Mechanics, Heat, Oscillations and Waves
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 03
Dimensional Analysis and Scaling

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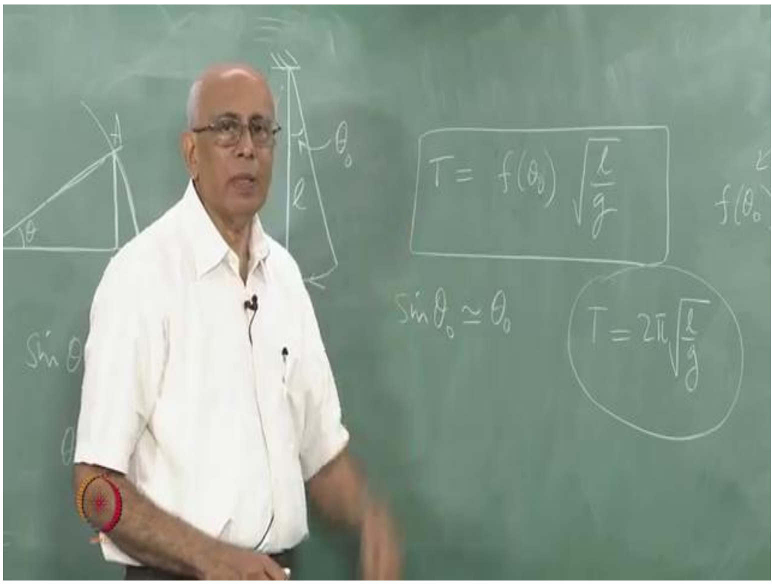

What can we say about this function theta naught? Well, unfortunately it terms out as you would expect, you cannot say anything much about it without actually solving the equations or motions, writing down the explicit solutions of theta as a function of T and so on. But, we can make the following statement, one thing is very clear that if the amplitude is extremely small namely if this thing is essentially in a straight line namely theta naught is so small that sin theta naught can be approximately replaced by theta naught in radian measure.

Recall for a minute, that you have an angle theta, sin theta is found by dropping a perpendicular here and it is equal to this divided by this base, the vertical divided by the base that sin theta. On the other hand, the arc divided by the radius, the sin let me write it properly here is the origin A, B sin theta is equal to A B divided by O A not the base, but the hypotenuse here this quantity. On the other hand, the angle theta is equal to the arc, so let us call this C arc A C is divided by the radius O A.

So, if theta is extremely small, then this distance becomes approximately equal to that distance along the arc and sin theta and theta become essentially in distinguishable from each other. On the other hand, the real nature of sin theta is that it starts with theta naught, but there are higher order corrections in theta naught which is the next leading correction is of order theta naught Q then so on which you will learn in mathematics subsequently.

But, right now the point I want to emphasis is that when this approximation is valid to whatever degree of accuracy you desire, then you can find this f of theta naught very easily and it terms out that f of theta naught equal to 2 phi in the limit in which theta naught tends to 0. But, when theta naught is not very, very small, but it is significant then you need to put higher order corrections and you can has the matter of fact in this problem. Find the exact answer in principle and the answer is going to be something which depends as you can see on physical considerations.

Whether I have a plus theta naught or a minus theta naught it does not a matter, the angle of amplitude is theta naught in magnitude and it goes from plus theta naught up to minus theta naught in this side. So, whether I call this the amplitude or the other one the amplitude is relevant, which means that this f of theta naught must be dependent only on the square of theta naught, it cannot depend on the sign of theta naught.

So, this is equal to some function of theta naught square in this case and it terms out that f of theta naught exactly is 2 phi into 1 plus the next term not surprisingly will be proportional to theta naught square over 16 plus higher orders, corrections of order theta naught to the 4 and 6 and so on and so forth forever. So, this leading term being 2 phi is the reason why you write T equal to 2 phi square root of l over g as the standard formula for the time period of a simple pendulum for very small amplitude oscillations.

Now, how small should this very small be, well there is no absolutely met to this, it depends on your accuracy. So, depends on what accuracy you desire, to what degree of accuracy is this true, well it depends on to what extend can you approximate sin theta naught by theta naught and generally if you wants like one percent accuracy are something theta naught must be just a few degrees, not more than that.

There is no specific number like 4 degrees or 6 degrees or something that which you might have seen some places which tell you under those conditions it is equal to this.

There is no absolute number of that kind of limit of that kind, it is a matter of the desire accuracy as you can see. But, that is the best you can do with dimensional analysis and you agree that is quite a lot, because I have this extracted this dependence on the parameters on the problem l and g.

And I have also shown by dimensional analysis, the very important conclusion that this time period is independent of the mass of the bottom, that is completely gone out of the equation for dimensional reasons and it has eliminated this parameters all together. Can we do this sort of trick and solve a little more complicated problem? The answer is yes and here is one more where this is illustrated and again it will also tell you what the limitations of the method are.

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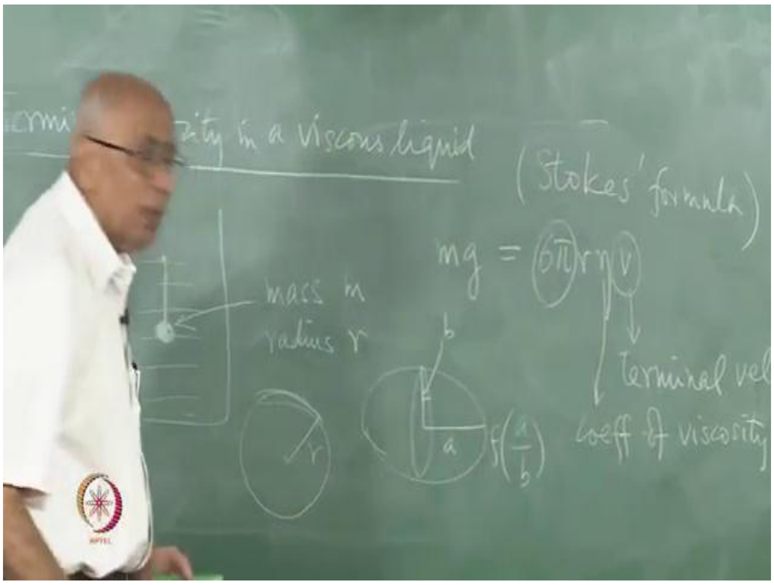

So, let us call this, this problem is the problem of finding the terminal velocity in a viscous liquid, now have in mind the following experiment. So, I take a viscous liquid, oil or water or honey or anything like that and drop a ball barring in to it, a heavy ball baring which of course falls in the water let us down this functions following down. And let say the mass of this object that is following down is m and for simplicity, let us starts by assuming that it is a perfect sphere of radius r ((Refer Time: 06:22)).

It is following under gravity, but as it falls there is also a drag force exacted on this ball by the friction from the liquid and it is called viscosity. As you can see intuitively honey or oil is more viscous than water, water in turn is more viscous than light a fluid some kind of petrol products for instance. And these are both more viscous than gas like air which also has a viscosity, every fluid has a viscosity. So, the question is what is the velocity that is reached when the viscous resistive force upwards on this ball is balanced by the gravitational force downwards?

We would like to know is there a terminal velocity, a limiting velocity here and the answer is yes and it turns out that in this particular problem, it turns out that the force of gravity downwards is m g and the force of gravity upwards is given by the following formula called the Stokes formula and we will come and talk about it again.

It is 6 phi r theta and then v, where this stands for the terminal velocity. This here is called the co efficient of viscosity of the liquid, r is the radius of this ball and 6 phi is you know is a number pure number. The question is where did this come from? So, the terminal velocity the limiting velocity of this ball is m g divided by 6 phi r theta and we would like to derive this from dimensional analysis. First, I have to define what is viscosity of the liquid and the idea is the following.

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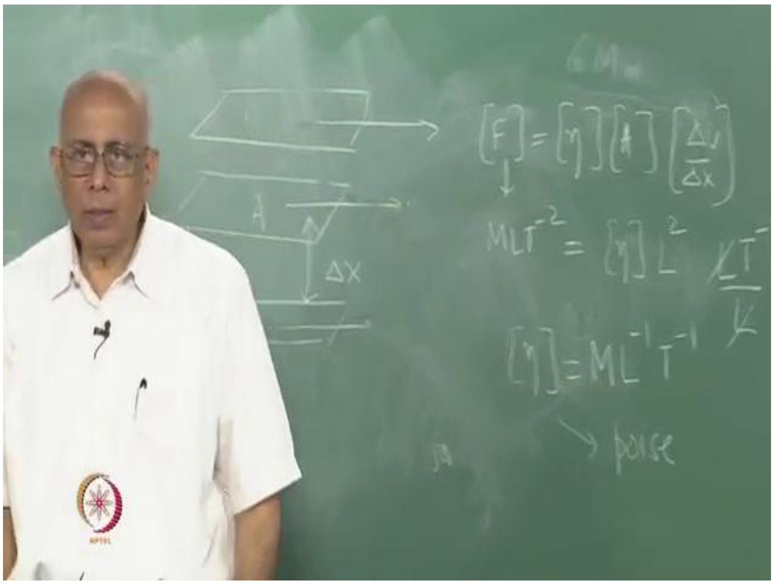

If you have a fluid flowing on a tray for example, in the bottom layer is moving at some velocity and as you go up, this fluid is moving slower and slower higher velocities. Because, this guy here the fluid sticks to this glass, glass plate for instance and then the fluid that is flowing slows down as it comes towards the glass plate till it fluid with a certain velocity v here and out here at any higher point, this fluid is flowing at a higher velocity.

So, there is a gradient or change in the velocity as you go higher up in this pool of water and this canal at the lower level, the once that I contact with the floor of this at flowing better much slower rate. Now, what happens due to this is that, there is a differential in the velocity as you move up in this direction and therefore, if you have a small difference delta v in the velocity, the velocity here is v plus delta v as you move up at a distance delta x, as you move up further the velocity increases even more and so on.

Because, there is less friction of the bottom here, this internal friction in the fluid is called viscosity and the force due to this internal friction is given by a law called Newton's law once again. It is not Newton's original laws of motion, but it is called Newton's laws of viscosity and the statement is as following. If you have two layers of fluid separated by a distance delta x, two parallel layers one of them flowing with a velocity v, this bottom one flowing with a velocity v plus delta v here, then the viscous drag force is given by a combination of quantities on the right hand side.

It is proportional to the surface area of the two planes such proportional to A, it is also proportional to the manner in which the velocity changes as you go upwards. So, it is the ratio of this difference delta v over this distance delta x and the constant of proportionality is denoted by eta which is the coefficient of viscosity of the fluid by definition. So, what is happening is that this faster layer is pulling this bottom layer in this direction. So, it exacts a force on this direction on this layer, this slower layer is slowing down the faster layer and this exacting a force in the opposite direction.

By Newton's third law of motion this forces are equal and opposite and what I have written here is the magnitude of this force, they acting different directions on the two layers. Now, given this what is the physical dimensionality of this quantity eta here, that is easy to find, because we know that dimensionality of force must be equal to the dimensionality of eta, dimensionality of A, dimensionality of delta v over delta x.

But we know the physical dimensions of all these quantities, this thing here is M L T to the minus 2 mass times accelerations. So, another illustration of the fact that a forces dimensions mass times acceleration M L T to the minus 2 independent of what kind of force it is. Then, in this case it is a force inside a fluid, we do not care it is still M L T to minus 2 dimensionally that is equal to the physical dimensions of eta multiplied by the dimensions of A which is in area and L square delta v is L T inverse divided by delta x which is L itself.

So, this cancels out and you are left with the statement that eta must have dimensions M and then this goes there becomes L inverse and T inverse and it is measured in standard international units in poise that is the unit. So, when M is 1 kilogram, this is 1 meter and that is 1 second, this is 1 poise and we have numbers we can measure what this eta is for various kinds of fluids.

It is a sort of intuitively clear to you immediately that oil must have a higher viscosity than water, honey must have a higher viscosity than oil and so on. Air must have a much lower viscosity, the several orders of a magnitude than that of water etcetera. So, this is the formula we would like to derive and let us ask what is the value, of about what is this v equal to.

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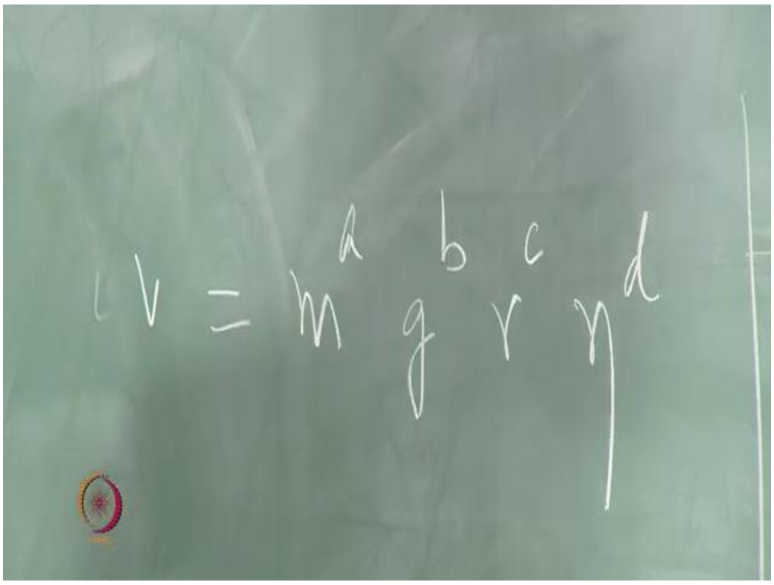

So, this physical quantity v can depend on the following arch quantity, it must depend on M. So, we do the same thing as before we say v is equal to m to the power a, g to the power b, r to the power c, eta to the power d, m g is a force. So, we have since m g is what is appearing down here, we should really say force to whatever a.

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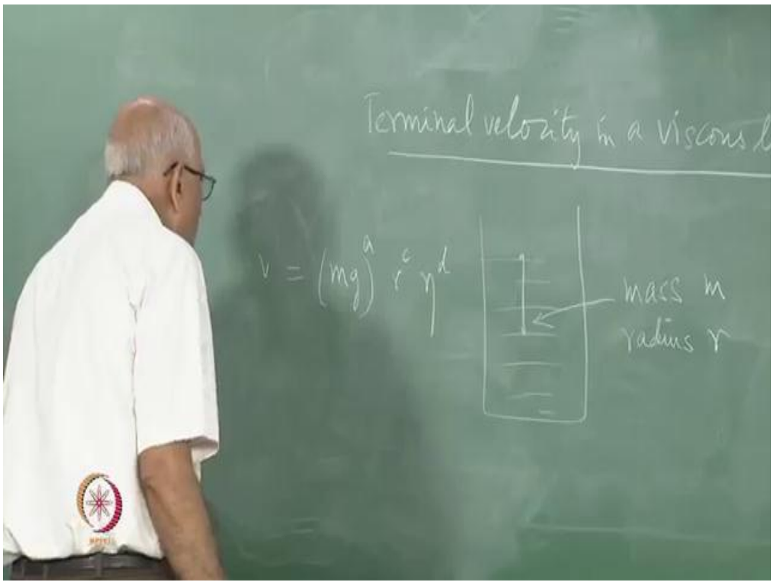

So, we should say m g to the power a, r to the c b eta to the d etcetera and now it is a very simple matter to put this M to put this expression M for eta the physical dimensions of eta.

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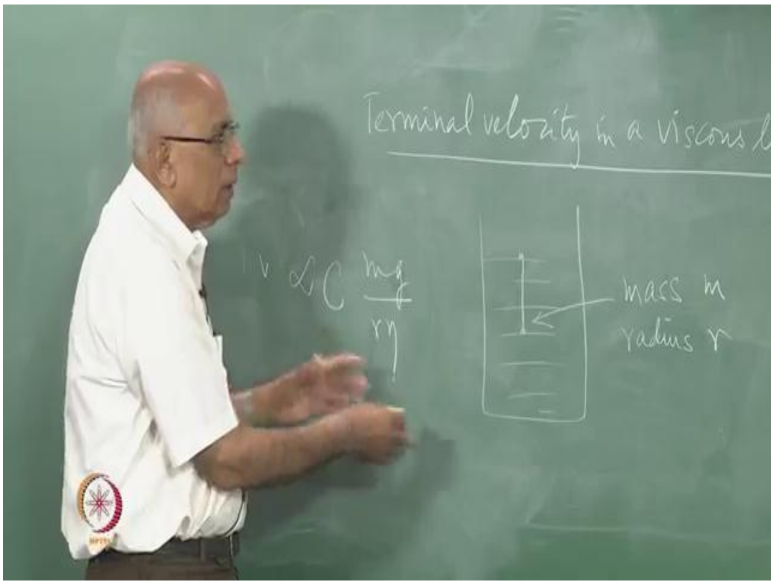

We know the physical dimensions of r and m g and show that this thing will indeed turn out to be v is proportional to m g over r times eta proportional to. Because, the constant of proportionality some constant here is something you cannot determine from dimensional analysis, we do not know what it is here. But, it is dependence on the radius and the viscosity is very obvious and by the way under physical considerations, you can see that if you have an object which is got, a fluid which is got low viscosity it will move faster and therefore v is going to increase, which is going to like one over eta.

And it so happens that the dependence on the radius also is sitting the denominator, the small of the objects later the terminal velocity in this case. So, this is the formula which is derived purely from dimensional analysis I am not going to do the exact derivation here because it is very, very straight forward. But, I want to make some remarks and what this constant is and etcetera. It terms out that the value of this constant 6 phi is quit tricky, it requires hydro dynamics to establish and it is not an elementary thing in that sense, we required much harder calculation to do this.

More over you could ask the following, suppose this has been not in a perfect sphere what would happen. Suppose, instead of a sphere you had a little ellipsoid of revolution, so that in this direction, this point this axis is semi major axis is minor axis is b and this is a. So, you have this foot ball shape make a foot ball shape object and it is ferried or an ellipsoid of revolution and this is got two lengths here a and b instead of a single r that we had earlier and I drop this inside the liquid and ask what is it is terminal velocity going to be.

Well, the point is immediately we are going to see that this argument breaks down and the reason it breaks down is that in this problem, there are two parameters with the same physical dimensionality. There is an a and b both are lengths, but as in the case of the sphere you had a single thing, a sphere you just add a single radius here and that is the end of it. But, here you got an a and b, whereas here I could say it depends on r and I put a power of r and try to calculate the power by dimensional analysis.

You cannot do that here, because of the fact that you have both an a and b and any ratio of a over b is dimensionless. So, this terminal velocity could depend on some unknown function of a over b just as in the case of the simple pendulum, the time period could have depended on some unknown function of the angular amplitude which was the dimensionless quantity. Here you have two dimensional quantities a and b, the ratio is dimensionless, the both parameters in the problem, and there is no reason why this velocity cannot depend on both of them.

But, they could in unknown dependence on some function of the ratio of these two parameters and that cannot be found by pure dimensional analysis. In fact, you would expect the thing like that happen for instance, suppose I said if a is equal to b, then of course you end up with the sphere as before and there is a dependence here on this r here. This could have been either a or b, but suppose one of them becomes much bigger than the other, a becomes much bigger, this gets much flatter here etcetera.

More approaching a disk for example, then it is clear that the behavior of that kind of the object in this fluid would be very different from that of his here. So, there is certainly n non trivial terminal dependence on this f on this ratio and it cannot be found by dimensional arguments. So, that is one point, the other point is that this 6 phi and this formula by the way is called stokes formula, this 6 phi also depends on some physical assumptions, it depends on assuming that this ball here as it is flowing down such that the fluid here is ticks to surface of that ball and this is called a stick boundary condition.

So, it is assume that the instantaneous velocity of the fluid act the surface of this ball is actually 0. So, it is stuck to it and that is the reason why relative to the ball and that is the reason why gets to this exact factor 6 phi, if you change this boundary condition as it is called and put in some other condition by saying that the ball slips to the fluid the fluid does in stick to it, then this 6 phi factor changes to something else in this case changes to 4 phi ((Refer time: 19:13)).

So, as you can there are limitations on this dimensional analysis, on the limitations of precisely that it cannot find with dependence on physical parameters if they are exist more than one physical parameter with the same dimensionality. So, that the ratio becomes dimensionless on the answer could depend on some unknown function of this ratio dimensionless quantity. So, that is the limitations of dimensional analysis and behave to be careful with that.

Now, my next illustration is going to take charge of precisely that point, we are going to look at what happen is when you have more than one line can that the physical arguments to get you the right answer, even if you have this situation and it has do with again fluid flow, but this time is slightly different let me talk about this, this is called a the discharge rate from phi.

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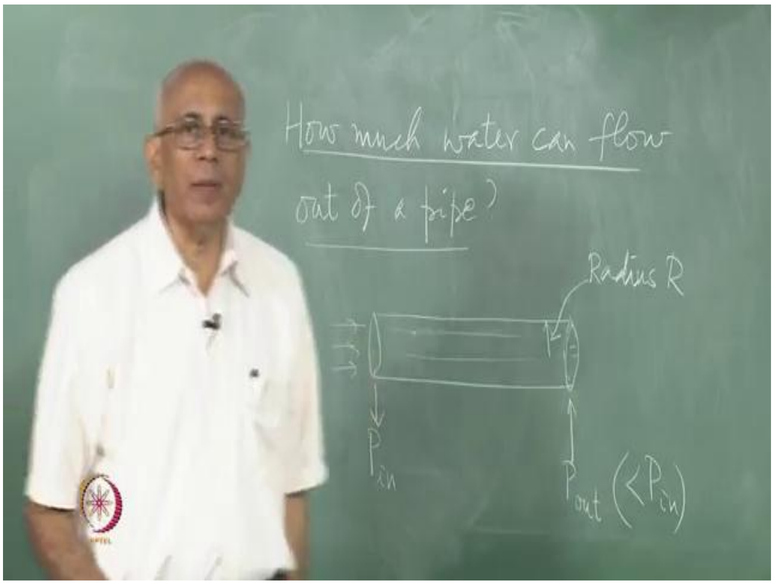

So, this ((Refer Time: 20:11)) how much water can flow power of a pipe. Now of course, this depends on this size of the pipe, it depends on the force that pushing the water through it depends on the speed ((Refer Time: 20:37)) what I comes out whether the flows move then uniform or weather stable and so on. So, to make the problem precise we are going to put in some conditions here and see what this flow look likes and other was ideal conditions.

So, I imagine having let us container observe over head water tank and then I imagine flow in a horizontal pipe. So, that we avoid the problem due to gravity being different in a putting the factor this what I will accelerate and the gravity you do not want to avoid that and write now I am asking what happens if there is an inlet here. So, what is flowing in here, the water flows through this pipe and that is the out that yet.

There is of course, the pressure difference between this inlet and outlet which is what is causing. So, water do flow this a pressure head bought in from some else. So, let us called this P in and the pressure here P out and this is less than P in. So, this it difference and the pressure between these two points which is what is causing the water to flow out. There is another physical parameter and this problem which is the radius of this pipe.

So, let us call that r radius we will make a term on the assumption which is that this flow is move then uniform it is call stream line flow. So, we will assume there is no turbulent there are no radius, there is no form, there is no bubbles this is none of that, but very slow smooth flow of this fluid. Such that, the path of any fluid particle is the straight line from this n to n ((Refer Time: 22:21)) goes out on this side and these conditions can be see how much flow this coming out per unit time out of this it is called the discharge rate and can be write down formula for it.

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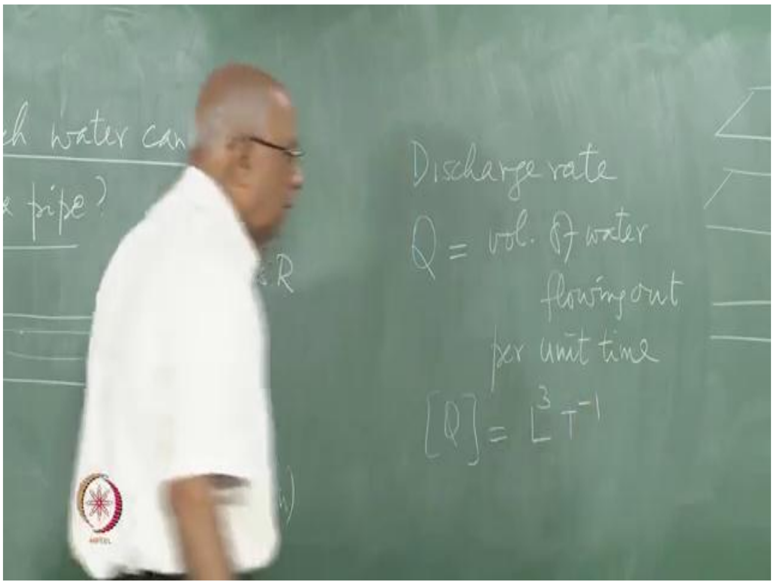

So, the discharge rate it is called Q this is equal to volume of water flowing out per unit time. So, write away tells you that the physical dimensions of Q is the dimensions of volume which is L Q per unit time, so with in this. So, the immediately know that is got to the some L Q T in this that is a volume per unit time. Now, comes to the following question, how does it depend on the radius r, but we can give a very simple argument for this radius r and it goes as follows.

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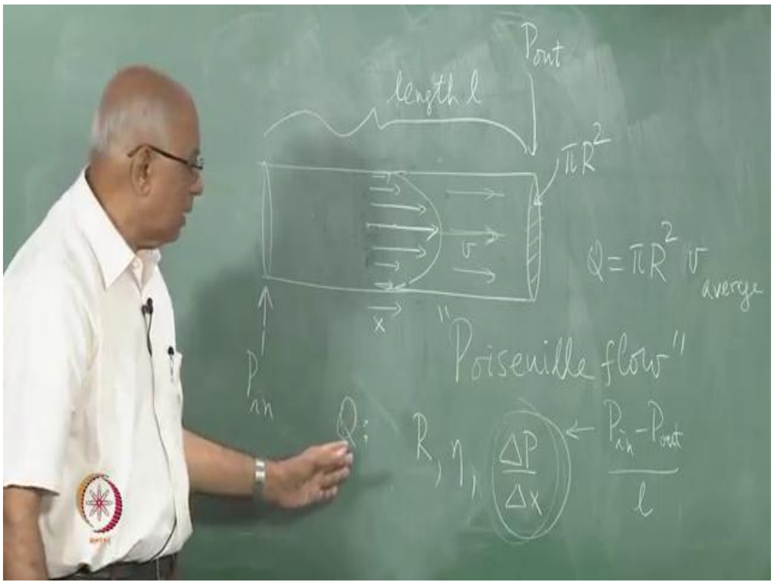

If here is the pipe and he is the outlet the out of marker is pipe, then if the flow this point with the uniform velocity as p v inside, then it is clear that all the fluid that crosses this area which is phi R squared in unit time will come out of this and that is going to be your Q. So, it is immediately clear that of this speed is uniform across Q must be equal to phi R squared times v and seems to suggest that for a given v, this is proportional to the square of this radius.

So, if you increase the radius by a factor 2, the amount of volume of water coming out per unit time increases by a factor of 4. But, actually the truth is well different from this it will turn out that the dependence on this radius is much more severe than just R squared it will turn out that it goes like the fourth power of this radius. So, if you double the size of the radius, then immediately the flow of waters increase the discharge rate is increase by a factor 16 proper then just 4 and let see how that happens.

Well, it is happens again due to viscosity, the same factor as before and it happens, because the fluid that is in contact immediate contact with the walls of the pipe are a practically 0 velocity this stack, because the previous viscosity. On the other hand, the fluid that is in the middle is flowing on humped it a much higher velocity. So, what really is happening is that you have velocity here in this fashion and have a velocity here which is less and then even small velocity and then even much less smaller velocity etcetera.

So, there is a profile this little profile here of this velocity as a function of the distance from the access of the tube, the velocity drops it is maximum at the middle and goes down to 0 radially and all signs. So, this formula as it times cannot be write and what you need is the v average. So, we have to compute what is the average velocity over these various different velocity as speeds of the fluid, some these average was properly some these different velocity appropriately weighted and then find out what the discharge rate is.

And that takes a little bit of calculations it takes fluid dynamics and it is not all to gather it trivial problem, this problem by the way is called poiseuille flow, the same poiseuille after which after whom the unit of viscosity poise is named a French physicist. And now, we would like to find out what is the dependence on this R of this fluid rate and how we going to do this using dimensional analysis alone. But, we need to put a little bit of physics.

By the way there is one more parameter in the problem and that is the point I want to illustrate earlier that this fluid pipe let say this is the horizontal section of it and let suppose the length of this pipe is l the inlet pressure is P in, the outlet pressure is P out it is flowing, because P in is greater than P out is coming out in this direction. So, given this what is going to happen, what is going to be the dependence of Q in various parameters.

Now, what are the possible parameters that we can write this down that we can putting here, if the first one is of course, R Q will depend on R of course, it will also depend on of course, the average velocity which I pointed out depends now on the viscosity of the fluid. So, certainly there is the dependence on the viscosity of the fluid and then it will depend on the force at pushing it and the force that is pushing it is P in minus P out, the pressure gradient at every point there is a pressure gradient this pressure here is higher than the pressure here is higher than the pressure here which is write and then pushed out.

And the pressure gradient is delta p over delta x, where x is the distance along the x axis along the pipe note here. Now, this delta p over delta x if the pressure gradient is uniform across the pipe, this quantity here is the know what it is, it is equal to P inlet minus P outlet divided by l the length here. If the pressure gradient is exactly the same at all points, then this must be equal to this difference divided by the length of the pipe l.

So, you see what is happen is that even though we have two quantities of dimensional length, one is the little l and the other is a capital R and that combination of these two the ratio of this two is independent of dimensions as no dimensions. And therefore, they could be some unknown dependence on a function of R over l or l over R we got over that by the physical argument by saying that the l dependence is going to come in the pressure gradient here in that sitting here and this gets. So, I treat this as a unit this as a unit, this as a unit and now asks what is this discharge rate from to be like and that is now attractable problem.

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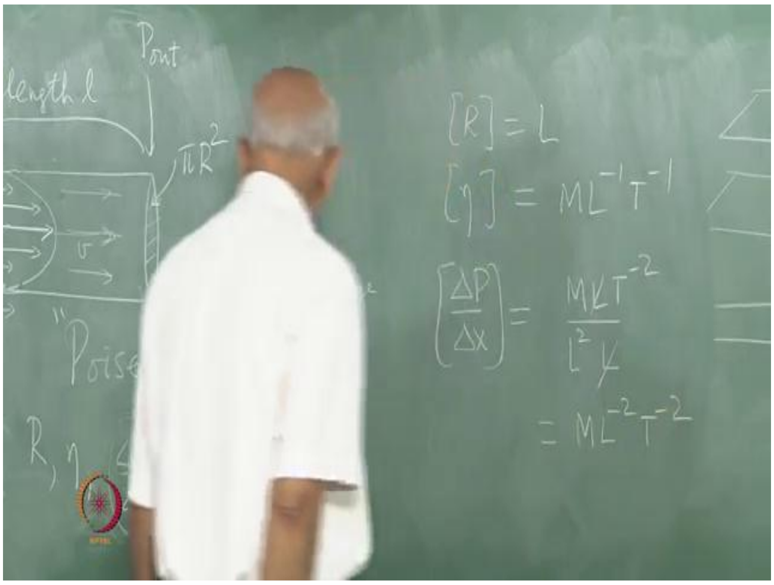

Because, I immediately see that I need the physical dimensions of all these fellows dimensions of R of courses length, dimensions of eta the rote that down earlier it is M L inverse T inverse here and then delta p over delta x, what is the physical dimensions of this quantity here , well pressure is force per unit area. So, this is M L T to the minus 2 over L square and then a delta x is a length. So, there is one more L out here this cancels and this is equal to M L to the minus 2 T to the minus 2.

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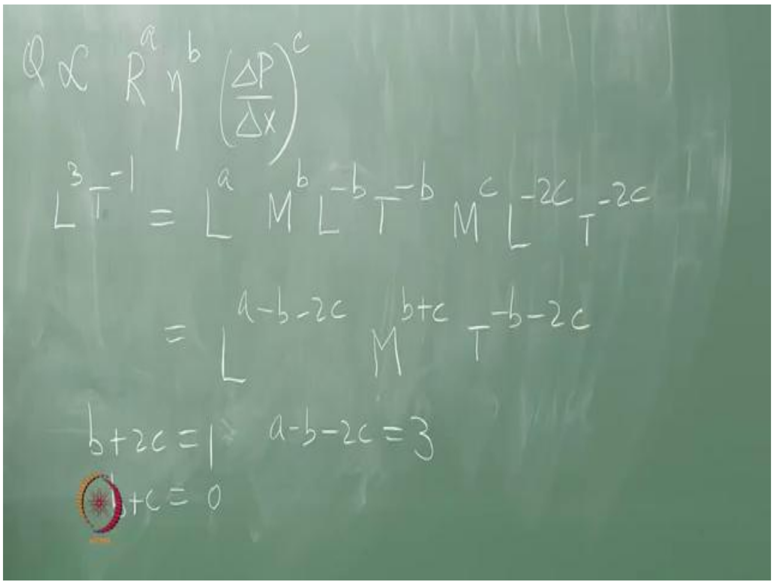

Now, we are all set because all we have to do now is to say that this Q is proportional to on this side R to the power a eta to the power b and delta p over delta x to the power c and write down all equations which is L Q T inverse must be equal to L to the power a eta to the b is M to the b L to the minus b T to the minus b and then delta p over delta x to the power c which is M to the power c L to the minus 2 c T to the minus 2 c. So, this whole thing combine in the everything together is L to the power a minus b minus 2 c M to the power b plus c and then T to the power minus b minus 2 c and that must be equal to this.

So, we have a simple set of simultaneous equations source b plus 2 c must be equal to 1 that this on here, when b plus c must be equal to 0 conclude and a minus b minus 2 c must be equal to 3 and all we have to do is to solve these three equations together and that is a trivial thing to do and you have unique solution and the solutions terms out b and this is something left to you as an exercise, the solutions turns out to do the following.

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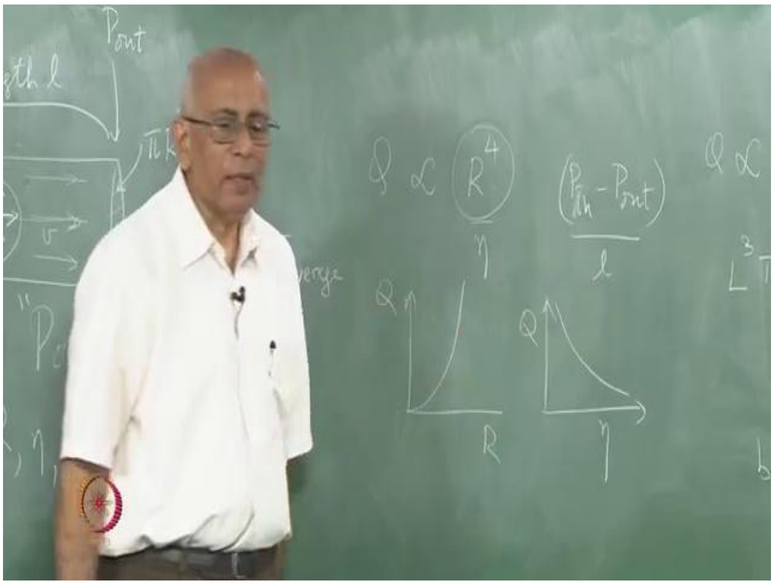

Q is proportional to R to the power 4, so this coefficient a this number a and that comes of the 4 and then is proportional to delta p over delta x. So, it will turn out be P in minus P out divided by lout here and then there is an eta denominate and that is the formula, this constant once again in this case it turns out to be something which is proportional to phi over eta something like that is some pure number, this quantity here this dependence here R to the power 4, says that if you give me a fix pressure head and a length of tube by fix length tube for given fluid under these conditions of was I flow.

Then, the discharge rate is proportional to the fourth power of the radius and we got this answer inspirit to the fact that you also had another length in the problem and therefore, potentially that could have been some function of R over l or l over R in the formula which would happen, we got this out explicitly by saying look the argument was a physical one with said dependence on what the pressure difference is pressure variant is and this appear only in the pressure gradient here.

So, in this sense we been able to sacrum went this problem of more than one parameter with the same physical dimensions an actually extract the formula out. When squared surprising formula that this the fourth power here, so really says that if somebody quietly doubles the let pipe radius, then he of being gets actually 16 times as much water, the discharge rate increases drastically. So, this is therefore the very, very sensitive quantity here.

Now, whenever you have formulas of this kind what you have to do is to try and graph these formula. So, you what to see if it makes the physical sense or not by plotting these things. For instance, if our plot it Q verses R as the function of R keeping the other things constant, this fourth part increases very rapidly it is very flat here and increases rapidly, if I plotted this Q as a function of eta for example, the viscosity for different fluids, then this thing goes like one over eta goes down in this fashion, similarly for the length and so on.

Now, again this formula is derived and a very, very specific assumptions, you want a length which is much greater than the radius, you do not want the sort pipe in turbulence sets and various other things happen, you want stream line flow, you want constant pressure gradient and so on. So, under all these conditions of course, I flow this is the formula for the discharge rate, which we form it without too much difficulty using just physical arguments. So, this is the power of the dimensional analysis and we will have vocation to look at other example has be go along.