Mechanics, Heat, Oscillations and Waves
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Lecture-26
Linear Elasticity of Solids

The next topic, we are going to look at is the Linear Elasticity of Solids. I assume that, you are already familiar with Hooke's law, which says that, the stress is proportional to the strain.

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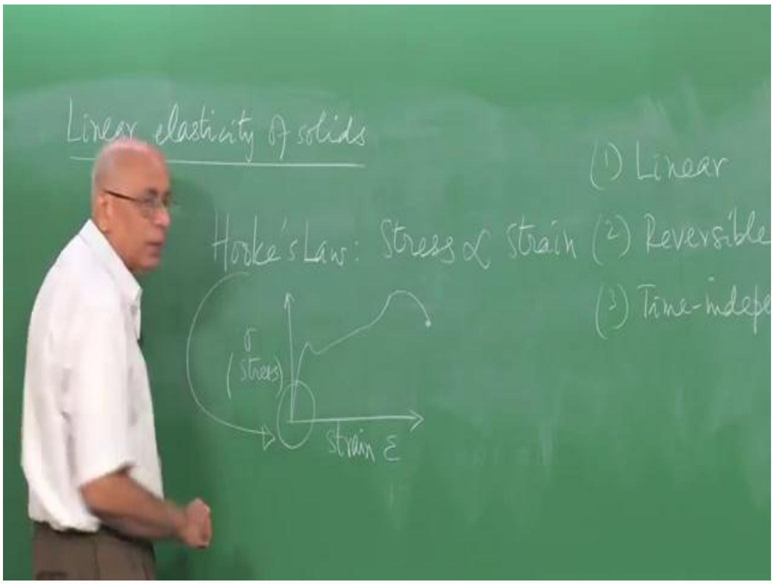

So, in it is simplest form, Hooke's law says that the stress on a solid is proportional to the strain, that it displace as a result of this stress, where stress is defined as force per unit area in some suitable circumstance. And the strain is defined as the deformation of this object relative to it is original this position. Now, this law as it is stands is extremely way, but we need to make it very quantitative, which we will do in a minute.

But, I am must say right away that this is a very restrictive law, because it essentially says that when you deform an object a solid to the specific, the response of this object, the strain has three properties. One, the response is linear in the sense that it is proportional to the stimulus that you apply. Two, that it is fully reversible. In other words, you remove the stress and the object goes back to it is original form, completely, totally reversible and three; that the response is time independent.

In the sense that, the response is instantaneous, you applied this stress and immediately, there is a deformation and you remove the stress and immediately, it goes back to it is original this position or configuration. None of these is true in practice really, except under very special circumstances, first for linearity, you need to have very small stresses in some sense.

For reversibility, you have to make sure again that there is no permanent definition of a material. If you take a paper clip and you bend it, we remove the bending force, it is reminds bend and that is the permanent deformation and that does not fall under Hooke's law at all. And thirdly; that it is times independent in the sense that, if I apply a stress to an object, they may be an instantaneous or practically instantaneous response, but if you wait long enough, this response will slowly saturates to some new value or go on forever.

For instance, if I pull an object, whether sufficiently high load, you can see that, if it is a thin rod of a metal, this metal, if the loads is high enough will stretch and stretch and stretch and little form a narrower and narrower mod neck and finally, it will break. So, this is clearly time dependent, now none of these things comes under the purview of Hooke's law. So, this is essentially a linear law for small deformations under very special circumferences.

And then, you can to plot and we going to tell say, what the stress and strain are in detail, the conventional symbol is Epsilon for the strain and sigma for the stress. Here and the statement is that typically, if you applies small strains, the stress is proportional to it, so it is linear and then, after while, it the leaves the elastic region, that is what called the yield drop. And then, it goes in for long time, it flows and then, takes of flows much more rapidly and finally, it breaks at that point.

By the time you reach this, this strain is like 20 or 30 percent or more, depends on a material. Now, what we are talking about, the Hooke's region, the region of applicability of Hooke's law is just here, this is where Hooke's law will apply in this region. But, now we have to ask, what do you I mean by this stress, what kind of stress is there, other different kinds of stress, different kinds of strain. And the answer is yes and this is what we are going to talk about.

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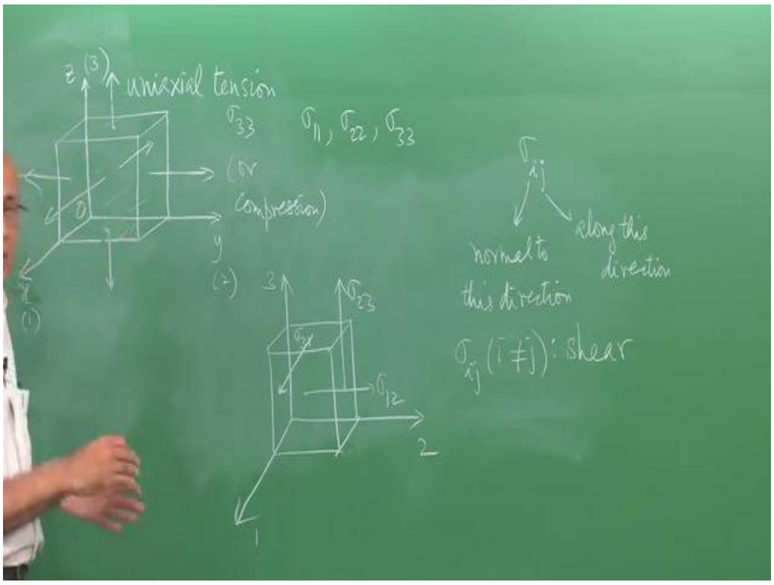

The simplest one in the way of looking at this is as follows, imagine in a unit cube of some material, so these are the coordinate axis, let us put coordinate axis here; that is the origin of coordinates. This is the x direction, the y direction and the z direction, I will find it convenient to call this, the one direction, this, the two direction and this, the three direction. This much more convenient to label them as 1, 2, 3 and then, we very clear as to what we mean.

Now, imagine this unit cube of this material and imagine holding three of the phases fixed. So, this phase, the side phase, the back phase and the bottom phase are struck to the coordinate axis, they are fixed. And whatever stress I applied will applicable on the other three phases here, then the following kinds of stress can be applied. I could go to the center of this and apply, solder a Hooke on to it and pull it upwards or pull this down words or do both.

Along the three axis here and I call that uniaxial tension, so this is uniaxial tension and the stress is uniaxial tension and this, can be measured by the load per unit area, the force per unit area on this unit cube. I can similarly push it inwards and that is uniaxial compress, all that will happen is that the sin of this stress will change. So, if it is plus, it is pulling and if it is minus is pushing inwards and of course, you immediately begin to see intuitively that material can have different responses for pushing and pulling.

There are materials like glass which can stand lot of pushing, but would not stand any pulling, will become brittle and break. So, there is some in homogeneity, but we want worry about that right now, we were talking about same metals, typical metals were these asymmetry is not present. So, you could have a uniaxial tension along with three directions or similarly, along these directions or for that matter, you can come out of the board and go in this fashion.

So, along the 1, 2 and 3 directions, you can have a uniaxial tension or you can have a combination of 2, biaxial, if it is along 2 axis, triaxial, if it is called all the three axis and so on. And the stress is corresponding to it, I am going to call sigma 11, sigma 22 and sigma 33. In other words, there applied on phases normal to the first index and in the direction of the second index. So, this here, this stress would be sigma 33, because it is in a phase normal to the z axis parallel to the x, y plane and it is along the z 3 direction, so it is sigma 33.

So, I hope that notation is clear, these are all tensions, uniaxial, biaxial, triaxial tensions or compressions. Similarly, one can now do the following, one can hold this phase fixed and shear it in this direction or shear it in this direction or shear it in the third direction outwards. So, again I am go to denote by sigma subscript i and j, this is going to be normal to this direction and it is along this direction.

Then, what do the various shears look like, when we got draw another figure for that. So, let us do that, this comes out, this is unique cube here, this fashion and now, what sigma 12 going to look like, for instance, it is got to be normal to the one direction. So, it is on the front phase and it must go along the two directions. So, I take that front phase and shear it like this. So, I keep everything fixed and shear the front of the cube in this direction; that is going to be sigma 12.

Remember, there is a one direction, two directions, and three directions, what about sigma 23, it got to the normal to this direction. So, it is on this phase, there is the back phase is fixed and along the three directions, so it moves of the phase 23. And sigma 31 will be on the top phase normal to the three directions, parallel to the x y plane and it should be 31 and therefore, it should come out in this function.

And this is sigma 31, these are shears, sigma i j, i not equal to j is a shear and there are three other shears, they are sigma 23 and 32 and 12 and 21 and the third one, which is 31 and 13. It is very convenient therefore to write this whole thing as a matrix.

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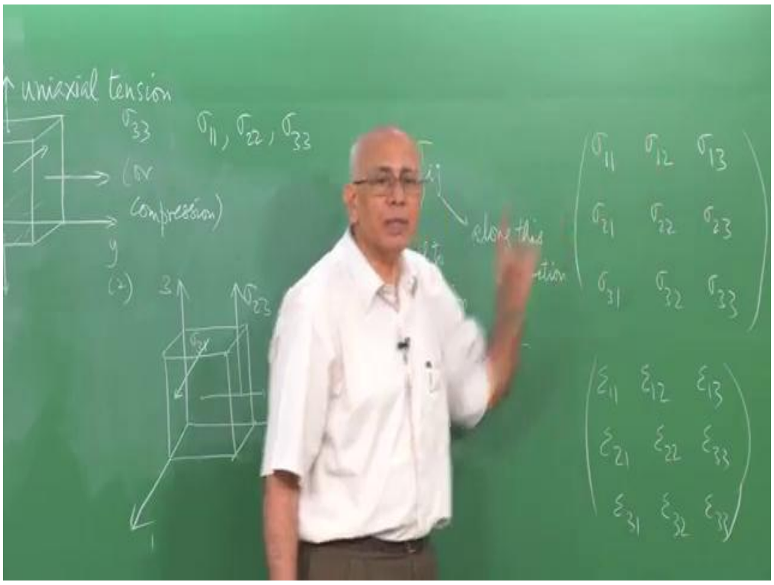

So, write this as sigma 11, sigma 12, sigma 13. So, you really have nine different kinds of stress that you can apply to unit cube of material and if the system is linear, if the response is linear, you are going to have nine different kinds of strain as well corresponding in to this. Some would be shear strain; some would be tensile strain or compressive strain etcetera, exactly as in the tension case.

But, if the material is nearly linear, ((Refer Time: 11:06)) then the strain to in general can be written as Epsilon 11, Epsilon 12, Epsilon 13. If the material is linear all it says is, Hooke's law says is, any given strain is linear in the stresses, proportional to the stresses, but there whole are of stresses and you can apply all of them simultaneously. So, it immediately says, that each component of this strain, each of this nine components is a linear combination of all these nine out here, which some coefficients.

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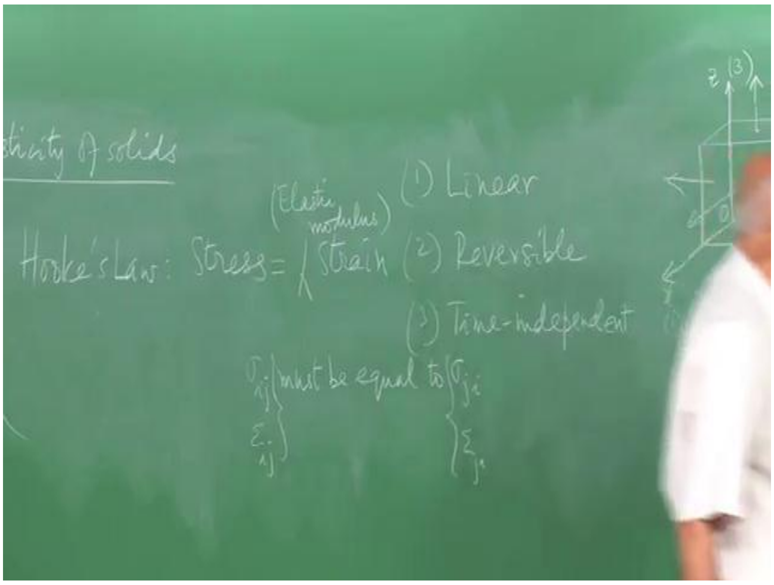

So, how many coefficient of or module i of elasticity do you have, whatever you called this, this stress is equal to elastic modulus multiplied by strain and how many such coefficient of elastic modulus can find. Well, you have nine strains and each of them is a combination of nine stresses. So, in general, there are 81 elastic modular, but now this a huge amount of reduction as follows. First of all, one can show that for thermo dynamics stability of this system, it terms out that sigma i j must be equal to sigma j i.

And similarly Epsilon i j must be equal to Epsilon j i, this means you do not have nine independent quantities, you have 1, 2, 3, 4, 5, 6 and these are the same as those. So, you have only six independent stresses possible, three strain, three tensions or compressions and three shears and you have six strain components. And therefore when you write each of these as a linear combination of all the others, you do not have 81 coefficients, but you have only 36 coefficients, which is already a big reduction.

But, there are further symmetries in the problem and it terms out that in almost all cases that we no off that we can conceive; the number is reduced enormously much more. And in fact, in the case of media, which are isotropic, which is the only thing, we are going to look at. Same properties an all directions not like a crystal, which is an isotropic or not like a composite material, we sometimes as different properties in different directions.

All of you know that, when we take a ruler and you bend it one direction, it bends easily, it is very flexible, but we bend another direction, it cracks. So, these are composite material, which are non isotropic, but for isotropic media, it turns out, you have only two independent elastic module just two. So, for such media, those are the ones we going to look at.

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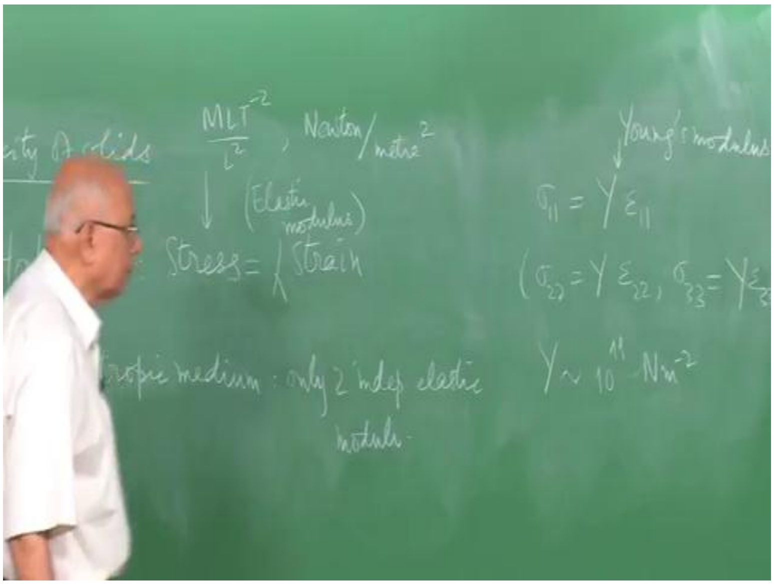

You have only two independent elastic constant, I need be write this down, isotropic medium material only two independent elastic moduli and you are familiar already with what this are and let us write it down. And at tension, you know that sigma 11 equal to the Young modulus times Epsilon 11 and this is called the Young's modulus. And therefore, since it isotropic, sigma 22 is the same Y times Epsilon 22, sigma 33 equal to the same Y times Epsilon 33.

So, much for one elastic modulus, now when it to put in some numbers, how big are these fellows, well the strain is dimensionless, physically dimensionless, because the strain if you pull it for instance in the original length is L and it is pull by delta L, the strain is define as delta L over L, which is dimensionless, no physical dimensions. Stress is force per unit area, so this is M L T to the minus 2 over L squared; that is the dimensions of stress and it is measured a Newton's per meter squared.

And typically, for most metals that we know of this Young's modulus is of the order of 10 to the 11 Newton's per meter square. So, that is the order of magnitude we are talking about, if you look at soft a materials like borne or something like that, it will be an order of magnitude small and so on. So, this is typically a figure here. So, stresses of the same dimensions as the elastic moduli and they typically of the order of 10 to the 11 Newton's per meter squared.

Now, one can compress the system in all directions and then, you apply what is called the hydrostatic pressure and the system will undergo change in volume and that is measured by the Bulk modulus.

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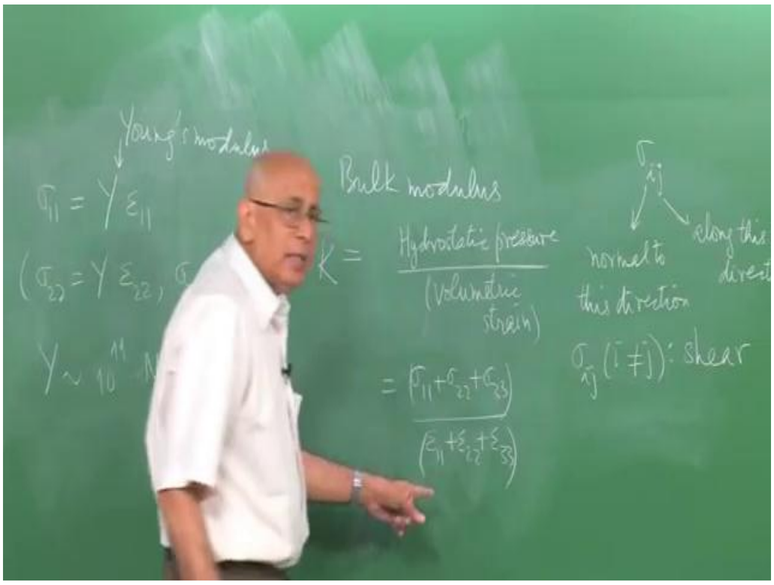

So, the Bulk modulus is we normally define it, Bulk modulus this is equal to the stress, which is the pressure, hydro static pressure. This would mean that you apply sigma 11, sigma 22, sigma 33 and push them all in all directions divided by the strain which is the change in volume over the original volume. So, this is volume metric strain, write K and the symbol generally used for it is K.

And finally, thus is shear modulus which says shear modulus which says G, I should also say how it is related to the stress tensile here; this here is nothing but, sigma 11 plus sigma 22 plus sigma 33. It is the trace of this matrix and that is what you normally called the pressure. The hydro static pressure defines as the trace of this matrix, which you write down for all the strains, no shears involved and that is divided by the volumetric strain, which is 11 plus Epsilon 22 plus Epsilon 33. That is reasonable too; you already know this, because you see, if you have a cube or a rectangular parallelepiped of size x, y, z.

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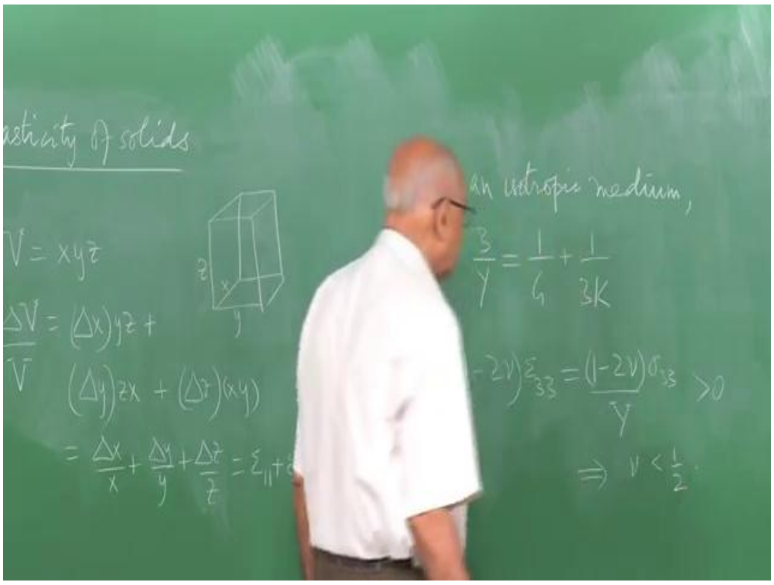

So, let us suppose you have an object is look is like with the three dimensions being this is x, this is y and z. Then, as you know the volume of this object this x, y, z and then, it is immediately clear that delta V, if I differentiate it is delta x multiplied by y z plus delta y multiplied by z x plus there it as a multiplied by x y. And therefore, if would divided by V, which is x, y, z; you end of it delta x over x plus delta y over y plus delta z over z. But, this is precisely what will you called Epsilon 11 plus Epsilon 22 plus Epsilon 33 strain in the x direction, y direction and z direction.

So, this is our definition of the Bulk modulus here and the Shear modulus is again the shear strain stress divided by the shear strain and as I set for the nice topic medium, this is one Shear modulus, one Bulk modulus and one Young's modulus and the three are related. So, the relation between them goes like this for in isotropic in medium 3 over y equal to 1 over G plus 1 over 3 K does not very hard to prove that will partially to very short while, but there is a relation between the three elastic moduli.

So, you really have only two independent elastic moduli, any two that we choose by the way, this also tells you that, gives you a hint that all the elastic module are of the same order of magnitude. So, for a material which is got a Young's modulus of 10 to the 11 Newton's per meter squared, the bulk modulus will also we have the same order of magnitude as will the same modulus give a take some factors.

So, this is a crucial relationship prove fairly easily, but more important than these elastic moduli, I set there are two independent ones in the question is which wants to you choose. Actually, these are not the best one to choose, the ones to chooser perhaps these Young modulus and one more modulus called the Parson's ratio and Parson’s ratio is as follows.

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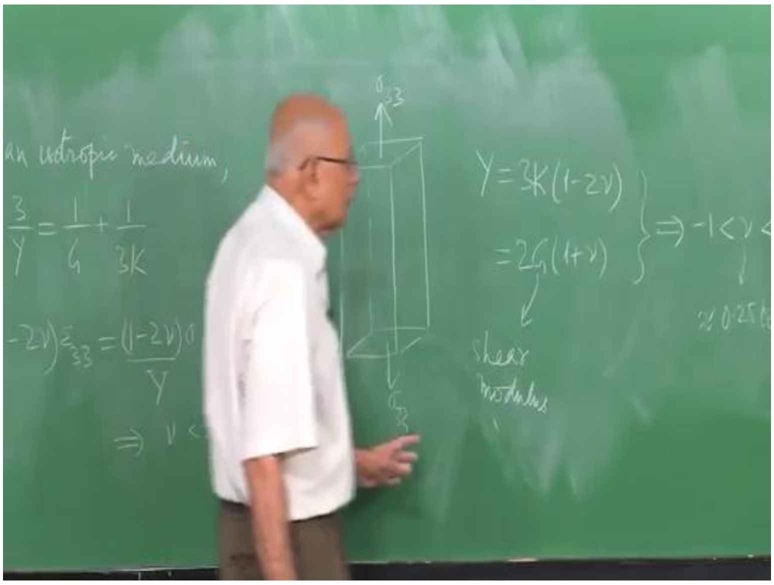

It says, if I take an object like a rod for example, it with rectangular cross section, this way properly and pull it in this direction by applying a load, then you know that lateral directions go inwards, they get compressed inside. And if this is the three direction, this is sigma 33 is stress applied in a three direction, then, parson's ratio nu is defined as the lateral strain divided by the longitudinal strain. The longitudinal strain of courses a Epsilon 33 and the lateral strain as a minus sin, because is going inwards Epsilon 11 orin this case of isotropic medium 22 over Epsilon 33.

So, that is what Parson’s ratio is, but there is a limit on this here, the very interesting limit on this ratio, it terms out that for most material it a familiar with like a metals, steel, copper etcetera. Parson’s ratio is somewhere between one-forth and one-third, so between 2.5 and 0.3 are so in general. But, there are strong bounds on it, which you can get from this relation here, because all you have to note is that delta V over V and let say we have uniaxial tension pulling outwards in this case.

This quantity is equal to when you pull outwards, we can ask what is the change in total volume, well delta z over z is high here is going to be Epsilon 33 , but this is going to be the ratio Epsilon 11 is minus nu times Epsilon 33. So, this is minus nu Epsilon 33, this is minus nu Epsilon 33, this is Epsilon 33. So, this thing becomes 1 minus 2 nu time

Epsilon 33, but that is equal to 1 minus 2 nu times sigma 33 is equal to sigma 33. The tension that you apply divided by Young's modulus, because Y times Epsilon 33 is sigma 33.

Now, when you applying in tension, which pulls the system outwards by stability, you cannot have a decreasing volume, it violates less certainly as principle, violates stability. So, this thing has to be greater than 0, which implies immediately that nu must be less than a half, you cannot have a material with in simple linear elasticity with a Parson’s ratio, which exceeds a half. In practice you find of the order one-third, but this is strong bound for such materials, you cannot have a Parson’s ratio greater than half.

And in fact, this is a relation this tells you the following and you can prove just by extending this little bit that Y can be written as 3 K times 1 minus 2. So, this is the fundamental relation between the young's modulus the Parson’s ratio and the Bulk modulus as I said there are only two of this constant, which are independent. You have four of them now, you have Y, the Young's modulus, you have K, the Bulk modulus, you have G, the Shear modulus and you have nu the Parson’s ratio.

So, that have to be relations between them and stability dedicates what these relations are, one of them is this. And similarly, this can also be shown equal to twice G and this is the Shear modulus times 1 plus nu and the Young's modulus must again be positive and that implies immediately these two relations immediately imply that minus 1 is less than nu less than half.

So, Parson’s ratio cannot like less than be less than minus 1 and cannot be greater than plus half and I has said in most cases is of the order of 0.25 to 0.30 very roughly. Of course, there are materials with this is not true, but they may not line within the probably of linear elasticity, they may be composites, they may be complex materials and so on. They may have very loose structure before as per instance you may ask why Parson’s ratio could ever become negative; it can when you have a material that is sufficiently network in very loose bonds.

Here in example either sheet of paper and like come to it up and imagine for a moment this is the solid. Now, what happen on it pull in this direction, you see when I pull in this direction and the transverse direction it also goes up. So, that is crew way of seeing that when you have an objects which are not simple looking, simple objects, which are not the homogenous, the very heterogeneous, the structure is like a composite material. then, you can have negative Parson's ratio in pull check in mesh for example, going to the direction Styrofoam.

But, then in most in normal circumstance is a Parson's ratio is indeed between 0.25 and 0.3 and one can show a little bit more work that the between minus 1 and 0, you have one cross material between 0 and one-fifth, you have another class. And between one-fifth and a half could be the traditional class or simple materials for which linear elasticity theory holes in it is present form.

On the other hand, I should immediately say that, this is only in the elastic region and only in the case of instant in a response completely reversible time independent response and real materials when the strain becomes of is stress becomes of efficiently large. You end up with the region of permanent deformation and that a rises you to very different mechanism all together than what an elastic deformation arises due to.

By the way, even elastic deformation is not instantaneous, nothing can be instantaneous is very fast compare to the time skills you are interested in. If you would take a thing like this and you clamp this end and you apply a load on this end the supplying uniaxial tension. This stress propagates has to propagates from here to the rest of the medium has to go all the way here and it cannot do so faster than the speed of sound in the medium and that might happen in milli seconds for normal laboratory conditions.

So, you thing it is instantaneous response, but actually this not instantaneous in the technical since in the words, but for all practical purposes it is instantaneous. So, the thing about elasticity that works emphasizing is that, one should understand the region are regime in which linear elasticity is valid and all this nice properties are valid. But, then given this given Hooke's law, one can go a great distance in understanding definition behavior of materials in this region of applicability.