Mechanics, Heat, Oscillations and Waves
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Lecture – 32
Wave Motion
Wave Equation, General Solution

We have come to a stage, where we would be talking about transverse waves on a string. And I pointed out, that the boundary condition which says that the two ends of a string which are clamped.

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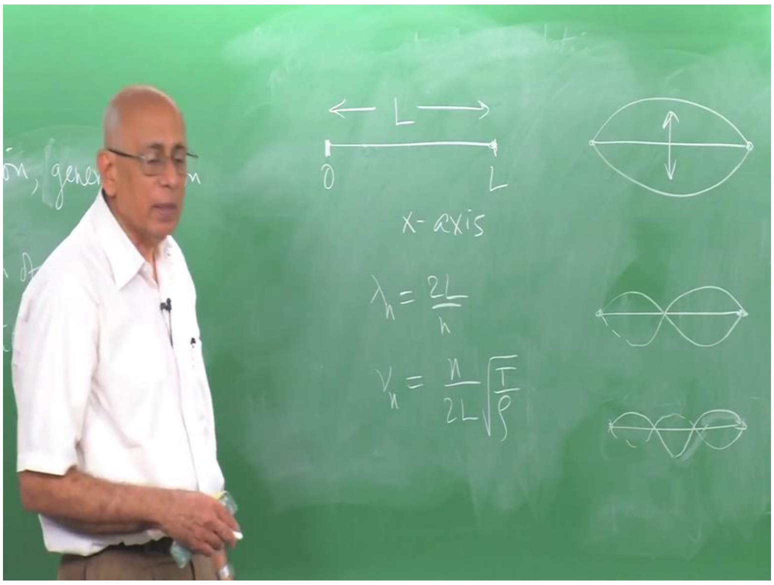

So, here is a string of length L, which is clamped at this end and this end. So, here is 0, here is L that is the x axis start with, I pointed out I mentioned that the kind of waves you can have on the string the standing waves, which would correspond to specific wavelengths. All those wavelengths are allowed, which have nodes at these points and it is so called fundamental harmonic or the first, the longest wave length oscillation that you can have would correspond to the following.

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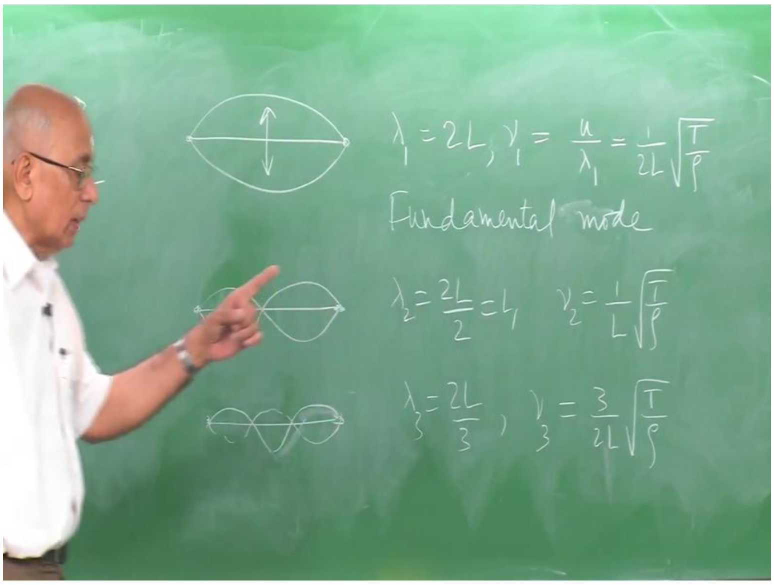

Geometry, it would correspond to the wave going up the string going up and down in this particular fashion. So, every point on the string goes up and down executes harmonic motion, executes oscillatory motion in this fashion going up and down; such that these two points they may nodes. So, the actual wavelength of this wave would be twice the length of the string in this case lambda and that is the fundamental notes.

So, lambda 1 is 2 L and the frequency nu 1 is equal to it is; obviously, equal to the speed u divided by lambda 1, which is equal to 1 over 2 L square root of T over rho. So, that is the fundamental mode, it would actually correspond to a super position of two waves, one of it is travels in the positive x direction and the other in the negative x direction. And we will see a little later when I talk about super position that they would leave these two traveling waves would add to each other, in such a way that the result is a standing wave, which goes up and down.

So, nothing in this, nothing moves forward up, backwards; every point oscillates up and down in this fashion. Well, the next oscillation that is possible has a geometry like this, here is a full wavelength and then, at the other end of the oscillation it look like this. So, here lambda 2 equal to 2 L over 2, which is equal to L itself and nu 2 equal to 1 over L root T over rho. And a next harmonic would correspond to something, which looks like this not drawing this too well.

But, this would correspond to a wavelengths, which is two third, so lambda 3 equal to 2 L over 3 and nu 3 equal to 3 over 2 L root T over rho. And the series goes on forever, so you are actually can excite on this waves are very, very high frequency, they would not normally get excited, because we are not talked about energy considerations yet. But, the frequency the lambda n, the n harmonic would have a wavelength, which is equal to 2 L over n and the corresponding frequency would be equal to n over 2 L square root of T over rho, this fashion.

Now, we would like to understand from more basic considerations, what governs the waves on a string of this kind. What kind of equation it governs in? Is it Newton's equation? Well, the answer turns out to be yes, it essentially Newton's equation, but written for a continuous medium; unlike a set of particles were you have independent particles perhaps interacting with each other and you write Newton's equation for each of them.

In this case, we have to pay attention to the fact that this string at every point, these points are connected to each other, the string does not break, does it tension on the string, which keeps it together. And we have to pay tension to the fact, that the motion of this particular point and the motion of neighboring points must be related to each other in such a way that the string maintains it is integrity. So, this leads actually not to differential equations for the equation of motion, but an equation which is in two independent variables for a given T, you have to say something about, what happens for different x is and for a given x you have to say, what happens to this point for different values of T.

So, not surprisingly the equation of motion that we get called the wave equation is actually a differential equation with two independent variables x and T in this case and these are called partial differential equations. Although we do not really gone into mathematics of partial differential equations, I will still mention this and derive this equation, because it is fairly easy to understand, how this happens let us take a string example itself.

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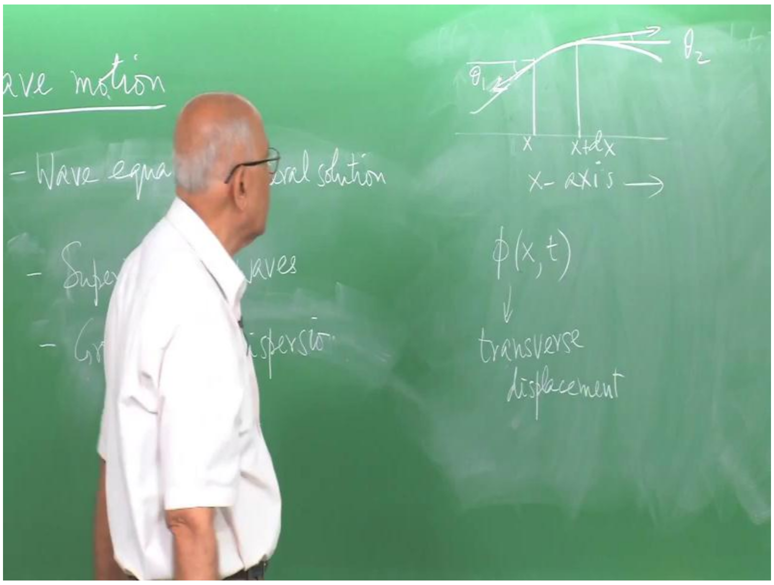

So, we have x axis here in this direction and let us suppose that moment tell you the string at something points looks like this. So, consider a point x and consider a point x plus d x, I am shown this exaggeratedly this is just an infinite decimal differential increment. The tension at this point acts tangentially in this fashion and the tension at this point acts tangentially in this fashion and what we are interested in is the transverse displacement.

And let me call the transverse displacement at any point x phi of x comma t, this is the transverse displacement of the string at the point x at the instant of time t. So, we have a function here and since it is just a displacement in the transverse direction, it is a scalar function. So, this function is a function of two independent variables of both x and t and if the string is clamped at the ends, it will satisfy some boundary conditions, but we are not concerned with that right now.

We are concerned with what sort of equations does this phi satisfy, well looking at this element of the string here, we are concerned with what happens in the vertical direction in the transverse direction. So, if this angle here is theta 2 and this angle is theta 1, then we concerned with what happens to the vertical component of the force and we equate that to the mass times the acceleration in the transverse direction.

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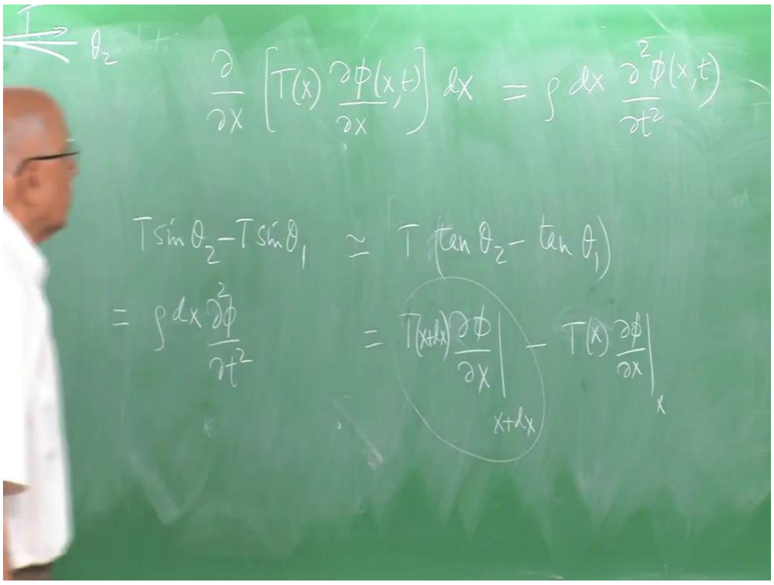

So, this thing here if this is tension T ((Refer Time: 07:17)) and this is tension T, so you have T times sin theta 2 minus T times sin theta 1, this is the force in the vertical direction here. This is equal to, well we need to know the mass per unit lengths that is rho and then there is d x, which is the length here of the string in between, that is the mass multiplied by the acceleration and the acceleration of course, is equal to phi delta 2 phi over delta t square. So, at this midpoint for instance you ask, what is the acceleration of this mass for this string, this incremental portion of this infinite decimal portion of the string.

Well, if the thetas are small, if the displacement is not, is if the oscillations are small, oscillations in a technical sense namely these things angles are very, very small, then you can approximate these angles by writing this as approximately T times tan theta 2 minus tan theta 1. But, remember the tan theta is a slope at this point, so it is the derivative of the function phi, so this is equal to T times delta phi over delta x, because you differentiate with respect to x keeping t constant ((Refer Time: 08:57)) and that is delta phi over delta x at the point x plus d x, which is this point ((Refer Time: 09:06)) minus the same thing at the point x.

So, this is T times this minus delta phi minus T times, this is T at the point x plus d x. So, we will align for the fact that the tension itself could change as a function of x t of x delta phi over delta x at the point x here. And what is that give you, this is approximately if I do a Taylor expansion, it says take this function, it is T delta phi over delta x evaluated at x plus d x, subtract from it the same function at the point x.

If you do a Taylor expansion of this complicated function about the point x, the leading term cancels and what remains is the derivative with respect to x of T, which is possibly a function of x delta phi over delta x. And that is the function of x and t on this side, this quantity here should be equal to multiplied by d x. So, we always have a d x sitting everywhere times d x that is what the Taylor expansion gives. The values of the function at that point, which is this cancels it out multiplied by d x times the derivative of this function with respect to x, which is what I have written here.

And this must be equal to on the right hand side rho times d x times d 2 phi over d t to at x and t. So, it looks complicated, but it is actually fairly straight forward under this approximation and then, canceling the d x out.

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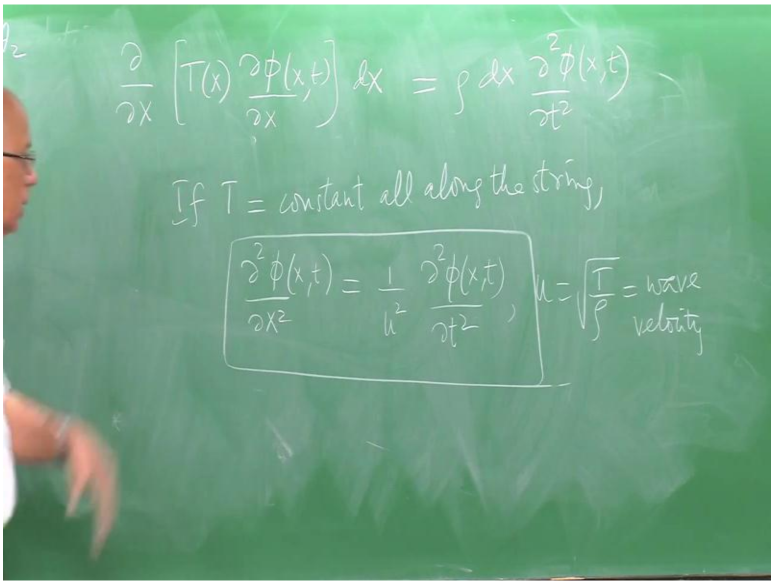

If T is a constant, if T equal to constant all along the string, when bringing t out it says delta 2 phi over delta x 2 and that is the function of x and t equal to rho divided by t. But, the square root of t over row is, what we identify with this speed that is emerging here automatically. It is 1 over u squared d 2 phi of x comma t over d t 2, u equal to square root of T over rho equal to wave velocity. So, this equation in general, although we have derived it here for transverse waves on a string, this equation is called the one dimensional wave equation.

It is the independent variables are t and one the spatial variable namely x. And this quantity here is needed for a dimensional reasons, because this is 1 over length squared d t over d x 2 and this is 1 over t square. So, you need a speed square to make the dimensions right and it is associated with this quantity is associated with the square of the speed of this wave here, of the wave of the wave velocity here. So, that is the exact wave equation satisfied by this transverse wave. Now, one can ask given an equation of that kind, what is the general solution to it and not surprisingly the general solution is of the form.

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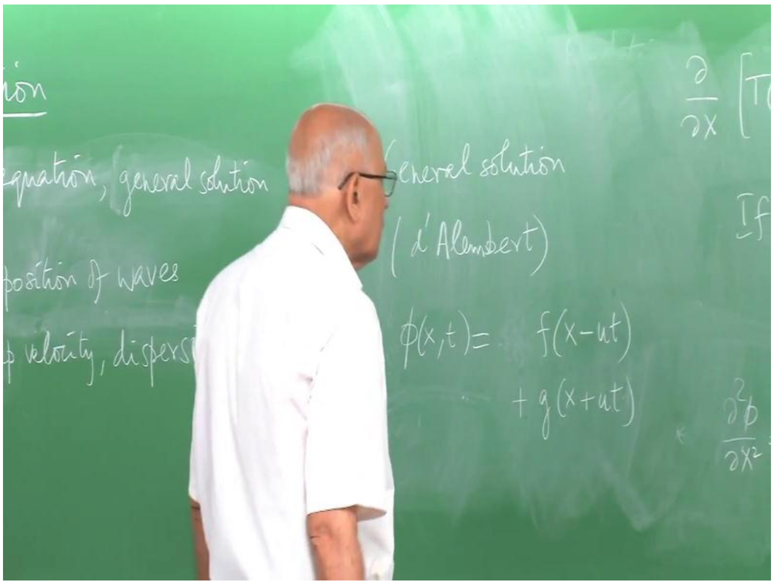

So, the general solution to this wave equation without putting anything else like boundary conditions or initial conditions or anything else, the general solution is due to d'alembert and it is not surprisingly of the form phi of x comma t equal to some constant say c 1 times. The general function you can observe that constant in this function x minus u t plus some other general function x plus u t and easy to see where this comes from it is a second order equation.

So, you would expect the loose way you would expect that there are two independent solutions. And you can see where this comes from if I took this function and I differentiated with respect to x keeping key constant partial derivative with respect to x I would get an x prime of whatever is inside if I differentiate twice I get f double prime of what a combinations here.

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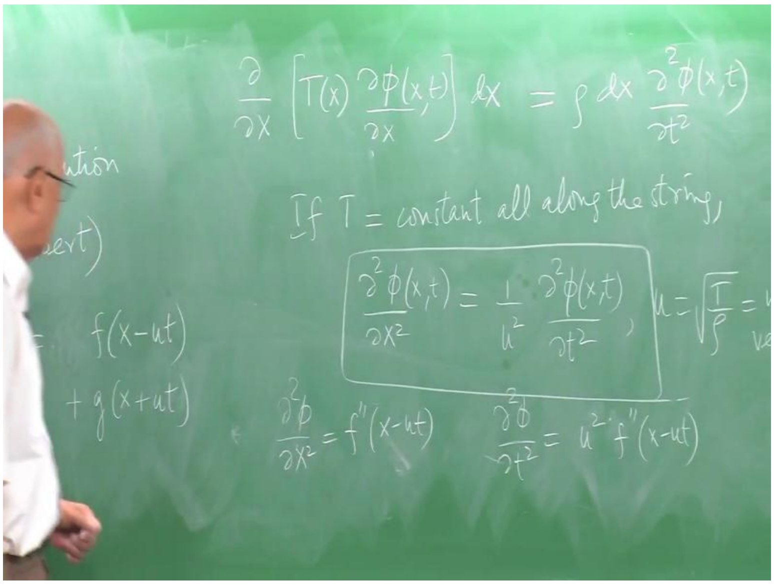

So, d 2 phi over d x 2 would be f double prime of x minus u t, where by a prime, I mean that the derivative of this function with respect to arguments whatever it be. So, for a fix t this has an change at all shift in x and its equal to f double prime. On the other hand, if I keep x fixed and do a partial derivative with respect to t every time I differentiate with respect to t I get again f prime and then, a minus u out here. So, second derivative would produce a minus u whole square times double prime, so this delta 2 phi over delta t twice equal to c square u square f double prime of x minus u t for this portion.

Similarly, for this, because when I square take a second derivative whether it is minus u square or plus u square does in matters exactly same thing. So, this also satisfies exactly the same relation and if I therefore, divide by this 1 over u square I get this is equal to that and that is, what the wave equation same. So, it Alembert solutions is obvious more or less obvious from the structure of this equation, as soon as equation of this kind we are guaranteed that the solution will be super position of these functions here.

Now, why super position, because these are linear equation this is the linear equation and phi. So, if you find two different solutions, which are linearly independent any super position of the two is also the solution of the equation precisely, what super position is determined actually by the boundary condition. So, that is for waves on a string, what about the sound waves? What about sound waves, in three dimensions in high dimensions, well not surprisingly, similar to similar kind of derivation shows us that.

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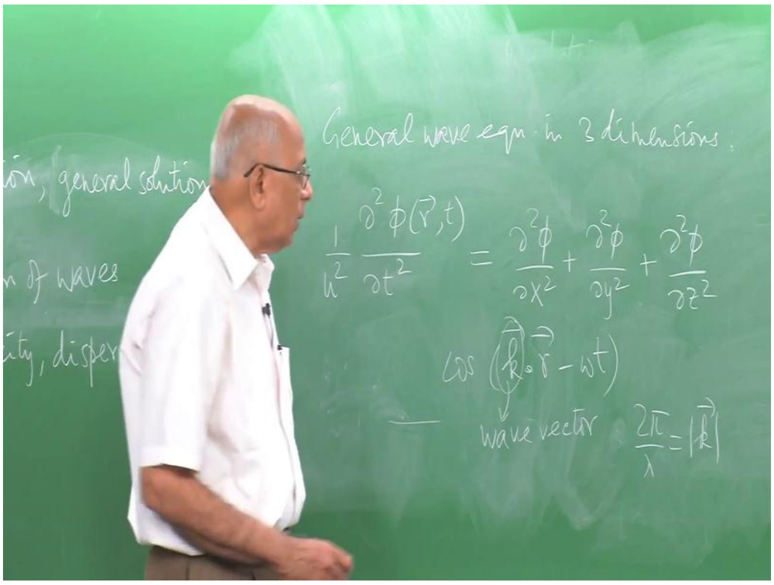

The general wave equation in three dimensions for any disturbance you continue to call it phi is of the form d 2 phi. And, now function of r and t over d t 2 and then, there is speed square 1 over q square this is equal to d 2 phi over d x 2 plus d 2 phi over d y 2 plus d 2 phi over d z 2 in three dimensions in Cartesian coordinates this is the function of x y z and t. You could write similarly, an operator at differential function here are differential some kind with respect to other coordinates system like spherical polar coordinates systems or cylindrical coordinates whatever this function this whatever appears here these derivatives could be a little more complicated.

In that case precisely, because unlike Cartesian coordinates the unit vectors in other coordinates system change as a function of position. So, there are the line element there are not just d x square plus d y square plus d z square, but there are functions of r theta and so on, sitting there, we are not going to get into that now. But, this, what sound waves in three dimensions for example, were this would be the displacement of air some given mass of air would satisfy n equation of this kind here.

In particular we could ask, what happens if you have waves, which are propagating out from a source. So, I start with the source at the origin and it is putting out waves, which are isotropic go out in all directions. What kind of wave would that be? What kind of surface would that be? When in that case it turns out that unlike the sins and cosines we have been talking about here you would have waves of the form cosine or sin of not k x, but k times r minus omega t for instance, where k is, so called wave number as before 2 pi over lambda r is the radial coordinate from the origin, but now energy considerations come into play. So, from this source as things goes outwards certain amount of energies pumbt into this medium as the source eliminates waves. Then, since the surface area at a distance are goes like r square it is clear that if the total energy is constant, then the amplitude itself must decrease in such a way that the intensity multiplied by the total surface area gives you exactly the same amount of energy.

So, that requires that this thing be proportional to 1 over r in three dimensions. So, that this square of this amplitude the amplitude must decrease like 1 over r, so that this square of it goes like 1 over r square compensating for the increase in the surface area by a factor r square. So, this thing is called a spherical wave this kind of thing called this spherical wave.

You could also have other kinds of solutions again, similar to the one dimensional wave we talked about earlier and a more general form for this would be something like cosine of k dot r minus omega t, where this vector k is called a wave vector whose magnitude two gives you 2 pi over lambda on this scale. And whose direction gives you the direction of propagation of the wave we looked at in the x direction, so this k was this was just scattered out just k time x. But, you could have a wave, which is propagating it some arbitrary directions. And then, we can ask, what kind of an object is? What kind of wave is this, is not a spherical wave well it would corresponds putting k dot r minus omega t equal to constant some given phase.

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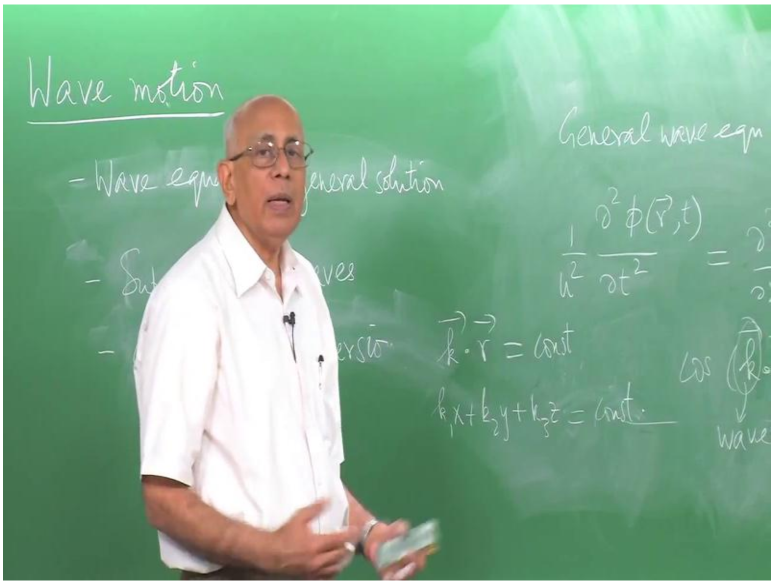

And, what kind of surfaces is it for fix t you need to ask, what kind of object this k dot r this equal to constant. And since, this stands for k 1 times x plus k 2 times y plus k 3 says if k 1 k 2 k 3 are the Cartesian components this equal to constant it is a plane this is plane and infinite plane in some direction the direction of this vector k. If, you of course, had no third dimension here, then it is a straight lines as you can see and it is a plane in three dimensions.

So, this is called a plane wave something like this is called a plane wave when we study three dimensional waves One more generalization, what about light? What about light propagating in vacuum we know that light consist of electromagnetic waves, which means it is the electric and magnetic fields that oscillate and there is medium here this speed of light is the same no matter, which direction you are in. In fact, it is independent of the source state of motion of the source are the observer according to special relativity. So, what kind of equation satisfied by the electric and magnetic fields in that case?

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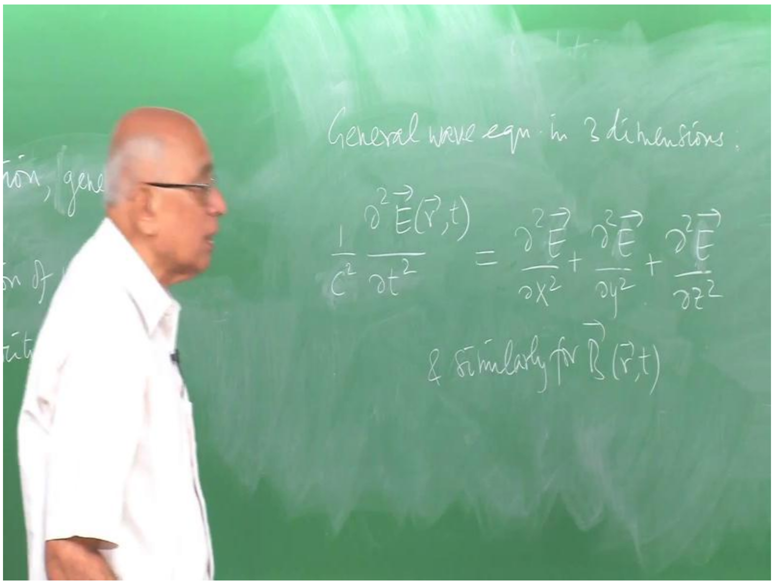

The equation is exactly the same thing except that this becomes a vector they electric vector field vector. And, similarly for the magnetic field B r and t and this u in vacuum is replaced by speed of light in vacuum in that is the wave equation satisfy by the electric field in a electromagnetic wave of, which light is an instant. So, visible light is one such a instance light of any frequency is an electromagnetic radiation and in vacuum it is got this invariant speed of propagation c here. So, as you can see the general wave equation as the large number of a masses in a very, very comprehensive kind of equation. It can have all possible kinds of solution depending on this situation much more complicated in the simple transverse wave in a string that we talking about here.

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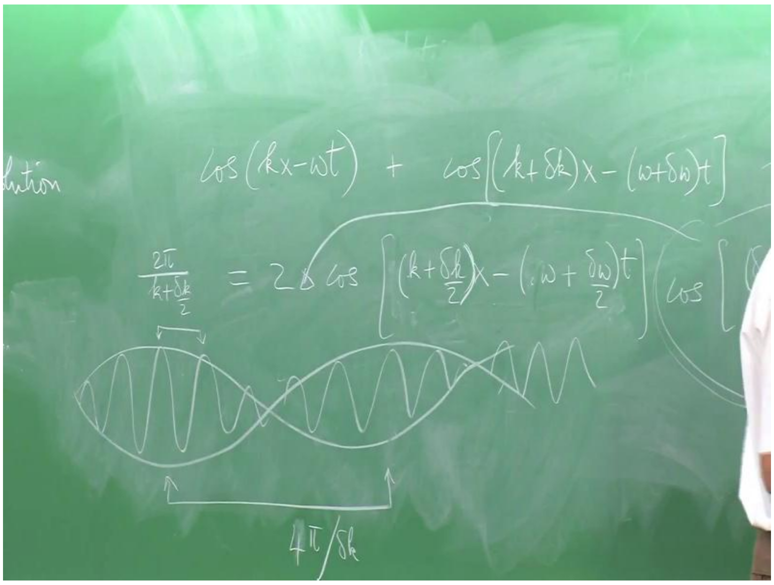

Next, let us look at super positions of waves, so let us suppose that we have two waves. one of, which is a cosine of k x minus omega t a traveling wave in the x direction a one dimensional wave in the x direction with a velocity of wave velocity, which is omega over k. And we also have along with if the same time a medium as a disturbance, which is k plus some delta k x minus omega plus delta omega.

So, we have two waves one of its travels in the x direction with the speed omega k in the other has slightly different frequency are different wave length here or wave number. So, what is this super position give us well we need to be simple use the identity that this cosine of the sum of these two twice is equal to twice the cosine of the sum of the 2. So, this is a 2 k plus delta k x minus 2 divided by 2, 2 omega plus delta omega t divided by 2 that is a cosine of this the sum of 1 2 here times the cosine of the difference of the two, which is delta k x minus delta omega t divided by 2.

So, we could write this as k plus delta k over 2 x related up this. And, what is the geometrically look like, well you can see that this quantities are infinitives decimal quantities you say that delta k and delta omega very small compare to k and omega their infinitives decimal quantities. So, the variation mostly comes here due to this and once you put that envelope in and you can bring that factor around down to this portion and considerate part of the amplitude.

Then, this phase anything look like this whereas, originally you had a single cosine up and down and another cosine going up and down when the join together when you super pose them, then the whole system looks like this schematically and so on. And the actual disturbance looks like this and this wave length here is much larger, then this wave length here this wave length here is given by wave number, which is k plus delta k over 2. So, it is the mean between k and k plus delta k, k plus half the difference here, so this is a 2 pi over the corresponding k, so that is k plus delta k over 2.

On the other hand this is 2 pi divided by that wave number, which is the 4 pi divided by delta k. So, I given instance of time with disturbance looks like this, in other words there is this original oscillation whose frequencies average whose wave length this average of the wavelengths of the two that we super pose the two waves super posed. Multiplied by and envelope function, which is got a much slower variations it is also traveling sinusoidal wave, but it is much longer wave length than be original two waves individually had here. At the same time this is also traveling along and, what is the speed of travel here is the speed of this cosine here.

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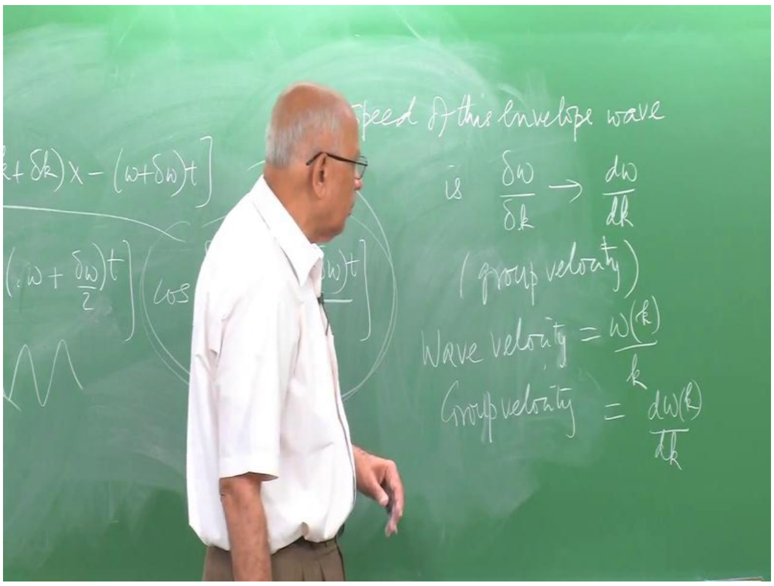


So, this could correspond to a wave a velocity or a speed of this envelope wave is delta omega over delta k or this tends is the limit to d omega over d k, if omega is a function of k. So, the point is I super posed two of these waves and I found that there is an envelope, which is traveling at a speed, which is given by d omega delta omega over delta k from this portions here, so much longer wave length this thing. So, this looks like this wave original to waves are broken up into groups there is a group here and the group has a whole this traveling with this speed here and this is called the group velocity.

In general, what happens is if I super pose a large number of these case a whole set of these case; such that omega is a function of k. So, for each case there is a corresponding omega, then, what I have is a wave packet and this wave packet will travel as a group the group velocity of this wave packet is given by the d omega over d k. Whereas, for a given k and a given omega of k the wave velocity equal to omega is a general function of k of over k.

And the group velocity I should really says speed, but this is a standard terminology is d over omega k over k and these two could be very different from each other as will see. So, if you got a single wave this no problem, but the moment you have a whole collection of waves are different frequencies super posed, then you have both a way velocity individual wave velocities as well as a group velocity d omega by d k. Of course, it might turn out that omega is the same for all case.

In other words, the frequency is independent of the wave length this speed of the wave is exactly the same for all wavelengths. I should say this in frequencies independent of the wave length, what I should really say is that this speed is same for all wavelengths, which is, what happens, so light propagates vacuum. In that case it is clear that the frequency and the wave length are related by mu times lambda equal to c r, if I write it in terms of k etcetera omega equal to ck, so exactly the same thing same relation.

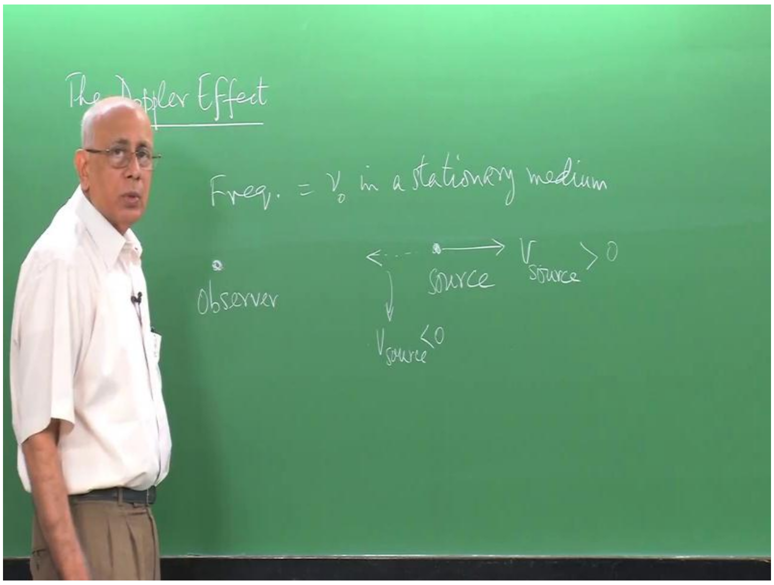
Once a function is a linear function of k this will imply that d omega by d k equal to omega by k equal to c. In other words about for light propagating of different wavelengths propagating in a vacuum in a ((Refer Time: 30:11)) wave packet the group velocity and the wave velocity and exactly this same. The wave velocity the same for all the wave lengths and the group velocity is the same as that wave velocity.

What is that mean? It means that this group does not lose it shape this wave packet does not used shape it does not disburse. So, the origin of discursion lies here it lies in the fact that different wavelengths have different traveler different speeds and d omega over d k is not the same as omega over k. But, in this special case of light in vacuum it happens to be show it also happens to be approximately show when you have wound waves propagating in terms out that and a normal conditions the dispersion is very minimal is very little.

And therefore, all wavelengths of sound propagated exactly the same average speed and therefore, you do not see the disbursing. But of course, the moment like for instance travel to a medium you see this immediately, which is why represent bake supplied into different colors, because is speed increase different wavelengths travel is different in the medium. If we consider wave motion rather briefly we are going to do a lot more offered when the context of light and optics when we will talk about electromagnetic waves at greater length.

But, one particular topic within way motion, which is of great practical significance and great importance, has to do with the Doppler effect. Now, this is an effect that all of you are familiar with when you here a ((Refer Time: 31:43)) and approaching you for instance as it approaches you it appears as if the frequency of deicing increases and as it depart from you it appears as if the frequency goes down. So, the picture of this deicing increase and then it decreases as it passes you.

This effect is quantifiable it is a very important effect and its known as the Doppler effect after is discovered. What you are going to do? Is a simplest case and nearly I am not to derive this in detail here and simply going to point out, what the main point main effect is, so we will look at the situation when you have a waves in a medium. So, let suppose that the waves in a medium have frequency nu, nu naught is in a stationery medium. (Refer Slide Time: 32:24)



So, you look at sinusoidal waves of a given frequency nu in a medium, which is stationery. When, if you have a source here and let say we have an observer here let say observer and the source of waves is here, we will consider only the case where the observer of the source is moving one is moving, which respect together along a line joining them. So, we will not look at situations, where the source moves like this for example, on the observer here with transverse component.

We are only going to look at the simplest case whether observer and source ether move towards each other or a wave from each other. And then, ask, what is going to happen if for example, the source starts moving ether towards or a wave from the observer, where well we need a convention here for the velocity of the source or the observer with respect together person. So, let suppose that the source moves out in this fashion and let me call that v source with respect to the medium.

So, the source is starting to move with respect to the medium with a velocity or speed v source. And the observer is stationery the question is, what is the observer going to see as a frequency of the waves emitted by the source given that the frequency true frequency if you like is nu not that is one question the other question we could asks and of course, v source could be in this direction. So, this should correspond to let me say v source less than 0 and this should be v source greater than 0.

So, the contention is that it v source moves away in this direction v source is positive and it will move this way the source is negative. Well you could also have a situation in which the source is stationery, so let us write that down.

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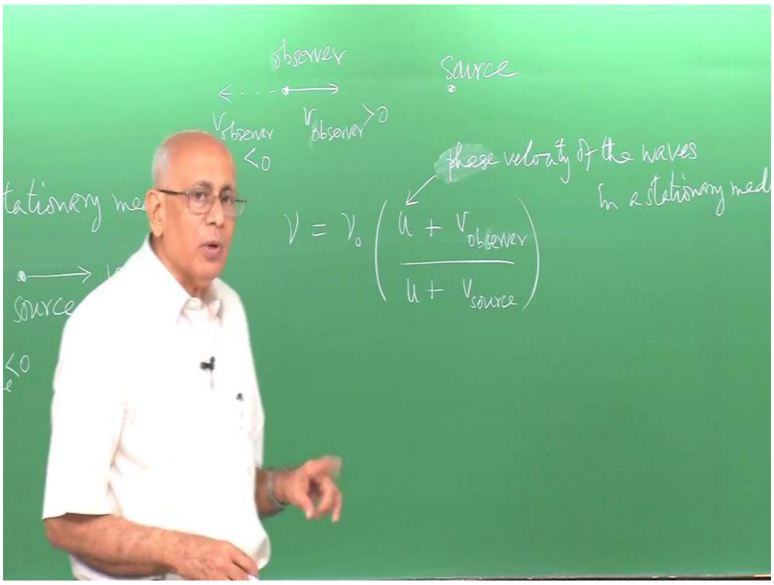

Here is the source and the observer either moves towards the source with the speed v observer with respect to the medium and this should be a positive speed on the other hand to moves away in this direction v observer less than 0 in this case. So, the contention is clear when the observer moves towards the source v observer I take to positive moves away is negative whereas, the case of the source moves away it is positive move towards it is negative.

So, this picture makes a quite clear as to, what are conventions for the signs of these velocity is r, when we have several cases possible. We have case, in which the observer is stationery and the source moves for the source is stationery the observer moves or both of them move with these given velocities here. The general formula is, what I am going to right down, now in the formula is that the observer is going to observer frequency nu, which is nu naught multiplied by we need information about this speed of these waves in a stationery medium.

And, if the stationery medium the speed is u this is speed of the wave and phase velocity of the waves in stationery medium. Then, the formula is this plus v observer divided by u plus v source that is the general formula with these sign conventions. So, you can see physically, what happens is if the observer starts moving towards the source this is a positive number here and the source was stationery that 0.

So, what does happening is that the frequencies increasing, because as it moves towards the source the number of crush the v observes per unit time is going to increase is going to look compressed exactly the same thing is going to be happen ((Refer Time: 37:14)) if the source moves towards the observer while emitting radiation or waves are a given frequencies. So, so many tuffs for tress per second and as it moves towards the observer the number of source observe by the observer is going to increase, so effectively the frequency increases.

On the other hand, if the source moves away this is going to look stretched out and therefore, the frequency observe by the observer is going to drop that is, what is going to happen if v source is positive well that is 0. So, this formula includes in it both cases when the source is stationery and the observer is moving and vice versa and this is the general formula. Now, what is interesting in this formula is that is derive under the summation of Newtonian mechanics.

So, there is no well to the sit mechanics here there is no special relativity here at all. And you can see immediately that it is not symmetric with respect to that is the velocity source is moving or the observer is moving. On the other hand one good ask, what happens, if you have special relativity, which we will talk about in the context of electricity and magnetism later on.

What happens, then after all then either of these frames of reference either the observer or resources its prefer there should be complete symmetry between these two and the formula, then changes in the relativistic case. But, this is the non relativistic formula which you can plug in numbers into it and then, we can find out what the actual frequency shift is etcetera, etcetera. But, let me make a mention of, what the relativistic formulas, because it is interesting to see, what happen two things happen.

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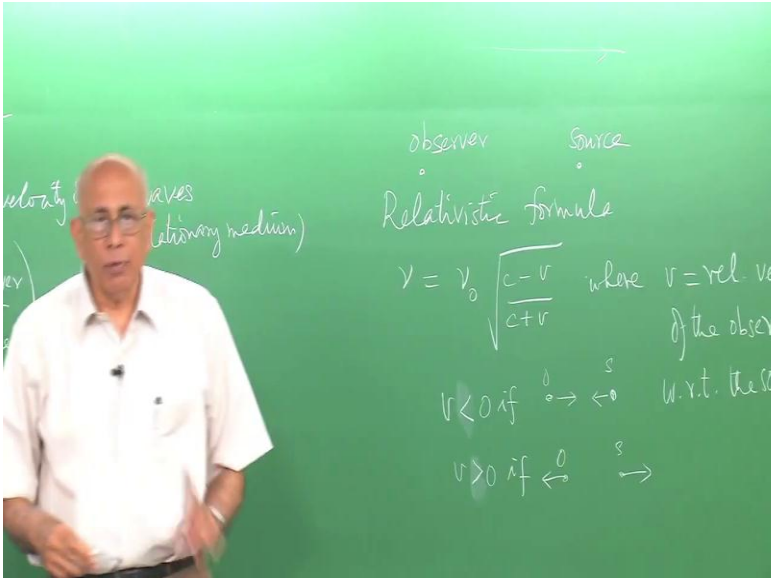

Here is the observer and here is the source once again we consider only the simplest case where there is a motion of either of these p 2 are both of them is in the line joining the two the observer and the source. And, let suppose the relative velocity between the two is some number v. Then, you should matter really whether it is the source that is moving or the observer that is moving besides, when we talk about light.

We have let another phenomenon to take care of, which is that this speed of light in a vacuum is independent of the state of motion of the source of the speed of the source or that of the observer that is the fundamental postulate in relativity. The principle of relativity is different and we will talk about it, but the physical postulate of special relativity is that the speed of light in vacuum is a fundamental velocity independent of the state of motion of the source. Therefore, that is pay the c, then the question ask is, what is this going to be, what is nu going to be nu equal to nu naught times.

And now, we have to pay attention to two things not only the argument about the number of stress or the number of traps actually in compress or decompress, but also the phenomenon of time dilation. So, when you have a moving frame time flows differently, then it as in the stationery frame provided these are inertial with respect to each other. When you put these corrections into account and put the special relativistic corrections into account this formula is not does not hope to anymore.

We have a formula that is symmetric about whether the source moves or the observer moves and this form rise c minus v divided by c plus v, where v equal to the relative velocity of the source or of the observer with respect to the source and all we need to know there is we do not care, which of the two is moving v is greater than 0. If the source and observer here is the observer here is the source are moving towards each other in this fashion does not matter, which of the moves and v is less than 0.

When they move towards each other if v is positive, this going to give a compression here, so the thing decreases so it is the other way about; v less than 0, if you are moving towards each other and v is greater than 0, if these are moving away from each other. And when v is positive you can see that the increase in the frequency and the decrease in the frequency comes about for two reasons. One, because this factor becomes smaller than c the numerator become small and the denominator increases meaning that there is decrease in the frequency.

So, in the moving away from each other there is a decreasing frequency, but if the moving towards each other, then this is negative. So, you have an enhancement a frequency both, because numerator is increasing and the denominator is decreasing in proportion. So, this is the relativistic formula which includes the correction due to time dilation and again I emphasize it is the longitude in a Doppler effect in the sense that the two source and observer are the motion of either of them or both are them is along the line joining the two of them.

If it were a transverse effect namely if you have a source here and you have an observer zipping along in a line is does not pass through the source then the formula changes and you to take up this is, what this happen, what happens then when we study optics when we study the propagation of electromagnetic waves.