Mechanics, Heat, Oscillations and Waves
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Lecture – 27
Simple Harmonic Motion

Today, we will talk, take up the topic of a Simple Harmonic Motion, as you know in nature things change with time, dynamical variables change as a function of time and very broadly speaking, you can distinguish two different kinds of change. One which is completely irregular in some sense, un practicable and the second one shows a lot of regularity in particular very often you see periodicity, things come back to the starting point after sometime and the cycle is repeated on and on.

So, this periodic motion is what we are going to talk about today and simple harmonic motion is the fundamental paradigm of periodic motion. So, it is so important and appears in so many places and it is so common that it is worth paying some attention to and understanding in some detail. It is the very model of periodic motion and all periodic motion in some sense is build up on making simple harmonic motion more and more complicated as one goes along.

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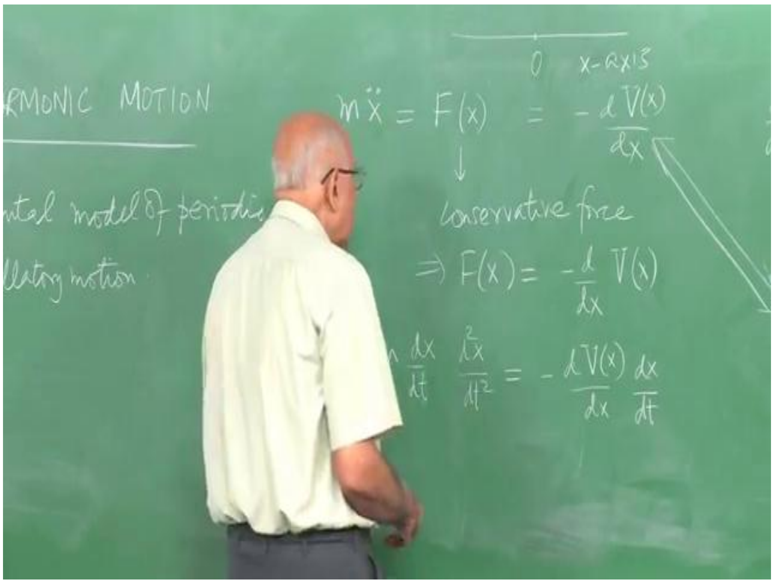

So, it is the fundamental let me write this down, it is the fundamental model of periodic or oscillatory motion. So, let us begin to analysis this simple harmonic motion and let us do this by starting with what you already know, namely the mechanics or the motion of a particle moving in one dimensions say the x axis and let say the particle as mass m and I write Newton's equations for it. So, I start by writing mass times the acceleration equal to the force on the particle and the particle moves on the x axis in a manner prescribed by this force here.

And in general this force could be a function of where the particle is. So, let say this F of x of course, you could look at the more complicated situation where this is also a function of t explicitly. I could shake the particle by hand as a function of time that leads to a different kind of motion we will not get into that, it is called force in motion and this will study a little later. But, for the moment you look at cases where F is independent of t explicitly, but depends on it implicitly, because x depends on t, x will changes a function of time and the task is to solve this equation in some sense.

Now, we have seen that when the force is conservative, I do not put the vector symbols, because we have one dimension and we take the convention then when F is positive it acts to the right, when it is negative it acts to the left and here is the origin. Then, in the case when F of x, if it is a conservative force implies that F of x can actually be written as minus the derivative of a certain quantity called the potential V of x.

So, let us consider the case, where F is a conservative force then this becomes equal to minus d V x by d x and the task is to integrate this equation. But, this is not such a trivial task, because this is the second derivative, had it been the first derivative I could have just written the integral down, but it is a second derivative and we need to find two constants of the motion in some sense when we integrated twice.

But, there is simple way of saying what the constant of the motion in this problem is as follows, multiply both sides of this equation in general by x dot. Just manipulate by multiplying by x dot on both sides, then you have m the x over d t that is x dot and then d 2 x over d t 2 this is equal to minus d V x over d x times d x over d t. But, we recognize that the left hand side is the derivative of d x over d t whole squared apart from a factor half.

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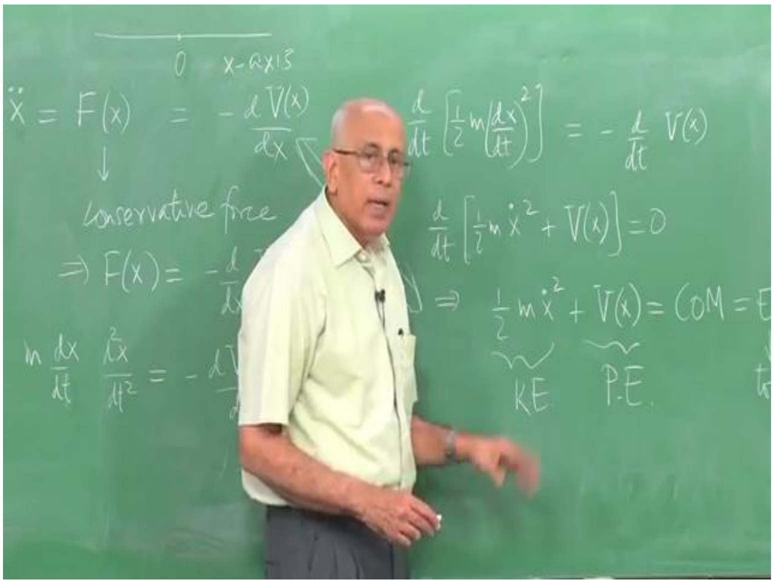

Because, this thing here is a same as saying d over d t of one half m d x over d t whole squared, if we differentiate that then you get a factor 2 first which cancels again this. The m comes out in the constant and then you have twice d x over d t times the derivative of d x over d t which is the second derivative. So, the left hand side is the time derivative of this combination, but the right hand side is equal to minus d over d t V of x it is the time derivative of this function of x, where x is a function of t that is the chain rule for finding the derivative of V of x function of a function, you differentiate first the function with respect to it is argument and differentiate the argument with respect to time.

But, bringing this to the left hand side it says d over d t one half m let me put a dot just for convenience, over head dot to denote the time derivative that is a standard notation x dot squared or the velocity squared plus V of x equal to 0. So, it says no matter how x changes with time, this particular combination of x and x dot does not change with time, this combination here which will of course, immediately imply that this combination half m x dot squared plus V of x equal to a constant of the motion.

Because, the derivative of a constant is 0 by definition and let us call this constant E say and I have a reason for calling it E, because you can see that we can identify we already can see what half m x dot squared is, it is just the kinetic energy of the particle and V of x is the potential energy of the particle which is the reason I called V of x the potential.

So, this portion is the kinetic energy and this portion is the potential energy and this E now has the significance of total energy.

So, you can now see that whenever you have a conservative force, this equation Newton's second law written down for this conservative force leads to this and conversely from here, you can go back to that by differentiating. So, these two are completely equivalent to each other in some sense. So, I from now on, take it that if you are able to write down the total energy of the system in this form of a particle moving on the one dimension in this form, then I take it that it is equivalent to writing this equation of motion and vice versa in both cases.

Now, how do we determine the value of this constant, it depends on the initial conditions. You have to tell me what is the initial value of the position and the initial value of the velocity x dot. Because, this is the second order equation in time therefore, you have to tell me two initial conditions and once you tell me that I plug those in here and then this remains the same at all times, even when x becomes the function of t, it has the same numerical value that it had a t equal to 0. So, that is what is meant by saying it is a constant of the motion.

So, once we accept this is so, the next question is what sort of V of x can we look at and solve for. Well, the most trivial case is V of x is a constant then of course, there is no force, because V of x the potential is a constant the derivative of the potential with respect to x is 0 and therefore, the force is identically 0. So, that is the very trivial case.

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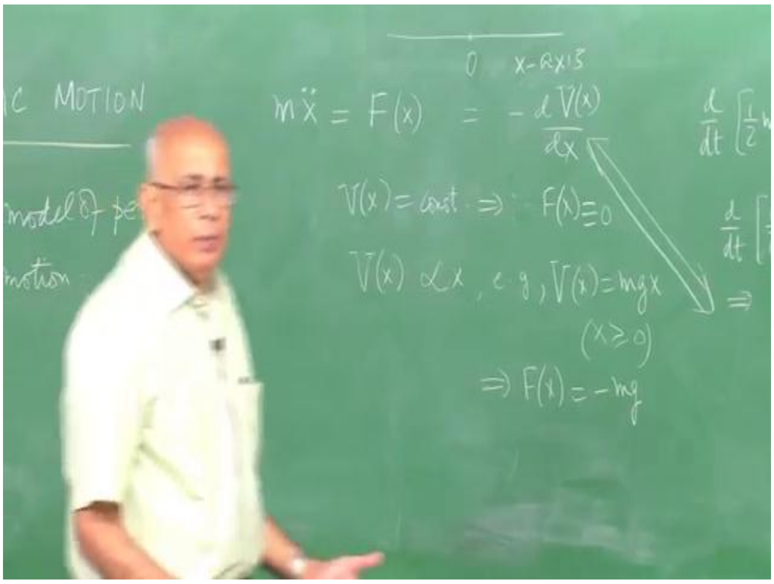

The next non trivial case could be to say that F of x or V of x equal to constant implies no F of x or rather F of x identically 0. The next case would be when V of x is proportional to x itself, the potential is propositional to x that is the first thing you can think of, the function is not a constant, but it is linear in the variable x. For example, V of x could be equal to m g x where x is greater than equal to 0, this would correspond to saying that the particle is moving in a gravitational field, constant acceleration due to the gravity being minus g and the positive x direction being taking to be the vertical direction from say the floor level.

So, that would be instead of writing m g h I have written m g x, but it is just the change of variable. So, this would corresponds to motion in a constant force field this of course, would immediately imply that F of x minus m g the derivative with the minus sign. So, it decreases x this force, that two we would study, we know how motion occurs under the constant force field they acceleration is constant and therefore, the particle moves. Such that, it is displacement at any given time t is proportional to t squared in the leading term. The next in order of complication would be when V of x is a quadratic function.

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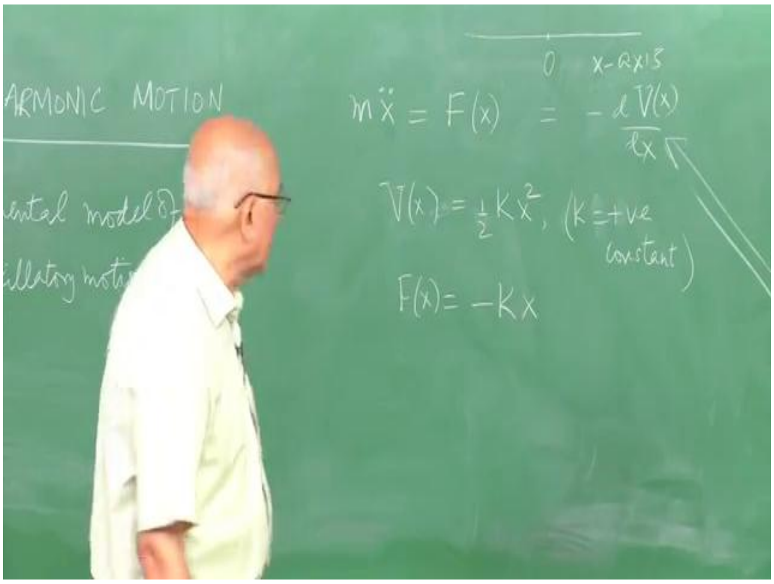

So, what happens if V of x is proportional to x squared, so let us write it as constant times x squared and for reason, which will become clear in a minute. It is easy to write, easier if we write half times the constant times x squared, where k is a constant k equal to positive constant. What happens now, the force F of x equal to minus k x, because it is the derivative become minus sign and now you see the reason by putting a half here, because it cancels the two here and it makes the algebra little simpler to write down.

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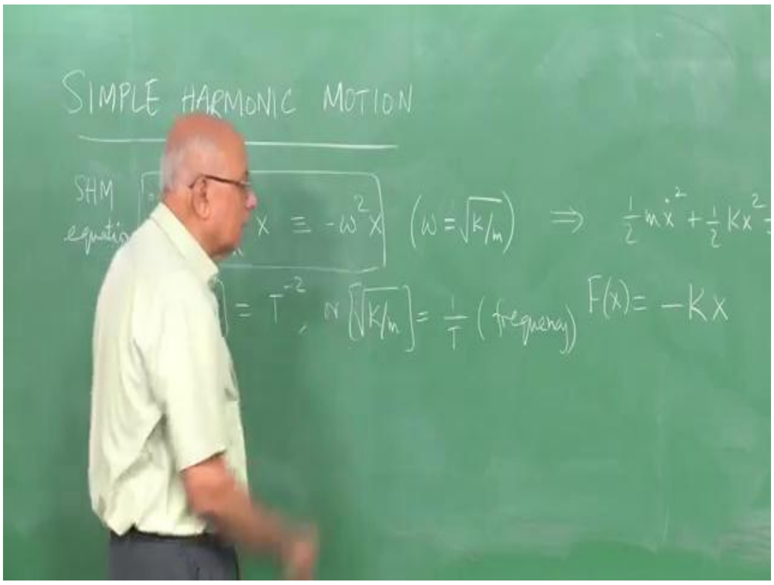

What happens now is that the equation of motion becomes x double dot equal to minus k over m x. Now, what are the physical dimensions of this quantity k as you can see from here the physical dimensions of k over m this is got dimension length and that is length then the second derivative puts a 1 over t squared here therefore, the physical dimensions of k over m must be t to the minus 2 it is got to be 1 over the seconds squared in units or square root of k over m physical dimensions equal to 1 over t which is got the physical dimensions of a frequency.

So, we see how the frequencies emerged here, how periodic notion is immerging from this equation of motion, this quantity here has the dimension of the square of a frequency and k is positive m is positive. So, let us right this as equal to minus omega squared x, where we identified omega to be equal to square root of k over m square to remain myself of the fact that this is the positive quantity.

So, once I have this equation and this is the equation of simple harmonic motion, this equation defines simple harmonic motion. Let us call it S H M equation, the next task is to ask what are the solutions of this equation here, and to see explicitly there is a periodic solution. But, even before that we begin to see something else I have set that once you have an equation of this kind one has an identification with the constant of the motion of this kind.

So, let us go back to that language and ask what does it say now, what is this equations say this thing will imply that the following quantity one half m x dot squared plus one half k x squared equal to E and this is the total energy. So, as soon as you have this equation or you have a quadratic potential a potential propositional to x squared you have the statement that one half m times the velocities squared, this kinetic energy plus this potential energy is a constant and the potential energy has a shape of a parabola because it is one half x squared.

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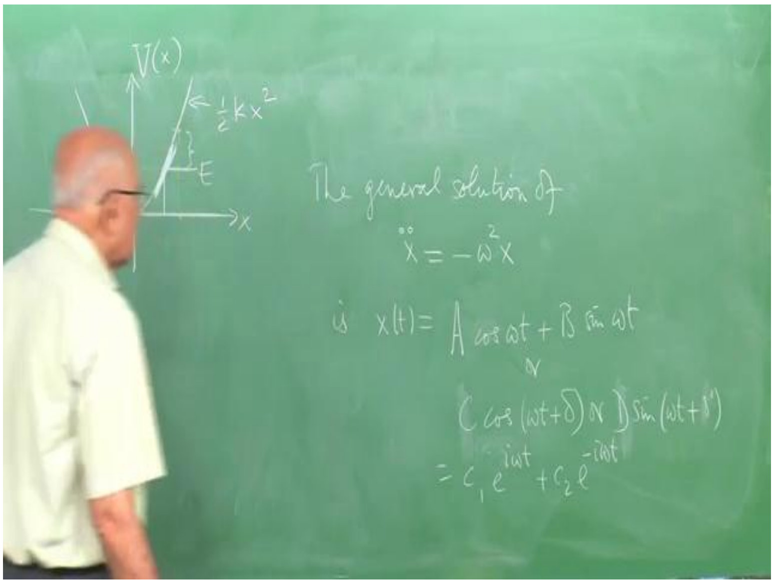

In fact, if you plot it is very simple, if you plot this potential has a function of x you have v of x it is a parabola that passes through origin in this fashion and this is one half k x squared with the single minimum at the origin. And now you can see what is going to happen even without solving the equation of motion, first of all E cannot be negative, because it is a sum of two's squares here and m and k are positive constant. So, E has some value like this for instance and this is the constant for a given initial condition, the total energy is given some positive value it is a constant.

Then, this particle cannot be in this region at all, because it is ((Refer Time: 15:22)) region that is the potential v of x at that level and the total energy is this much therefore, this must be my kinetic energy with the minus sign. But, the kinetic energy cannot be negative. So, it immediately says that if a total energy of the system of this particle is some given number then the particle cannot go to the right of that point or to the left of these points it is restricted to this region about the origin. And what it actually does is to oscillate back and forth between this point and this point.

Such that it is total energy remain constant, but when it comes here it is clear that all it is energy is potential, because the potential energy is equal to the total energy here and when it goes there all it is energy is potential is 0 kinetic energy, but it has kinetic energy here, because the potential energy 0 and the kinetic energy is a maximum. So, the kinetic and potential energy is of this particle or oscillating back and forth such that the sum is the constant.

And the question is what is the frequency osculation and the frequency is; obviously, omega for at least the angular frequencies omega. Because, it is immediately follows from this equation we can solve that equation and write the general solution of x double dot equal to minus omega squared x is x of t equal to there must be two constants of integration in this problem, because you are integrating the second order differential equation.

So, let us call those constants A and B then actually one can solve these equations by inspection. Because, we know that if you differentiates sin omega t you get cos omega t with the factor omega, you differentiate it again you get minus sin omega t. So, we know that both sin omega t and cos omega t have second derivatives which are themselves times and constant and therefore, they are solution. So, the general solution is of the form A cos ((Refer Time: 17:40)) cos here A cos omega t plus B sin omega t, where A and B are arbitrary constants.

In other words, independent of t and x of course, how do you find these constant, you have to tell me the initial conditions, you have to tell me what is x of 0 and what is x dot of 0. For instance, if you start at this point and let go then the particle will start moving backwards to words the origin, because the slop of V is positive here. So, V of x has a positive slop therefore, minus d v over d x is negative and negative force means it point towards the origin backwards from where ever you are returns to decreases.

So, the force restore you to this point here, but it over should because of conservation of energy till it comes here and here the slop is negative and therefore, minus slop is positive which means the forces directed to the right and therefore, it moves back in this fashion and indeed you see that cos and sin are periodic function with the time period which is 2 pi over omega.

So, you can write this in many ways, but here is one way of writing it there is an equivalent way of writing it or there always has to move two content of integration. So, you could also write this as a cosine or sin it does not matter depends on how you choose here you are constants omega t plus some delta or B sin omega t plus some other delta prime these are all equivalent ways of writing this same solution, because you could also write this has A cos omega t cos delta and then sin omega t sin delta.

So, you can identify A sin delta and A cos delta with these constants, let us not even use this same symbol here something else say C of D they are all equivalent ways of writing exactly the same solution. There is one more way of writing it using complex numbers, what would that be we know that cos omega t plus i sign omega t is E to the power i omega t. So, you could also write this as equal to some constant C 1 E to the i omega t plus C to E to the minus i omega t that is yet another way of writing exactly the same solution.

But, there are advantages to writing this solution this exponential form, because the derivative of exponential is a again the same exponential, apart from a constant. The disadvantage is the since x is the real number it is real displacement in this problem, the complex constant the constant C 1 and C 2 will have imaginary parts they will have to be complex. So, that the final x of t is real in this case. So, this is the completely real representation, but this is a complex representation of the same solution within matter, but we have a general solution and these solutions are periodic. So, now, we have a first important result.

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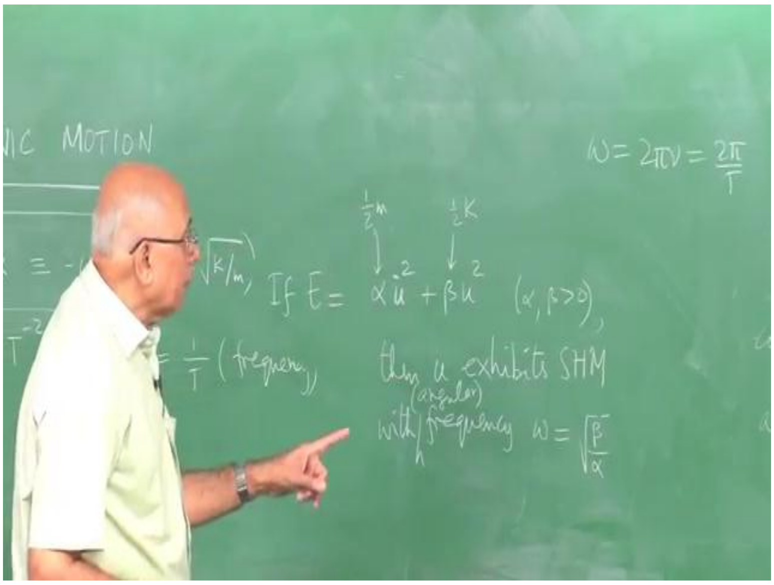

We can also write cos omega t or sin omega t could also be written as cos 2 pi mu t which is the same as writing cos 2 pi t over capital T. Because, this is the frequency, this quantity is the frequency this quantity is the time period, this quantity is sometimes call the angular frequency. But as far as concern there all very, very simply related to each other, we have omega equal to 2 pi mu equal to 2 pi over t explicitly I am going to use lose terminology which is done physics very often and just call omega the frequency it should really be called the angular frequency.

But, I will call the frequency, because it is related to the time period as 2 pi over t is on it or t is 2 pi over omega. So, the motion is periodic, the solutions are periodic as we see and the time period there is 2 pi over omega and curtail question is what is this omega and now comes the point, the moment I have any dynamical variable, any variable at all we saw this in example of the particle moving and the action this potential.

But, if you have any dynamical variable u such that the total energy of the system which describe by this variable u has the form some alpha times, if the total energy has form alpha time is u dot squared plus beta times u squared. If the total energy has this form that it is some code and code kinetic energy which is depended on u dot squared and a potential energy which is depended on u squared with some positive constants alpha and beta.

So, if this happens then u exhibits simple harmonic motion that is granted by the same argument that we went through here, it does not matter what this u is, it could be the displacement of a mass which is attached to the spring, it could be the angular displacement of a simple pendulum, it could be the level of water in a you tube which is going up and down, in each arm in simple harmonic motion, it could be the charge on a capacitor in an electrical circuit and so on. The lots and lots of instance is very have simple harmonic motion we will see some of these things as we go along I will give some physical examples.

But, all that we need to do is to identify the total energy to be of this form and the job is done with frequency and this is crucial with angular frequency of course, with the angular omega equal to the ratio of these coefficient is with the square root and that is it. In the simple harmonic oscillated case which we looked at the particle this quantity was one half k and this quantity was one half m and then the square root of k over m was you are frequency. And now it is square root of beta over alpha that is it.

So, this is all you have to do in any physical problem, if you want to find out the motion is simply harmonic right down the expression for the total energy by whatever physical argument and if it turns out to have this some of these two squares with positive coefficients you are guarantee that the dynamical variable whose energy is given by this expression. We will very simple harmonically with time, periodically with sins and cosines has the functions with an angular frequency which is the square root of this ratio here. There is another way of seeing this and asking what is the total energy well the this example tells us what it is.

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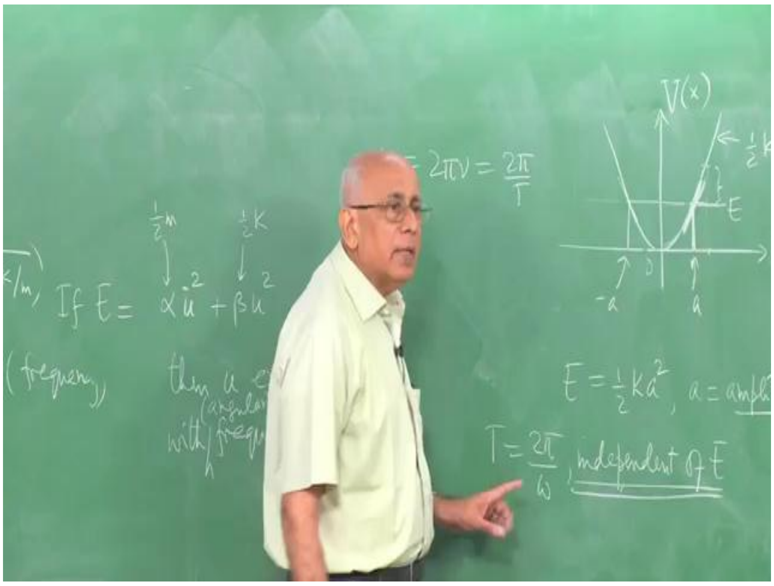

For instance, this point here for a given energy is the amplitude, so let us call that A and this minus A and it is fashion, then it is immediately clear that the total energy of the system is equal to the potential energy at the point A. So, it is immediately clear that E equal to one half k a squared in this problem, where A is the amplitude. So, we also have another feature of simple harmonic motion, namely that the energy is propositional to the square of the amplitude in this form and that specify either the energy or the amplitude is completely equivalent.

Once you tell me what the constant k is, you could either tell me the energy are the amplitude if you tell me the amplitude I tell you the energy and vice versa that is the input. So, this is another important feature of simply harmonic motion very obvious one, but the most crucial feature of all is that fact that this time period does not depend on the amplitude at all. The amplitude is determine by the total energy or vice versa, but T is independent of that T equal to 2 pi over omega independent of E I cannot emperies this strongly enough.

Because, this periodic motion which is called simple harmonic motion has this defining property is very crucial property that the time period of this periodic motion is independent of the energy of the system are to gather or independent of the amplitude of the system. And the next thing we will do is look at several physical examples of simple harmonic motion and you will see this feature emerging you want it to imply.