Mechanics, Heat, Oscillations and Waves
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Lecture – 29
More on Simple Harmonic Motion

We are worked out some features of a Simple Harmonic Motion and looked at some physical examples of simple harmonic motion. What I like to do now is to gone and introduce a concept, which is not quite in the curriculum of the high school physics that way use to, but it is, so important and so basic and the study of dynamics that I thought it is worth introducing it if and only in the context of simple harmonic motion itself.

And so very simple idea, but a very powerful one like many simple ideas are and it is about face phase face plane phase trajectory and so on. So, let me do this with the help of an example.

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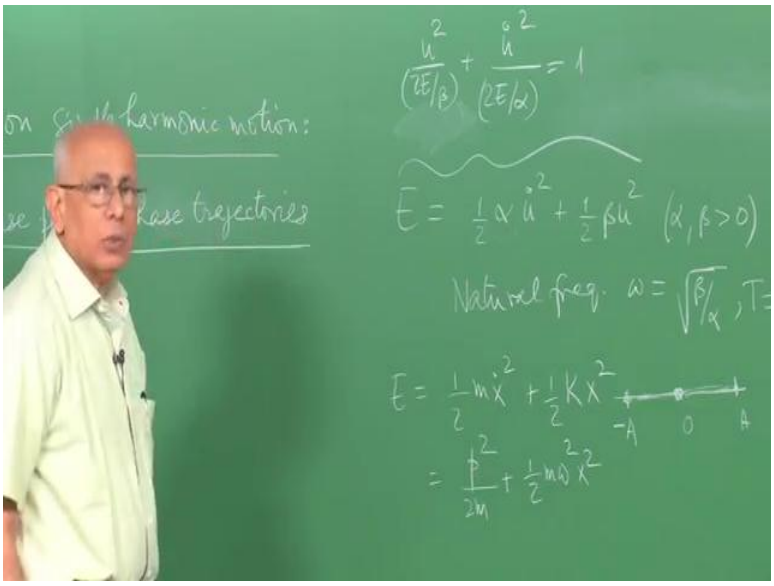

Now, to recapitulate what we have done earlier we found that if you have the variable u of t whatever, it be a displacement or a charge or a liquid level, whatever it be it displace simple harmonic motion if the following thing happens. If, the energy of the system of this is of the form some constant times u dot squared plus some constant times u squared. So, if the energy is a quadratic function of this kind be the u squared term and u dot squared term, where alpha and beta a positive constants.

Then, u executes simple harmonic motion and time it various sinusoidaly are like some super position of cosines and sins with a certain natural frequency omega, which is equal to the square root of beta over alpha. In the case of the spring and mass system we saw that the energy in that case was E was equal to 1 half m x dot squared plus 1 half spring constant times x squared. So, in that particular problem x add the physical meaning of displacement from some center of osculation and the coefficient alpha was just the mass of the oscillator and the coefficient beta was the spring constant here.

And the ratio of these two ((Refer Time: 02:29)) the square of the natural frequency, the time period is of course 2 pi over omega T is. So, that was one basic characteristic of simple harmonic motion, you could rewrite it in an equivalent way, which is the terms of the equation of the motion which is a red u double dot equal to minus omega squared u, where omega is given by square root of beta over alpha. So, whether one uses this form of simple harmonic motion or this form he does not matter it conveys exactly the same information.

So, that is the first point about simple harmonic motion, whenever you have a total energy that is the quadratic function of the velocity corresponding time derivative of the variable. And the variable in this form you and up with simple harmonic motion, a small extension of it is that the center of oscillation need not be yet u equal to 0. So, if for instance I modified this and wrote the energy as 1 half alpha u dot squared plus 1 half beta times u minus a whole squared, where a is any real constant.

Then all it does is to shift the center of oscillation, instead of oscillating about the value u equal to 0, this variable u oscillates about the value a here. So, it does not matter in either case, this is a trivial change of variable one can always called u minus a equal to u prime. And since u prime dot is a same as u dot you get exactly the same physical motion about the new center of oscillation, so that is a very, very simple generalization.

Now, given the fact that you have an energy, which is looks like this it is clear from here, that the energy cannot be negative, because these are squares and these are positive coefficients here. So, the least value of the energy is 0 when there is no motion at all when the system does not move you remain 0 at all times. But, for all other values of E all positive values you have solutions here, which are non-trivial motion about some center of oscillation.

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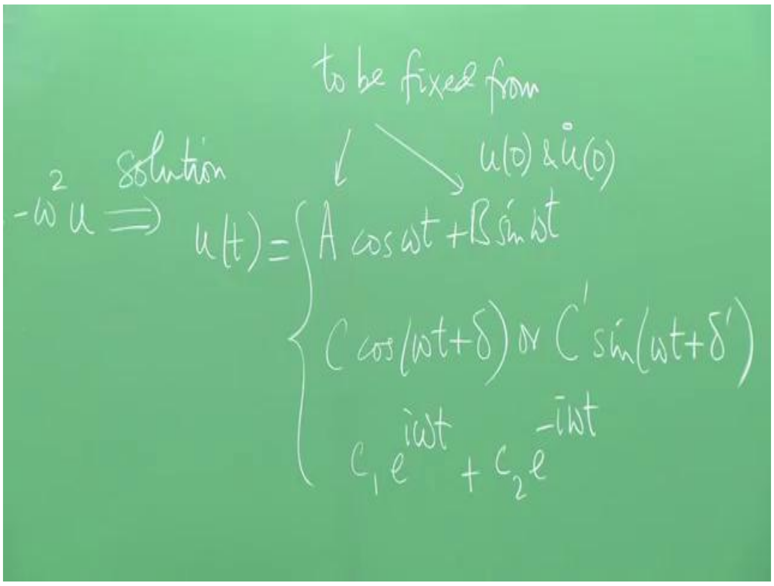

Now, one way of understanding this motion is to explicitly solve this equation. And we saw that the solution of this equation could be written implies solution, which looks like u of t in the general case is of the form A cos omega t plus B sin omega t, where a and b are constants to be fixed by specifying the initial value u of 0. So, these constants are to be fixed from u of 0 and u dot of 0.

So, if you tell me, what the initial velocity code and code u dot being velocity is and the initial position is, then I tell you what these constant is a and b are and this is the general solution. But, you can also write this general solution is another way you can write this as c cosine of omega t plus delta r for that matter some c prime sin of omega t plus delta prime these are all equivalent to ways of writing the solution. Because, this thing here, when expanded will give you two constants this is c cos delta and there is c sin delta, which are two different constants.

Similarly, when I have a sign here or is yet another way, which is to write it as c 1 e to the i omega t plus c 2 e 2 the minus i omega t. And this is going to involve complex constants, if you want to solution to the real, then these constant are necessarily got to be complex. But, again as you know since e to the i omega t or minus i omega t is a linear combination of sin and cosine omega t he does not matter.

So, these are different ways of writing this solution down, what is clear; however, is that, if I imagine in this problem for instance that this represents of particle, which is oscillating about the point 0 with some amplitude. Let us call this amplitude A and some amplitude here minus A, then it does not really matter, what the initial conditions are as long as the amplitude is the specified or the total energy is specified. Because, I could start with the oscillated are the mass pull to the right extreme, let go in which case spring pushes it back here, over shoots goes here goes back it should be write at the center here.

So, that is, what the motion does oscillatory motion about this point, but I could also started from this point I could have stretch it to the left and let go in which case it moves back and forth and exactly the same path. Or I could have started from the center by giving it a flick with the right amount of velocity in order for it to reach the point A and come back and ((Refer Time: 07:42)) such path and go on this oscillatory path once again. So, it is quite evident that an infinite number of initial conditions or accommodated in a given motion with the given total energy. How do we take that in to account? How do we put that in to understanding of the osculated? In the way is done is a as follows.

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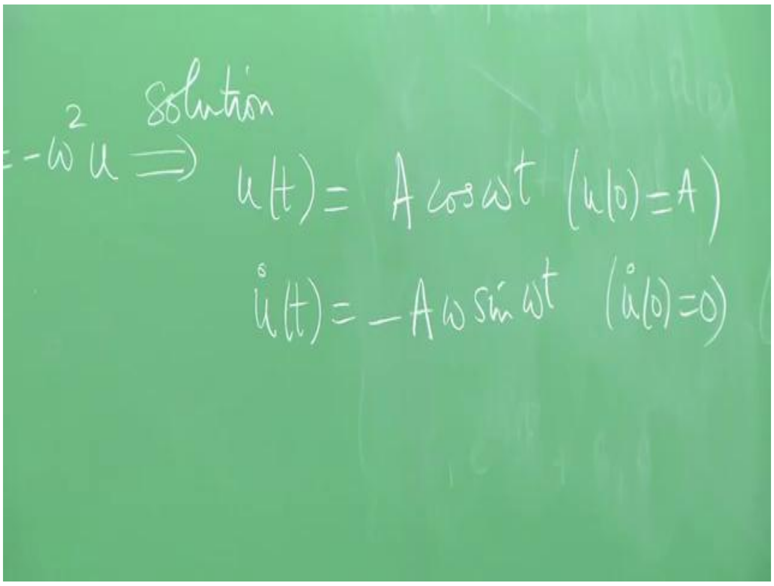

Suppose you solve this equation and you discovers just to be specific that the solution is A cos omega t this means that at t equal to 0 at time 0 u of 0 is A. So, we started here at this point and then, u dot t equal to minus A omega sin omega t this means at t equal to 0 you starting from rest, because u dot is 0. So, it is equivalent to stretching the spring to it is right most extreme to the end of it is amplitude and let in go with the spring in the mass initially at rest, so u dot of 0 is 0 and u of 0 equal to A.

So, for this particular solution this solution corresponds to those specific initial conditions. What happens next? Well, as you can see physically this is going to be move backward here. So, u is going to decrease, but u dot is going to increase let it is a largest value here, negative large value goes all the way here tops and goes back in this fashion here.

Each time it crosses 0 it is kinetic energy is at a maximum and potential energy is minimum each time it heads the end point, the potential energy is that the maximum and the kinetic energy is 0 and this turning points.

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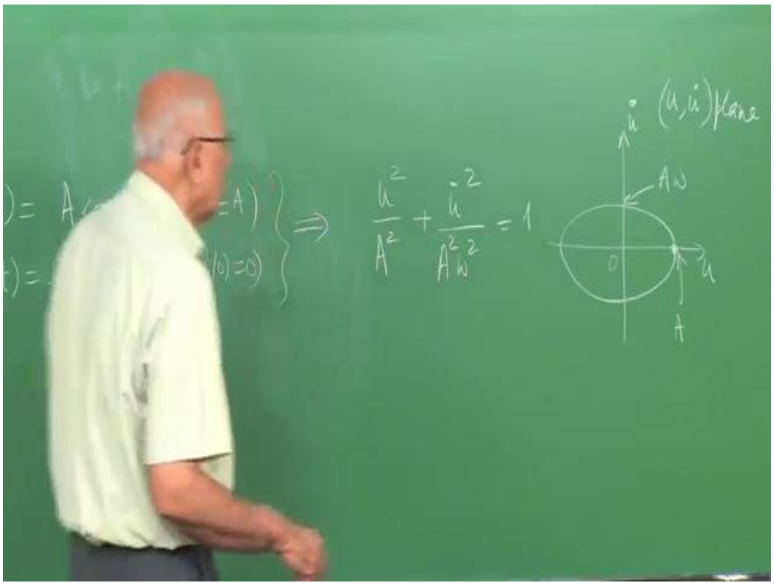

So, how is that taken in that account can be done by just looking at the solution and it immediately follows this solution here. That u squared over A squared plus u dot squared over A squared omega squared equal to 1. What we done is, to eliminate the time between u and u dot and we that in equation with actually connects u and u dot in this fashion with no reference to time at all, which means it must be two at all times this thing here must be all times.

And you see the already present here in this equation independent of, what the particular solution was this of course, you will immediately say where. of I plot it u verses u dot, so I plot u here and u dot here. Then, this curve is an equation to ellipse a well known equation to an ellipse; such that, looks like this I will come an minute to this direction this ellipse is traverse this here is the semi major axis, which is a and this here is the semi miner axis, which is A omega note here.

And the ellipse is center about the origin in the u u dot plane, so I should call this, the u u dot plane and it calls the phase plane in this problem. But, will come to that the second and this tells you know that for this motion at any instant of time the system or the particle is on this ellipse and no other place at t equal to 0 u is a and u dot is 0. So, you are at this point here and the question is, what happens a t increases to positive values.

Well physically you can see that, if this for a mass attach to the spring and stretch as soon as the let go kind of it amplitude the velocity is going to be directed towards the negative side. This means, u dot is count to be negative on this side and u is decreasing and does one you one way that can happen here. ((Refer Time: 11:57)) And u decreasing the comes back here, but u dot is a negative goes here..

So, that direction of the arrow is fixed by the simple physical consideration when it reaches the lest most extreme it is like this at this point. And then, u dot the comes positive it is not going back towards the origin goes through the origin here and u and goes back and replaces this path over and over again. So, it remains on this ellipse and that ellipse is nothing but, this thing here already you can see that from this equation here.

I could write this u squared over 2 e over beta plus u dot squared over 2 e over alpha is equal to 1 the same ellipse. As before, we already know that they energy is related to the amplitude this case by e equal to 1 half beta A squared. Because, from the kinetic energy 0 the potential energy is the potential energy is that the maximum and happens when u in the maximum which is a.

So, this equation that I have return here is nothing but, this is the same thing all you have to do is to identify e with half beta e squared and omega with omega squared with beta to the alpha this as same ellipse as before. This ellipse is called a phase trajectory and this plane is called u u dot phase plane is called the phase plane. It is customer plane form technical reasons to plot not u verses u dot in many cases, but if it is a particle moving for example along the x axis.

Then, it convenient to write is energy I terms of the momentum p squared over 2 m plus 1 half m omega squared x squared I want remove this k in write it in terms of the natural frequency omega. And then, the equation to the ellipse in this case would be x squared over 2 e over m omega squared plus p squared over 2 m e equal to 1 and the phase trajectory in this case this equation here.

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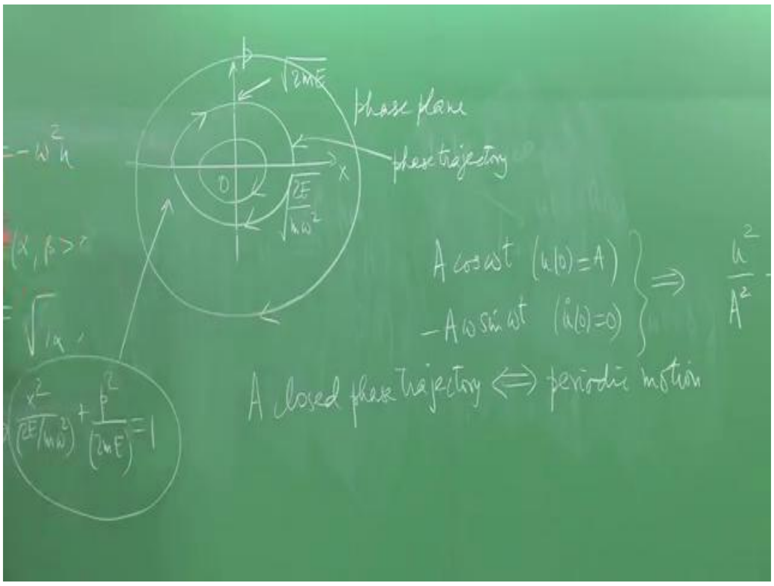

This ellipse here would look like this here x here is p and the ellipse looks like this travesty in this direction for reason, which have already plane this is 2 E over m omega squared square root and this point is 2 m E. So, this is called a phase plane and this is called the phase trajectory. And, what we see immediately is that this trajectory connects, now immediately motion for different initial condition, but the same total energy.

If, you started here it would correspond to pulling the oscillated to left extreme and let in go from rest. If you started here same thing pulling it to the right extreme let in go, if you start at this point it corresponds to flicking it to the right with the right velocity from here flicking it to the left, left velocity. But, all these motions like an exactly the same pace trajectory, what happens in the total energy changes when you get another ellipse as you can see the semi major, semi minor axis depend on root E.

So, you have another ellipse for this kind for a large value of E you have another ellipse for a smaller value of E the all must ever in the same direction. So, the phase plane is a beautiful way of eliminating time. Understand from the way the behavior of the dynamical variables and understanding in motion in terms of a trajectory in the phase of a variable and it is a velocity or momentum corresponding moment. And this concept is generalize when you have more decrease of freedom when you have a higher dimensional motion and so on.

Because, it kind of a it helps us to appreciate the important crucial fact that in all these cases the dynamics id happening not in only in the space of the variables and serves, but in a space, which is that of the variables as well as the time derivative. Because finally, Newton's equations when you write it down in this case for instance we have due double dot equal to minus omega squared u. So, the equation is a second order equation time therefore, the solve I to unique you need knowledge of not only the initial value with the variable, but also the initial value of it is time derivative.

In other words, dynamics happening really in phase space and not in the space of the variables are alone. What you see in the space of the variables alone is only half the story the rest of it is determine by what happens in the velocity space in put those together you get the idea of the space plane. Now, one important observation is that when you have a closed space trajectory as in this case.

If I start at some point here I come back to it after certain amount of time implies that all the variable concerned has come back to the starting point. And therefore, the motion is periodic, in other words closed space trajectory implies periodic motion and vice versa. So, important that I write down a closed phase trajectory implies and it is implied by periodic motion.

Of course, more general periodic motion does not necessarily have to have ellipse is as phase in the trajectory closed curves in any other closed curve would do here is a simple example, which brings this point home immediately. Let us, consider of ball bouncing of the plot up and down.

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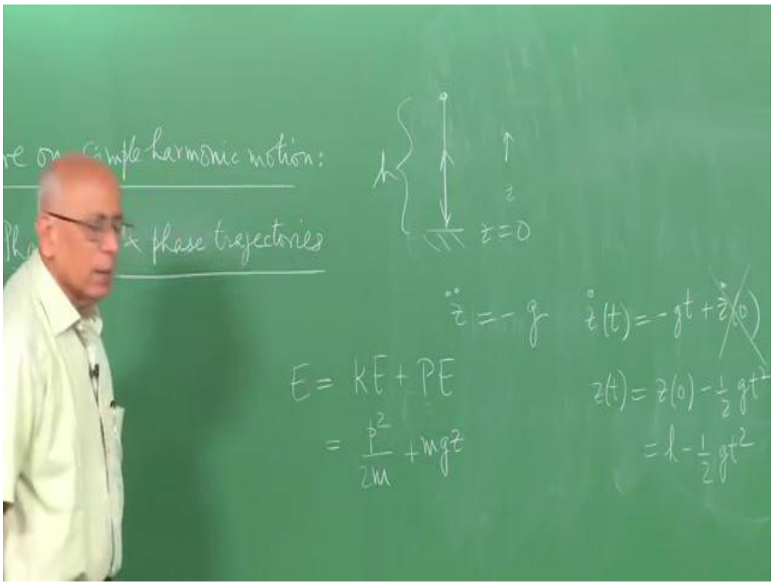

So, here is the flour level, let us call it z equal to 0 and I drop ball from a height h when in drops down hits the ((Refer Time: 18:32)) ground and goes and bounces up and down and let say the z coordinate is measured from the ground level upwards. So, that the maximum value is h for this given initial condition. The flour is assume to be perfectly region and the ball is suppose to Suffern no loss of energy is a goes up and down this is no air resistance this is periodic motion, but it is not simple harmonic motion can be describe it phase trajectory.

Of course, we can solve for this problem in very, very simple way if z is a vertical coordinate I know immediately that z double dot equal to minus g, because the acceleration due to gravity is constant. If, I solve this equation here I get z dot of t equal to minus g t plus z dot of 0 and if I drop the wall ball from rest, then this goes away. And, if I integrate this equation I get z of t equal to z of 0 minus 1 half g t squared and if I set z of 0 equal to h, then this becomes h minus half g t squared.

Of course, when it is hits ground the velocity or momentum is instantaneously revered same magnitude, but it reversed up. So, in this sense the flour is acting in a barrier and the motion in velocity space is discontinuous because a velocity changes discontinuously minus some value, because plus some value the same value instantaneously. Now, how do we write the phase trajectory well I can eliminate t between these two, but I do not need to do that I can do this little more simply by simply writing down the total energy of the system this is the kinetic energy, plus the potential energy, in the kinetic energy at any point is p squared over 2 m.

So, let me write it an terms of momentum rather than z dot become as I said, but technical reasons in more complicated problems, that is the more appropriate thing to do. So, p squared over 2 m plus the potential energy, which of course, is m g z. Now, all I need to do is the fact that is no loss in the energy in, therefore this is equal to constant of the motion. And were all set to write the phase trajectory down in any initial condition for any specified initial condition.

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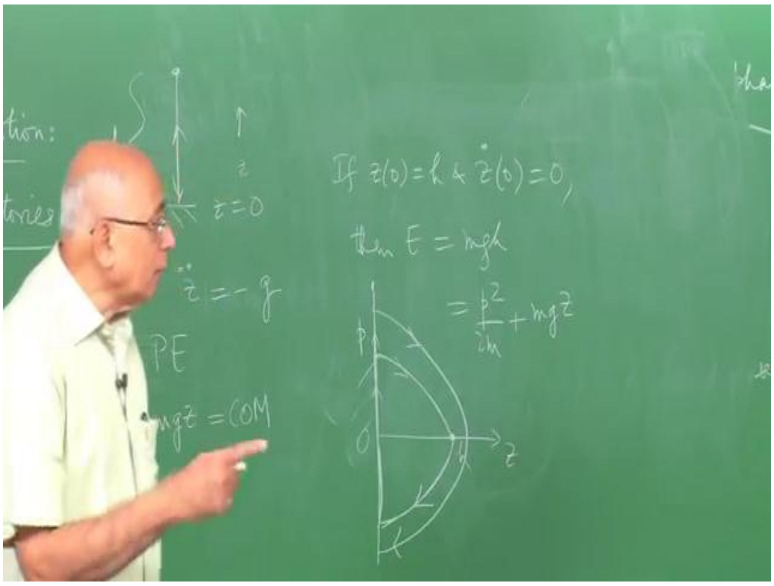

IF z of 0 equal to h and z dot of 0 equal to 0 when the total energy is m g h, because of start here and at rest thus no kinetic energy the potential energy is m g h and I let go that is the total energy and this must be equal to p squared over 2 m plus m g z at any point in the trajectory. Now, what is this look like this is variable in the y axis the p axis, which is square here and you got linear term here.

Therefore, this answer must this curve must is some kind of parabola and indeed it is, because, if I plot this, let us do that here is z I do not plot z negative, because it cannot go through flour, let us a barrier this point. And, if I started height h at rest and starting here, and then, what do I do the ball accelerate downwards, so the moment is directed downwards till it is the flour and it is in a parabola.

Because, this is quadratic and that is linear and that is the constant there and this is the part of parabola it hits this point when comes here. This distance is h from the origin and it moves in this fashion till hits the flour and, what is the momentum, where it hits the flour again that is the trivial thing to do. Because, when z is 0 all we got do is set p squared over 2 m equal to m g h and we know what p square root of whatever it is and instantaneously it is reverse to it is plus value.

Because now, as soon as the flour at the momentum is reversed without lose of magnitude, so it is some point, which is equiv. distance here and it traces than the rest of the parabola. So, this corresponds to the ball falling down and this portion corresponds to the ball going up and as it goes up it loses momentum. So, the momentum is decrease while it gains height till it h hits.

So, this is the parabola part of the parabola and then, discontinues to jump here, and moves to this part that is the phase trajectory and does not look anything like an ellipse. So, this motion is not simple harmonic, because the energy is not quadratic in both the momentum as well as the variable itself. But, it close trajectory and corresponds to a ball falling up and down and making oscillatory motion, but it is not simply harmonic.

And of course, we know that the time period for this is dependent on the amplitude are the height, to which from which you dropped. And therefore, it certainly not simple harmonic, what happens for drop it of greater height, well the space trajectory would be a bigger parabola and so on. So, for each time I have periodic motion I can associate with the close trajectory of the some kind and this trajectory will give me information on what the emotion is actually like without reference any specific initial condition.

So, that is the great advantage of the phase trajectory tells you something about the way this is behaves independent of the specific initial conditions. So, whole families of initial conditions the taken care of and that is the lest of the that is the main advantage is of using the space phase description I set phase space have set phase plane, because in this problem is only 1 degree of freedom u or z or x and does the corresponding velocity.

So, the phase space variable I just two in number the variable and it is derivative. But, if I liked at motion for instance of a plant around the sun when you have three coordinate in whatever spherical of polar or whatever coordination system. We choose to describe it in Cartesian was ((Refer Time 25:04)) does not matter you have three coordinate and you correspondingly have three generalize momentum or the last it is and the space phase would them be the six dimension.

If the particle moves on the plane the x y plane the corresponding space basis four dimensionally. Then, becomes harder and harder to draw pictures but one can always drop the directions of this phase trajectory on to specific planes to at a time this times; obviously, done. So, the moral of the story is that periodic motion corresponds to closed space trajectories and you can get information from the phase trajectory about the time period for the instance. For instance for a problem like it is clear by symmetry that the time taken to go that here to here is the same as the time taken for each of the quadrant.

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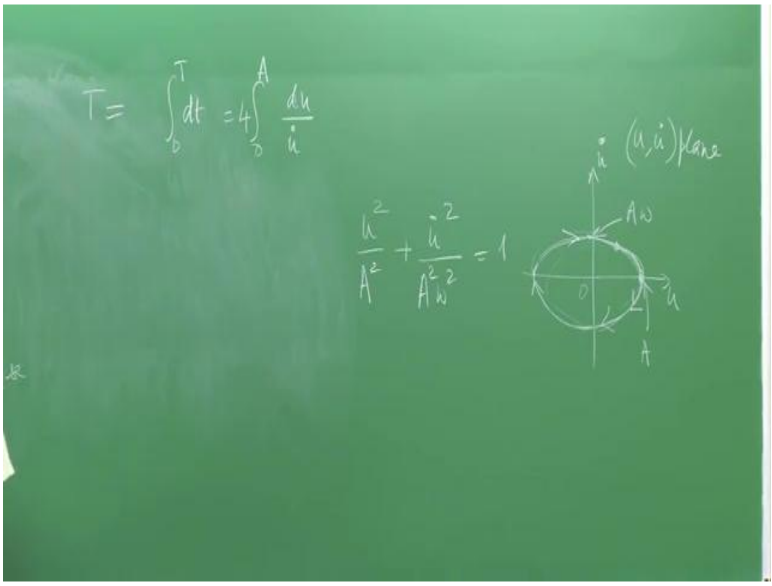

And therefore, the time period of this motion t should be equal to integral 0 to t d t, which can be return as which can be return as integral 0 to an amplitude d u over u dot by definition. So, that corresponds to starting here and going all the way up here for u dot positive and this is four times that, because it is equal to this time plus, this time plus, this time plus, this time dot full time period here. And for u dot all I have to do is substitute here from this expression, whatever this is in terms of u and do the integral here and in principle you have the expression the for the time period for given amplitude.

Of course, in the case of the ellipse it terms out very easily that the time period dependent of the amplitude. But, try this in this problem the time period to go up down is the very trivial thing the compute, but you can also do this by looking at the phase trajectory here by saying I start here and z equal to 0 and go to z equal to h and the time period for a full oscillation is twice that and you need to compute this going to be here which, you can easily do, because you can write down the this p in terms of z for a given value of the energy, which is, so I, which on this case and a simple integral tell you, what the time period is this is the little round about wave we solve the equation of motion explicitly in this case. But, in more complicated cases will do not solve the equation a motion this is will still tell you what the tell time period how this oscillated motion.

All this for motion, which is periodic strictly periodic, but in real life as you know as always combine is always friction, which cases periodic motion to come to halt represent time. So, the next thing we are going to do is to look at, what happens we put a damping and again will do this in the example of the simple harmonic capsulated.