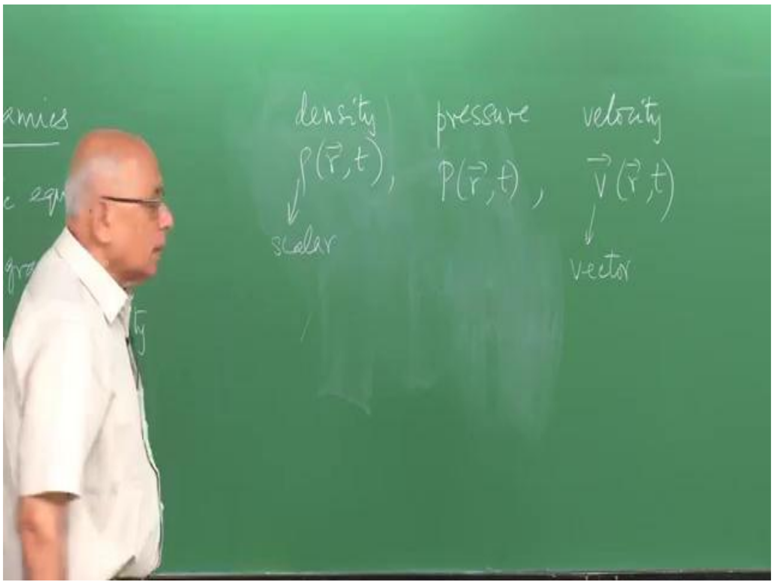

Here, what we do is to associate with any point in the fluid at density rho which could be a function of time, because the fluid could be compressible and the fluid could be in homogenous, in other words they could be different densities at different points in general. So, the density the pressure of the fluid at that point P of r and t and the velocity at that point and I will use this symbol v as a function of r and t. I emphasize once again that by this v I do not mean the velocity of any individual fluid element in particular.

I mean the velocity field of the fluid at the point r at a time given t. So, at different times different parcels or different elements of fluid would occupy the location r and I am taking about the velocity of anything that occupies the location r at time t. So, these are the dynamical variables in the system to describe a fluid and what we need to know is to determine, what we need to find is to find all these quantities given certain inputs which would be like conservation of matter, conservation of momentum, the equation of motion of a fluid element and so on.

So, we will do that systematically, but these are the unknowns in fact, in general one is told what the pressure gradient is or what the pressure is, if you given this field then the task is to determine this and that. So, that is the general problem of fluid dynamics in the Euler approach, namely given the external forces on the fluid, the pressure of the fluid, the pressure heads and so on determine the local density and the local velocity, you need a sufficient number of equations to do this.

Now, this is a scalar quantity that is important to remember and that is a vector quantity of course. So, the fluid state is determined by a space and time dependent scalar field rho and the space and time dependent vector field the velocity as a function of space and time coordinates. So, that is the general problem that we are going to tackle or at least formulate in solving, in trying to understand fluid dynamics, but let us first look at a much simpler case namely that of equilibrium, hydrostatic equilibrium.

So, I have in mind of fluid in some container which is completely at rest, nothing is moving, every element is at rest and I would like to know what is the condition for this hydrostatic equilibrium. You know that the condition for the equilibrium in mechanics for a single particle would be to say or a body, given body would be to say the net force on this body must be 0 and if it is initially at rest, then by the first law it continuous to be at rest.

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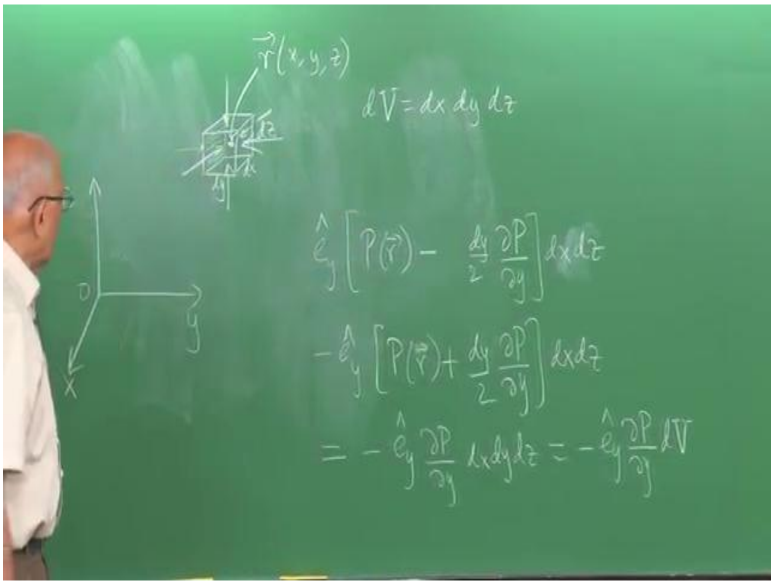

A similar constrain condition of hydrostatic equilibrium is what we see and for this purpose let set up a coordinate system somewhere completely arbitrary where you choose it. And let look at some given fluid element at the position vector r and let us take this element to be at this position r which is got Cartesian coordinates x, y and z and ask in this case what is the condition for this element to be a trust, well there is a pressure acting on this fluid. So, there is a pressure acting this direction, like this from all directions, there is a pressure acting on it.

So, from the front and from the real this motion and let us look at the y component of this presser, the pressure at the point r is p of r and that stands for x, y and z. But, if this line element is d y, this is d x and that is d z infinite decimal volume element of volume d v, so let us write d v equal to d x, d y, d z then from the left you have a force pushing in the plus e y direction. So, this is equal to e y from the left you have e y times p of r minus, because this phase is d y over 2 to the left of the point x, y, z. So, this equal to d y over 2 multiplied by the derivative of this pressure with respective y.

So, I do a Taylor expansion of the pressure on this phase about the pressure in a middle of the volume element. And the first term is the pressure at that reference point minus the correction which is proportional to the increment length increment which is d y over 2. Because, it is half here down and then multiplied by the derivative with respect to y of the pressure itself and this must be multiplied by d x d z, because that is the area of this element here, the vertical is d z and the horizontal going in to the board is d x.

And the pressure of course, is the force per unit area you can see that this multiplied by that is a force has physical dimensions of force that is the force appearing from the left. But, that is compensated partially by the force from the right here and that force is minus e y again p of r, but this time with the plus d y over 2, because this front phase is d y over 2 removed in the y direction positive y direction from the centre of the q times again the derivative d x d z and may you add these two the net force in the y direction of this element is equal to minus e y delta p over delta y times d x d y d z which is equal to minus e y delta p over delta y times d v.

So, that is the y component of this force and exactly the same short of argument will tell you that the z and x components are respectively given by the same things with e z delta p over delta z and e x delta p over delta x.

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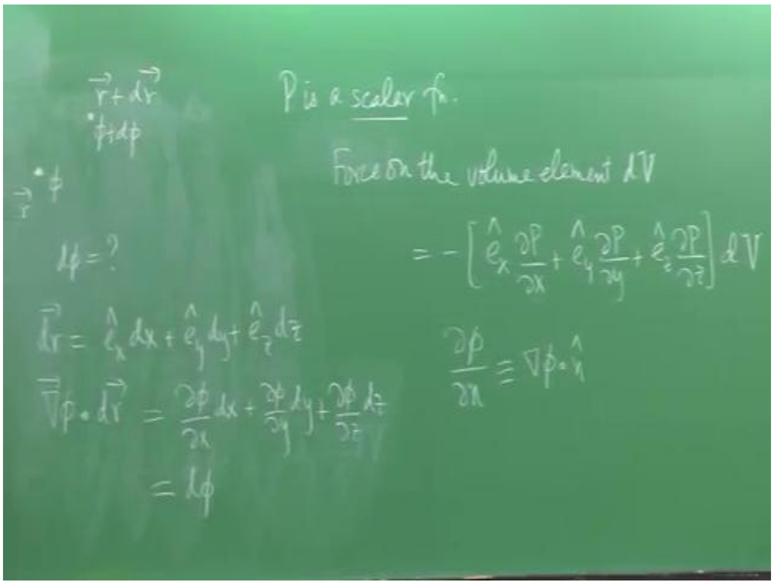

Therefore, the net force the force as say the force on the volume element d v is given by a sum of all these things quantities minus e x delta p over delta x plus e y delta p over delta y plus e z delta p over delta z times d physical dimensions are, because this is force per unit volume and this here is a gradient force per unit area and this gives you one over length once again multiplied by volume element gives you a force physical dimensions of force.

So, this quantity is the force on the volume element d v, now this looks very much like you take the quantity p which is a scalar function. So, remember that p is a scalar it is a scalar function the pressure is a scalar, but it take it is derivative with respect to each of the coordinates on which it depends and you multiplied by the corresponding unit vector in this direction. Now, this quantity has a physical significance which we will see in a second and it is got the name this thing is called the gradient of the pressure p, it is a vector and the idea of gradient this is a natural place to introduce it is. So, let us do that.

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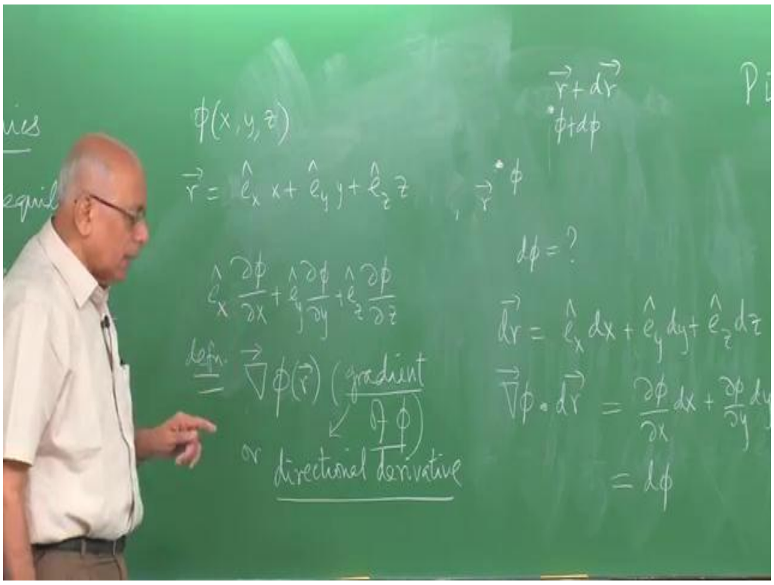

If you give me a scalar function phi of x, y and z any scalar function of the three coordinates then I can construct just as I define the position vector to be e x times x plus e y times y plus e z time z what I can do with this quantity is to construct partial derivative. So, I can find the rate of change of this function here at any point with respect to each of the three independent coordinates x, y and z keep in the other two constants.

So, I move in the x direction by d x and find out how much changes and so on and I get this times d x, this times d y that times d z and I can constructer vector from this quantity which is defined in this fashion and this is by definition. This is the definition of the so called gradient which is denoted by the symbol here the Greek symbol lambda of phi which of course, the function of r, this is called the gradient of phi.

So, the gradient produces of vector function from a scalar function a vector function of the coordinates is produce from a scalar function of the coordinates by this operation here of differentiating with respective each of the independent coordinates multiplying by the corresponding unit vector and then it is some kind of vector. Now, the question is what is the meaning of this vector? What is this gradient actually tell us? Well, it is not hard to see that it tells us what the rate of change of this function is in any direction what so ever.

So, if start with some point r in space and there is a neighboring point r plus d r in this fashion, then I could ask if you have a phi which is function of r and it has some value here and this phi has some other value at this point how much does this phi change by. So, this is phi and this is phi plus d phi I could ask what is d phi equal to clearly d phi going to depend on in which direction you move away from r. So, even for a given distance from r if you move in any direction in space, you may have a different d phi at each of these points.

But, you see if you give me a d r then I can always write it in the form d r is e x times d x plus e y times d y plus e z times d z and then it is immediately clear that if I define the gradient in this fashion ((Refer Time: 18:46)) phi dotted with d r. Well, one should strictly speaking put a vector symbol on this triangle here on this delta, but I want do so because it is a understood that this is a vector quantity. So, I do not need to put this delta each times, since I cannot write bolt phase and just going to leave it without an arrow.

This quantity here, if I use the rule for cross products the x of these with the x of that the y of this with the y of that and so on this as you can see is immediately equal to delta phi over delta x d x plus delta hi over delta y d y plus delta phi over delta z d z. But, this is a total change in phi by definition, if phi is a function x, y, z then the change in phi d phi along any direction is not in to components this is the change if you move only in the x direction, only in the y, only in the z direction.

But, to go to this point you have to do a little x x recursion, little x recursion little y x recursion to go there and this is the net change here. So, by definition this is equal to d phi, in other words the gradient of a scalar function dotted with the infinite decimal displacement d r gives you the change in the function, when you move from r to r plus d r. So, it actually gives you the change of the function in any direction in space which is why the gradient is called the directional derivative on directional.

If you need to find out what is the rate of change of this phi along any direction, then what you have to do is to dot this gradient of phi with the unit vector in the direction and you have the rate of change. In other words, if you have a phi and it changes along some directions specified by the unit vector n then this is equal to grad phi dot n. So, that is the meaning of the gradient of any scalar function.

Now, one could ask for a geometrical meaning in addition to this, one could ask if there are physical meaning of some kind and the answer is yes, and it is very simply understood if you look at not a function of three variables for which it hot from it to draw pictures, but let us look at a function of two variable, let suppose that you have a Helitran on the surface of this earth which I will assume to be plane for moment and I have a Helitran at which for each x and y horizontal coordinates I have a certain given height.

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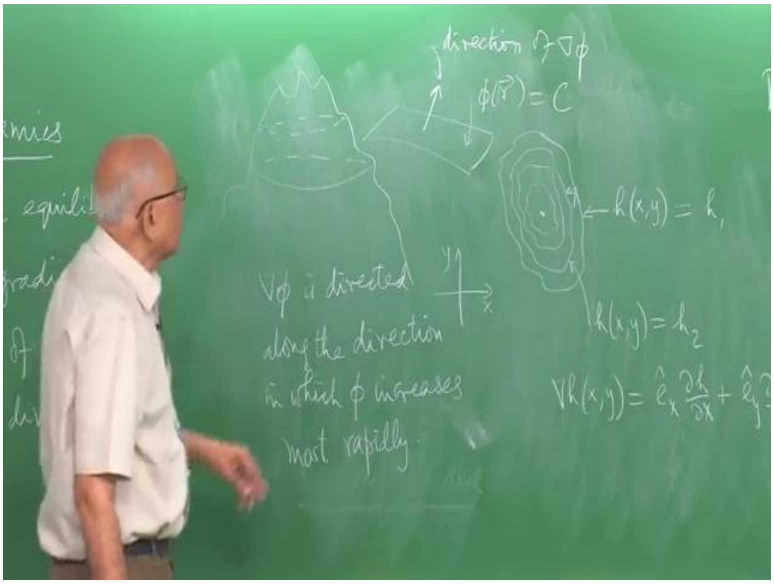

So, it is like mountain and range is like some kind of complicated mountain range of this kind and on this mountain range you would have seen there are contour maps which you tells you all points of the same height. So, all points of the same height same given height are form contour map and if you look at it from above the contour map is going to look like this. So, that is the peek and then you have points of equal height in the x, y plain is look like this etcetera and that is the peek looking at from above.

So, we are now in the x, y plain and z the height is coming out and this is an equal contour map it is says the height at a point x comma y along this curve is come constant equal to h 1 some height h 1 and this contour, the height as a function of x comma y is h 2 some other height and so on till you hit the pick here at this point, then you can ask what does the gradient of this h look like. So, it is clear that gradient of h of x comma y in this case is just e x delta h over delta x plus e y delta h over delta y at any point.

So, the question is what is the direction of this gradient, the magnitude is trivial it is square root of this squared plus that is squared. But, what is the direction of it well it is clear that if you move along this curve, the height does not change at all. So, it is not contributing to the gradient at all, on the other hand if you move normal to it that is what is going to contribute the maximum to the gradient. Because, if you move in any other direction like this you can resolve it into motion along the contour and perpendicular to the contour and along the contour there is no change at all.

So, it is only the perpendicular component that contributes the gradient therefore, the gradient of a scalar function is in the direction in which the function increases in magnitude, increases the most t p. In other words this is the steepest way to claim the mountain normally you take a winding path you claim up, but if you did this you go straight up the gradient, the maximum possible steepest. So, the physical the geometrical idea of the gradient of a scalar function delta phi in three dimensions for example, is directed along the direction in which phi increases most rapidly as a function of space coordinates for a given time if it is function of time as well.

So, that is the geometrical meaning of a gradient that the level surfaces of a function in three dimensions for instance, it would be some plane which say phi of x, y, z phi of r equal to constant, when along the surface phi does not change at all and the gradient points normal to the surface, this is the direction of a grad phi, just as if you had a function of two variables, the level curves of this function or h of x y equal to constant and the gradient points normal to that level curve, there is level curves at any point.

Similarly, the direction of a gradient phi in three dimensions is normal to the level surfaces phi of x, y, z equal to constant at any point and what we just is discovered is that the pressure the force on this liquid we have just discovered on a volume element piece.

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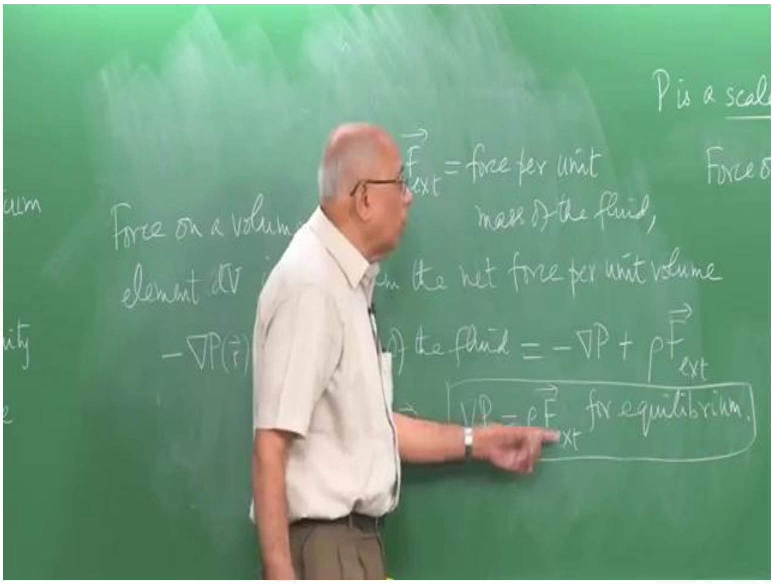

So, force on a volume element p a delta v d v is minus the gradient of the pressure at that point r could be a function of t. But, for the moment let us just look at like this, it is a static case times d v. Now, if there is an external pressure on the external force on this, which is the general case for instance, you put a fluid and the gravity there is in external force you to gravity, then you have if f external equal to force per unit mass then the net force on the fluid equal to minus gradient p, the force per unit volume then is minus gradient p plus rho times of external per unit volume of the fluid.

So, let us write that out explicitly equal to minus gradient of p plus with density of the fluid times F x time, which implies that the gradient of p equal to rho F external for equilibrium. So, this is the condition for hydrostatic equilibrium in a fluid, if f external which could be a function of space of the coordinate is the force per unit mass of the fluid, you multiplied by the density you get the force per unit volume the external force and you equated to the gradient of the pressure and that gives you the condition for static or hydrostatic equilibrium in this case. Now, immediately you see that when you have a gravity fluid and a gravity.

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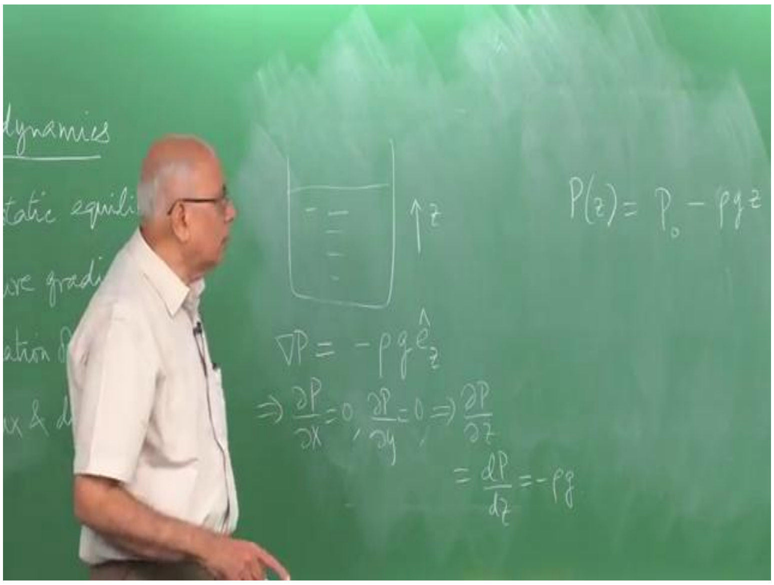

So, let us put a fluid in a container and this, the z direction and the gravity then we see that gradient of p would be equal to minus the density of the fluid times g. Because, that is the force in the vertical direction pointing downwards to the minus sign and that is it here. But, this quantity has a x component delta p over delta x delta p over y the y component etcetera and those are 0. So, this implies that delta p over delta x equal to 0 delta p over delta y equal to 0, which means that p is really a function only of z, because this says that no depends on x no depends on y at all.

So, this implies that delta p over delta z this same as d p over d z, because there is no dependence on x and y at all and this is equal to minus rho g and of course, if you can trivially integrate this to write p of z in this case equal to p 0 which is maybe the force or the floor of this resell minus over g, which is a statement which is we know immediately that the pressure decreases as you go up and it is the largest here and it decreases linearly as you go up here the fluid column. So, that is the elementary relation which are already familiar with, but it comes from the more general relation which says the gradient of p is rho times f external per unit volume.