Mechanics, Heat, Oscillations and Waves
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture – 28
Some Physical Examples of Simple Harmonic Motion

What we will do next is look at Some Simple Physical Examples of Simple Harmonic Motion.

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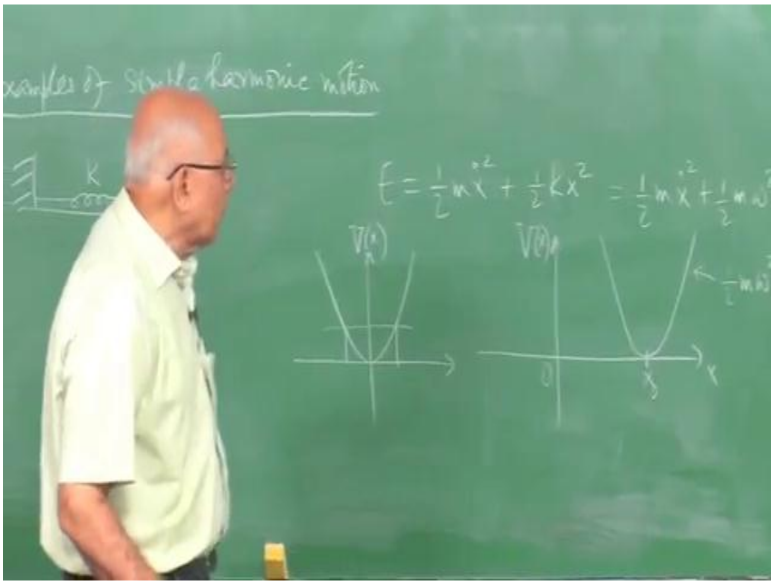

The most standard one of course is to say that, you have mass attached to spring of spring constant K. So, here is you mass m and it suppose to move on some friction less surface. So, here is m, this spring constant is K and you attach the spring to valve here and the mass undergoes simple harmonic motion oscillating back and forth. And this is the very model of a simple harmonic motion, which we wrote the equation of motion down form.

Of course, one could ask, what happens, if I took this mass and suspended it in the gravitational field of the earth, for example. So, I suspend this spring vertically instead of horizontal in, then it of course, bounds is up and down. The question is, is that also simple harmonic motion, answer is yes and actually the more formal way of seeing this is the following.

If I have simple harmonic motion for which that total energy is 1 half m x dot squared plus 1 half K X squared, I do not like to write it as you can see in terms of K and m, because the defining property of the simple harmonic oscillator is actually the ratio of these masses, the frequency. So, let us put that in and henceforth, let us write this as 1 half m x dot square plus 1 half m omega squared X squared, because K over m is omega squared.

There is a much better way of writing this, because it immediately brings out this quantity here and then, the ratio of the frequency squared is of course, a ratio of these coefficients and m cancels out and you get omega, omega squared. Now, what happens in this case is that the potential looks like this, V of x and the particle is oscillating for a given total energy, it oscillates between minus a and plus a, this is the amplitude.

Now, suppose I shifted the center of oscillation to some other point, so let suppose that the potential was like this, here is the X axis; here is the V of X axis. And is the same potential, but it is about some other point X naught here, not necessarily the origin. As long as this does not change or this does not change, the parameter do not change, this does not matter at all. It will now oscillate about the point X naught rather than the point X, but with the same angle of frequency omega.

Now, what is the potential energy in this case going to be equal to, it is equal to 1 half m omega squared X minus X naught whole squared. So, it is a parabola center at the point X naught here.

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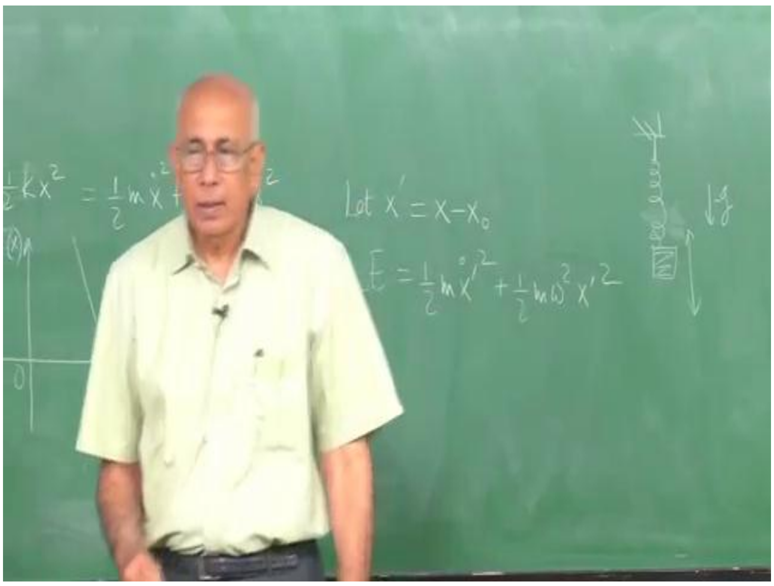

And again, it does not matter, because all I have to do here is to put, let X prime be equal to X minus X naught and since this is a constant, the total energy now becomes E is equal to 1 half m X prime dot squared. Because, whether I differentiate X or X prime with respect to t, it does not matter, because it just a constant shift plus 1 half m omega squared X prime squared.

So, that is the whole oscillator once again, except that the center of oscillation has shifted. And this is what happens for instances, if you put this osculated in, you attached the spring and then, the mass here m and you also have gravity acting downwards. So, there is an acceleration due to gravity acting downwards. What happens is that the equilibrium point is not at 0 extension of the spring, but rather, it is extendable under gravity and that is the equilibrium point.

And then, if you pull it a little bit and let go, it oscillates up and down about that new point, but the frequency does not change, exactly the same. Of course, if you change the potential by adding some other force to it, something that makes a depart from the X squared form, if the other terms here like X cubed and so on, then you are in trouble, always.

What I did by putting it under gravity was to change this by adding a X term m g X ((Refer Time: 04:56)) and then, I can bring it to this form by completing squares. So, if the new potential energy is this plus m g x, you can see that I can complete squares and make this X minus X naught squared plus some constant. And all that, the constant will do is to shift 0 level of the energy, but the physical oscillation will still happen about the new center of oscillation and that will be determine entirely by this coefficient.

So, ((Refer Time: 05:25)) to try this out an exercise by adding a force of gravity here, force due to gravity of potential here. And then asking, what happens in this problem, what happens to the frequency of the simple harmonic oscillator and in this case. But, the trick is very simple; all you have to do is square it and so on. But, if we make it a more series change, if you put X cubed, X 4 and other point here.

So, that the force is conservative, but this no longer a simple harmonic force, no longer proportional to the displacement, but to some higher power of x, then it is no longer simple harmonic motion. It may still be periodic motion, but it will not the simple harmonic motion. So, the mass in spring was a simplest case, well let us look at a slightly more complicated situation and this is the simple pendulum.

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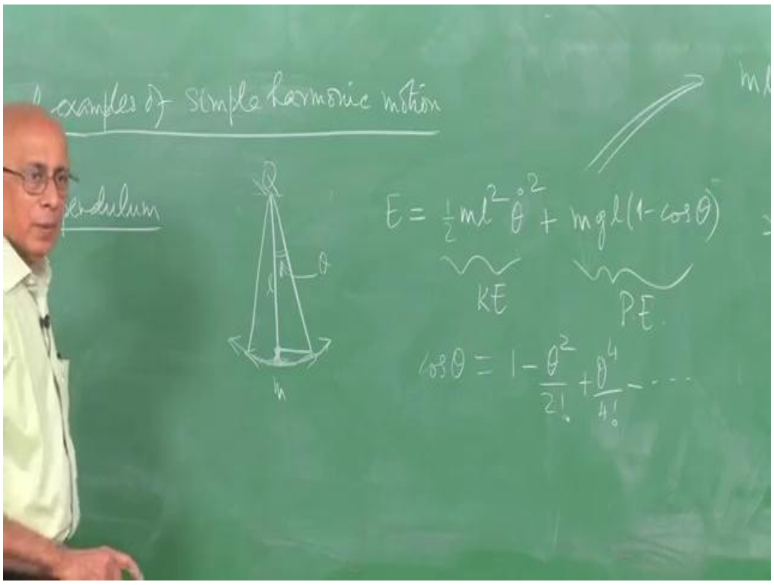

So, the simple pendulum in this case, the restoring force is actually provided by gravity and what happens is that, you have as point of suspension and you suspend a light mass less rod with the heavy bob here of mass m and this under goes oscillations. Angular displacement up to some point and an instantaneous angular displacement is some theta say and it goes up to this point and back in this fashion.

Then, the question is this simple harmonic motion or not, now what we have to do here is to write down the model for an expression for the potential energy. So, in this case that total energy E is 1 half m l squared, because this is l and the energy is a rotational kinetic energy as this bob undergoes an angular displacement. And that rotational kinetic energy is moment of inertia about this point of suspension o, which is m l squared multiplied by the square of the angular frequency, which is equal to angular speed, which is theta dot square.

That is the kinetic energy plus the potential energy and the potential energy comes, because when it is displace away from it is bottom most point, where we take the potential energy to be 0 here. Because, in the reference level is taken with respective this equilibrium point, then the amount to have raised it by is this much, this difference here and this difference is l minus l cos theta and therefore, this is m g l times 1 minus cos theta.

So, that is the total energy and it does not look anything like theta dot squared plus theta squared that would be simple harmonic motion, but this is not simple harmonic motion.

But, we can see physically what sort of motion it is by this procedure of plotting the potential energy, which is what you should always do.

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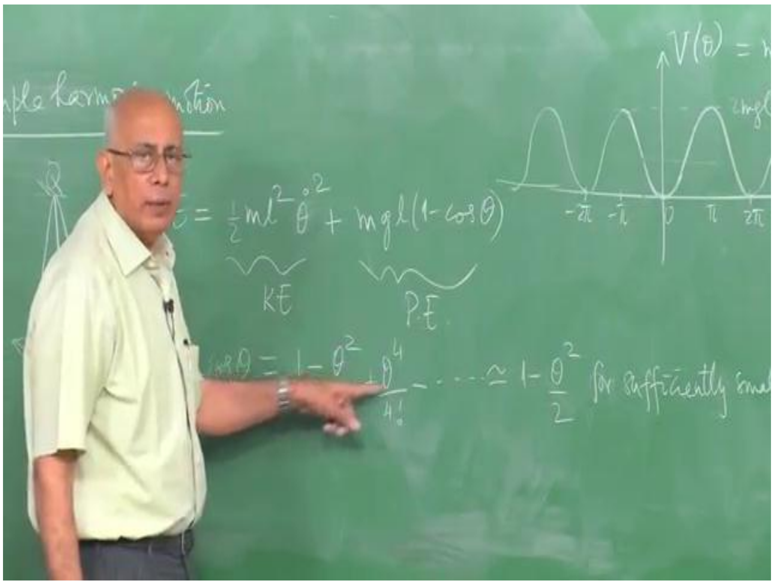

So, this is the potential energy and we should plot here is theta and here is V of theta, which is m g l into 1 minus cos theta and we should plot this function as a function of there. And it is clear that when theta is 0, the potential is 0 and it is maximum; when theta is pi, then it is twice m g l and then, it periodically oscillates back and forth in this fashion. And this point here is pi, this is 2 pi, this is 0, this is minus pi, this is minus 2 pi and so on.

So, for small oscillation this pendulum about theta equal to 0, you have something like oscillation in this potential here. Now of course, you could have a larger oscillation, you could go all the way up to pi on this side, up to minus pi on this side. If you go beyond that it escapes is well and false into rotation, if you increase the energy sufficiently. But, as long as a energy is below this value, this value is 2 m g l, when theta is pi, cos theta is minus 1. That is the largest value for which you can have oscillatory motion.

Then, you are in a potential which looks like this, but this potential is not a parabola, it is 1 minus cos theta and therefore, the motion is general is not simple harmonic. But, it is clear; it is periodic once again, what else will do, if this was the total energy, the pendulum would oscillate between this angle and this angle and so on. But, it is not simple harmonic, when does it become simple harmonic, it become simple harmonic, when you can take this expression for cos theta and approximated by the leading term in power series for cos theta.

So, we have cos theta in radian measure equal to 1 minus theta square over 2 factorial plus 4 factorial minus dot, dot, dot, add infinitive and this is approximately equal to 1 minus theta squared over 2 for sufficiently small theta close an up to 0. For sufficiently small theta, how small, well when you can neglect theta 2 to the 4 compare to theta square. So, you specified degree of accuracy and I tell you how big there should be in order for the approximation to be good.

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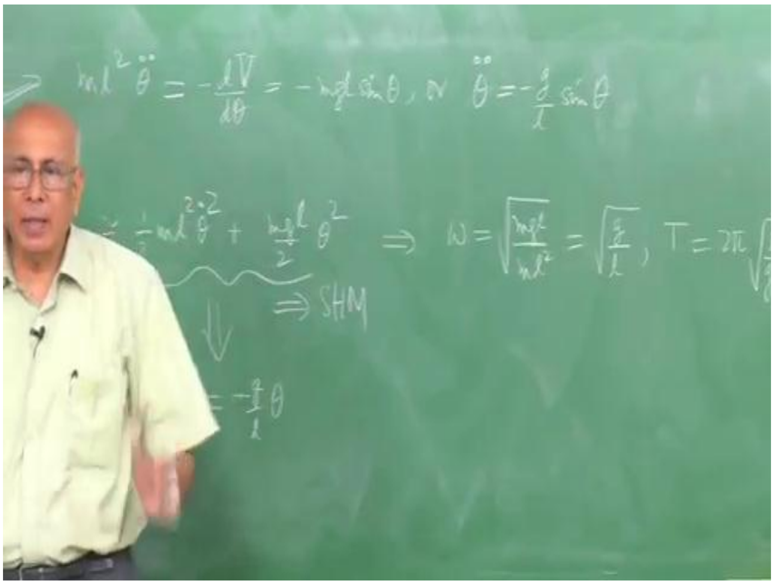

Then, in that case, for small oscillation, small amplitude oscillations, this thing becomes equal to 1 half m l square theta dot square and plus m g l over 2 theta square. The one cancels out and then, you have theta square over 2 and that is it and this is now in the form of simple harmonic motion. This will immediately imply simple harmonic motion, because now you have a dynamical variable theta, the angular displacement, which is got a kinetic energy proportional to theta dot square times positive coefficient.

A potential energy proportional to theta squared, positive coefficient the ratio of these two will give you the angular frequency and 2 pi over that angular frequency is the time period immediately. So, this immediately tell you that omega equal to square root of m g l over m l square equal to square root of g of l. So, the time period is 2 pi square root of l over g for small oscillations.

We do not have to solve the equation of motion for this, because from what we would seen, it is only the ratio of these two coefficient that determines the angle of frequency and therefore, the time period immediately. You could ask alright given this, I know that this will imply the equation of motion theta double dot equal to minus omega square x, which is minus g over l theta in this problem.

That how you get a simple harmonic motion, because we saw already that in equation of this kind, the total energy of this kind implies an equation of motion of this kind and vise versa here. But, now you could ask alright this is so, what is this exact equation imply, what kind of thing will imply; that is not hard to find either, because this thing here will imply that m l squared theta double dot this quantity here is equal to minus d v over d theta, because that is the potential.

And the force on the right hand side is this thing here and d v over d theta as you can see is m g l, you differentiate minus cos theta you could plus sin theta, then you put a minus sin. So, this is equal to minus m g l sin theta or theta double dot equal to g over l sin theta; that is the exact equation of motion for a simple pendulum and the simple pendulum become simple harmonic first small oscillations and then this is the approximate equation of motion.

And when is this valid when theta is so small that the next term here in the expansion of sin theta can be neglected compare to theta, the next term theta cubed over 6. So, you can neglect higher corrections in the power series expansions of sin theta, in powers of theta, then you get simple harmonic motion and the time period is 2 pi root l over g. So, here is a problem, which is actually non-linear, very difficult, because in principle the equation of motion is quite hard to solve with cannot be solve elementary means.

Because, it is not an equation; that is linear in theta, it is not a simple harmonic osculated equation; it involves sin theta here, which in principle has all powers of theta, all hard powers of theta. And yet in the small oscillation limit, the problem becomes very straight forward, it is not hard to see that, the approximation we made is equivalent to saying that sin theta is approximate theta, which is equivalent into saying that sin theta is approximately theta, which is equivalent to saying that the motion is approximately in a straight line which is essentially that approximation that not in here.

So, that is our second physical example of simple harmonic motion in the case of simple pendulum, where small oscillations become simple harmonic for the spring that was exact equation of motion. But, for the pendulum that is not true as soon as the amplitude exceeds a value such that you cannot make this approximation, the time period depends on the amplitude. It is no longer simple harmonic motion. Well, let us look at a example which is taken from say fluid dynamics.

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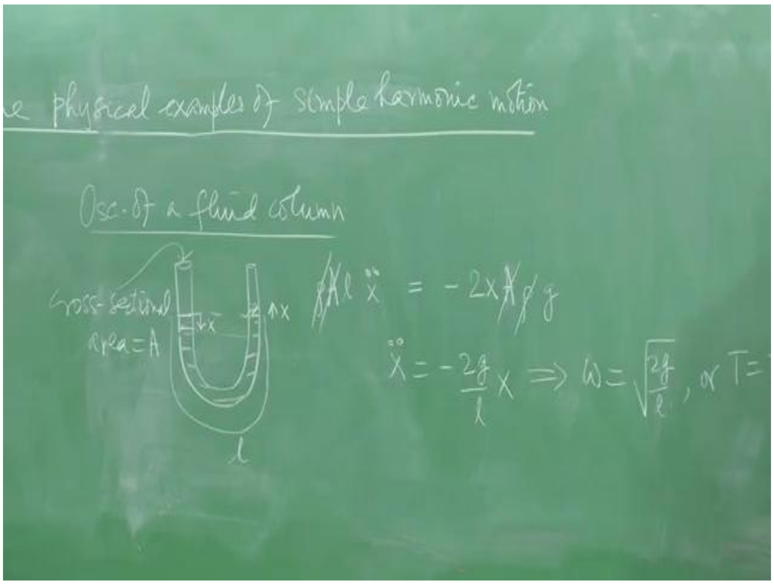

And here is another example, which is an oscillation of a fluid column, let suppose we have a U tube of constant cross section. Although is not essential what I about to say and fill it with what up to a certain level and of course, the level of the same on about sides in this fashion. And let suppose the cross sectional area of this tube cross section equal to A for instance and the fluid density is some rho.

Then, when I this flowed it in you can say that, if I have push this little bit harder here, a flowed goes up there and then, whatever is the exercise there comes down and then the whole thing oscillates and if there is no friction, this will go on forever. If there is friction and we will take about damn simple harmonic motion a little later, then of course, the oscillation subside, but otherwise the fluid in oscillate forever.

And the question asked is now, if I displace this little bit, so that I have a little excess x on that side, height x and on this side goes down by the same amount. So, this it is also x in this side, then the question is, what is the time period of oscillation of this fluid column? Now, that see is little found, because what is the force, the force appear, because this level is out here and the other column has a level, which is higher and the difference in levels now is 2 x.

So, you have something which is 2 x times the cross sectional area times the density multiplied by gravity and that is pushing down to equalize this. So, minus this is equal to the acceleration of whatever is being oscillated and that is the full fluid, the entire fluid. And let suppose the entire fluid, when it is an equilibrium, this whole thing has a length L, then the total amount of fluid is the mass, which is equal to rho times the cross sectional area, which is called A.

So, A here times A times the length times x double dot, so it says mass times the acceleration x double dot of this little element is equal to the restoring force out there. So, rho and A cancel out and I have an equation, which says x double dot equal to minus twice g over l x, this will imply that the angular frequency is equal to square root of 2 g over l or the time period equal 2 pi square root of l over 2 g.

So, we can compute the time period of the oscillation of this fluid element, assuming that is no damping. If there is viscosity, if there is stickiness and this damping of course, damp out. That will come when we discuss the rate at which dance will depend on the friction and this will discuss subsequently, when you take about damp simple harmonic motion.

But, you see in this case, I did not write the total energy down, I found it easier to write down the actual equal of motion. Paying attention to the fact that, what is offering inertia is the full fluid here and what is producing the x as force is the difference in levels, the fluid column which corresponds to the difference in levels and that give as this expression for the time period of oscillation.

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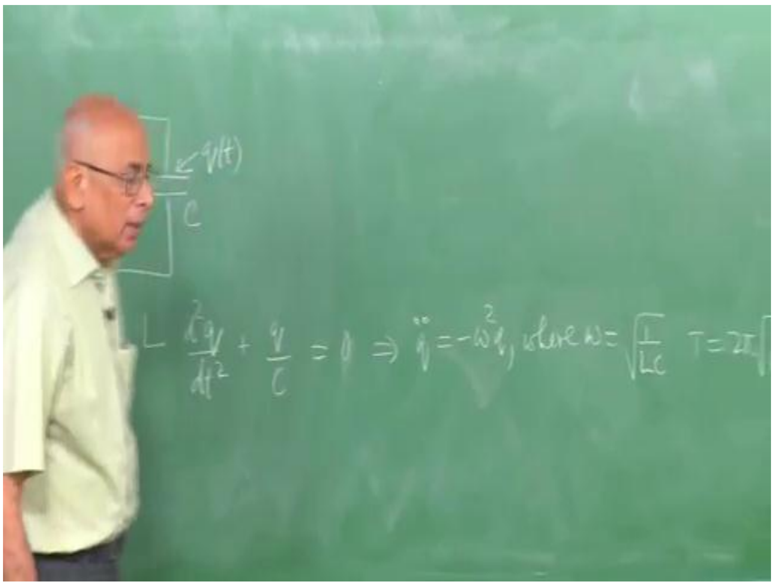

Let us look at an electrical problem in analogy you are familiar with an inductor and an inductance L, if you come if you connected to a capacitance C and an inductive L and you charge the capacitor. Then, what happens the charge get discharge through the inductive and as the current voltage flows as a current flows to this L. There is a back EMF set up and the charge will go from one plate of the capacitor to the other and osculate back and forth, if there is no resistance in the circuit.

A little later will put in a resistance and since, what happens and how it damps out, but in the options of the resistance. What you have to do is to ask, what is the voltage that you have the back voltage here is proportional to l times the charge on the capacitor instantaneously is q, which is a function of t. Then, this is l and d q over d t, well I should really write l times d 2 q over d t 2, because it is the rate of change of the current are the flex that we are talking about.

This plus q over c; that is the voltage across a capacitor; that is equal to 0, when you have free oscillations without connecting it to an external source. If you connected to an external source, the voltage on the right hand side, you would have a forced oscillator; it would have the voltage apply the EMF. So, here we are, this is the equation of the charge instantaneous charge on the capacitor.

But, it is exactly the simple harmonic oscillator form, because this says q double dot equal to minus omega squared q, where omega equal to square root of 1 over L C. So, the time period of oscillation t equal to 2 pi root L C in this problem. So, we do not care as you can see, whether it is an angular displacement or a linear displacement or a charge on a capacitor or the excess height of a fluid column, we do not care. The phenomenal is exactly the same.

Once you bring the energy down to the quadratic form or you bring it term in the equation of motion to this form, you can identify what the angular frequency this in the matter is over. Now, we already seen that, you can displace the oscillator by and identify the frequency, but you can do this in even more complicated situation. When you do not have simple harmonic motion, you can find out and what conditions the simple harmonic motions. So, let us look at another example were we have a particle moving in a potential, but the potential is complicated.

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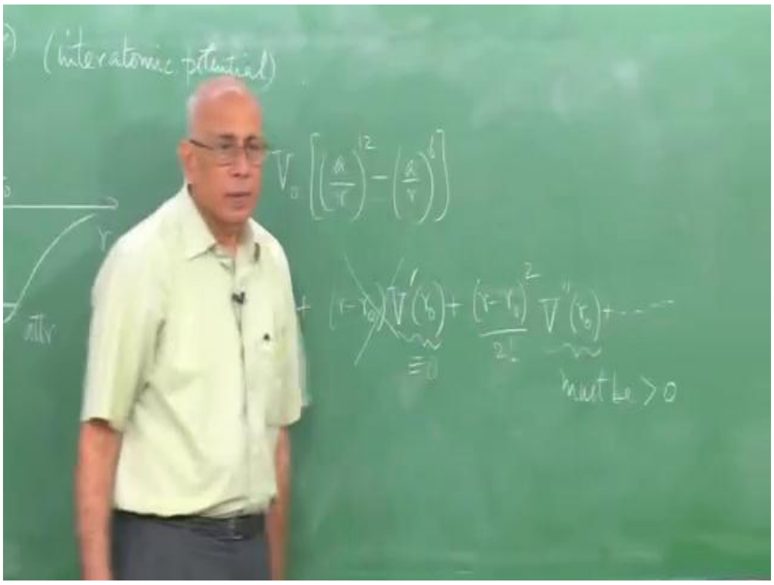

I mention that, if you took intermolecular forces, this central potential in this case the V of r, which is an inter atomic or intermolecular potential in suitable cases. Has the following shape as a function of the radial distance from one had taken to be the origin of coordinates, this potential has a shape which looks like this. And I wrote down a form for this, which was V of r was equal to some constant V naught times a length scale a over r to the power 12 minus a over r to the power 6.

To show that you have a very repulsive force for small values or less than a some constant times a and then, you have attractive force beyond the certain force point. So, till this point d v over d r is negatives from minus d v over d r is positive and therefore, the force pushes you outwards really, it is a repulsive force and beyond this point, wherever this point is minimum. This slope is positive, so minus is negative and the force is radially inwards, so you have repulsion and attraction.

So, this potential models a fact that when you build two atoms very close to each other, electrically neutral atoms force, but one the sufficiently for apart this weak attractive force in this form. Now, if you put the second atom in this potential well, it will undergo oscillations about that point. Now, this is not a parabola, so the motion is not simple harmonic in general.

But, if you call this point, where the minimum is, which is 2 to the 1 6 times a is you can find out from there. So, let us call this point r naught. When as long as you have very small amplitude oscillation about this minimum and this is a simple minimum, a simple minimum always looks like a parabola in between. So, as long as you have small oscillation about this point, the potential V of r about this point can be return is V of r naught plus.

And I do a Taylor expansion of this potential about that minimum. In the Taylor expansion if you recall says value of the function at point r is a value at some given value point r naught plus r minus r naught times the first derivative at that point plus square times a second derivative and so on. So, the first term is r minus r naught v prime at r naught plus r minus r naught squared over 2 factorial V double prime of r naught plus dot, dot, dot where a prime denotes the derivative with respect to r and then, you put r equal to r naught after you differentiate.

But, this is a point of minimum for V of r and therefore, at a minimum the slope vanishes, this term is therefore, 0 by definition, because V prime of r naught is identically 0 at the point. Then, the potential looks like a constant, this fellow is constant, we do not care, what it is, it does not play any role at all. We have seen that in simple harmonic oscillator problem, if you add a constant to the total energy nothing happens, frequency does not change at all.

So, we have a system, where you can write a kinetic energy plus a potential energy, which look like this quantity, but this is a constant, these quantities are constant. And therefore, the motion is simple harmonic provided this quantity must be positive, the effective spring constant must be positive, only then you have simple harmonic motion. Is it positive, well the answer is yes, because at a minimum the second derivative is positive, had this been maximum of the potential that would have been negative, but because it a minimum, we are guarantee this is positive.

Now, I ask, what is the frequency of oscillation of this second molecule about this equilibrium point and the answer is very simple, all you have to do is to put in the mass of that particle. And essentially, V double prime of r 0 over 2 factorial over 2 is the square of the frequency apart from that factor of this mass, because a ratio of coefficients determines what the frequencies is. So, in this problem hard as it looks is very easy to find, what will be the basic frequency of oscillation or vibration of one atom due to the potential of another and so on.

So, it is like a shifted oscillator about point, of course, higher amplitudes, it is no long it simple harmonic and have many more complicated phenomenal happen here but, in this allows level of approximation, it is oscillated.

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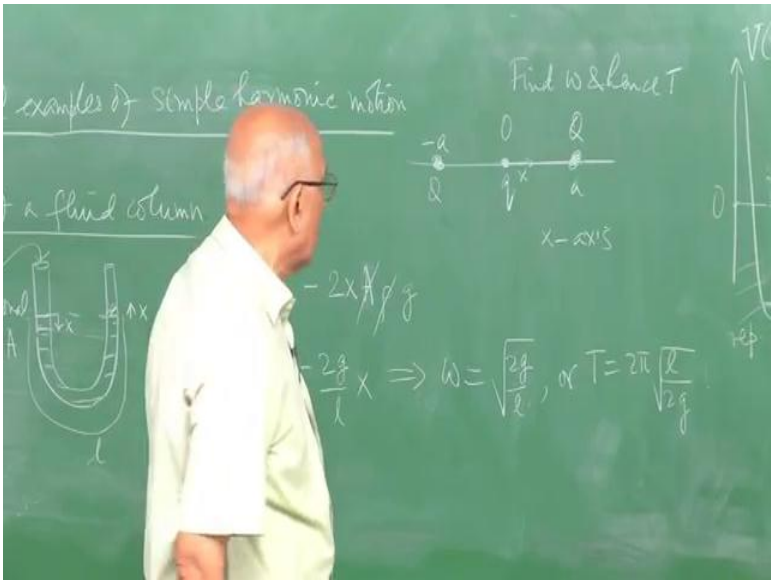

Now, I leave an exercise is an exercise the following problem, take two charges on the X axis put a charge q, a positive charge q at the origin and fix a charge capital Q at the point a on the X axis; that at the origin. And the another charge capital Q at the point minus a, these two charges are fixed and this charge is free to move. And now, imagine that ti displace it is slightly about an through an amount x on the positive side and let go.

Then, this charge is going to osculate, because the force due to, repulsive force due to this is greater than the attractive are force due to this, repulsive force due to this. As it moves here, the repulsive force pushes at backs. Similarly, when I comes here, the repulsive force due to this, pushing it to the right is greater than the force due to this pushing it to the left and therefore, in to oscillate. Now, what you need to do is to find the frequency of these oscillations and hence time period.

What you need to show is that, there is restoring forces proportional to x for small values of x, compare to a, small displacement compare to a and that should be help you to read out by writing the equation for the force for small x. It should be able to read out what the time period is from the frequency, angular frequency. So, I leave that as an exercise.