Mechanics, Heat, Oscillations and Waves
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Lecture - 25
More on The Kepler Problem Satellite Motion

In this last module on Mechanics of dynamics of particles or bodies, I would like to say some more about the Kepler Problem, namely motion in the 1 over r potential because of it is importance to us both in selective mechanics as well as in electrostatics many other parts of physics as well. The 1 over r potential, the coulomb potential or the Kepler potential is a very, very special kind of potential and it exhibits in motion in it exhibits some very, very lower properties should not exhibited in general, in other central forces either.

And this is something which is worth remembering and that is the reason, I would like to spend a little bit of time on this, before I do that you recall that the last time we talked about centrifugal and Coriolis forces as non inertial forces. Now, this slightly that this term non inertial might give the wrong impression that the somehow has something to do with the lack of inertia not at all.

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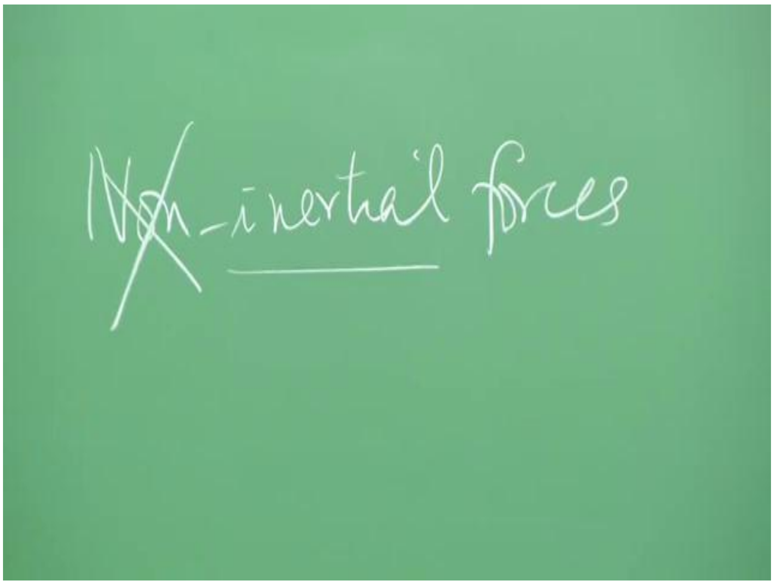

In fact, the right terminology for it for what we call non inertial forces should really we call inertial forces, because they arise in an accelerated frame of reference due to the inertia of a body. So, they should properly speaking called, we call inertial forces or if you like non inertial frames of reference have these forces appearing due to the motion, due to the fact that they are accelerated with respect to inertial frames.

So, the proper terminology would be inertial forces, but we understand what we mean when we say a pseudo force or a non inertial force, we mean inertial force like the Coriolis force or the centrifugal force that is said, let us go back and examine what the motion of a satellite does, how it taken to be described in two difference frames of reference. And there is a common error which is to confuse the two frames, namely the laboratory frame or the terrestrial frame which is fixed with respect to a point of the surface of the earth and the frame of reference of a person in an artificial satellite for instance.

The idea is the following, if you want to describe the motion of a satellite going around the earth, for instance like the moon or an artificial satellite, then from over point of view from the lab frame it looks like this.

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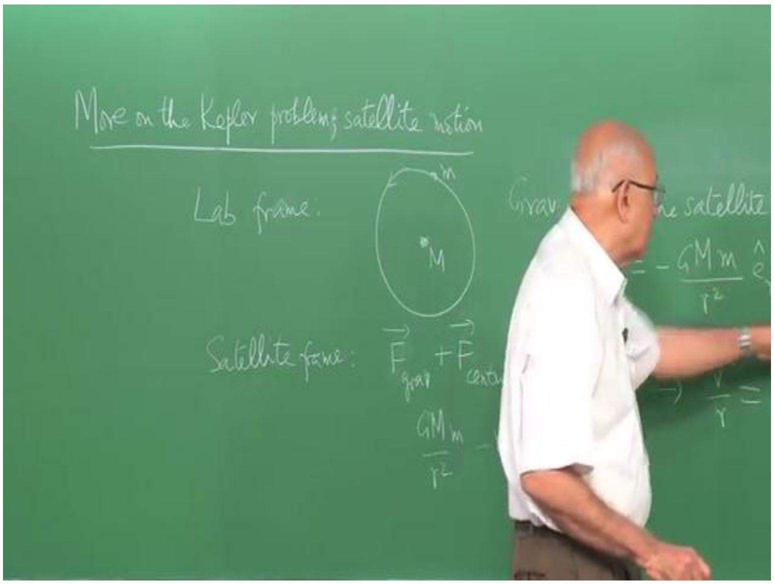

So, from the lab frame or terrestrial frame, here is the center of attraction and that is the motion of path of the satellite goes down in some closed orbit. Then, as first we have concerned the object here, the satellite of mass m attracted by a body of mass capital M sees the gravitational force.

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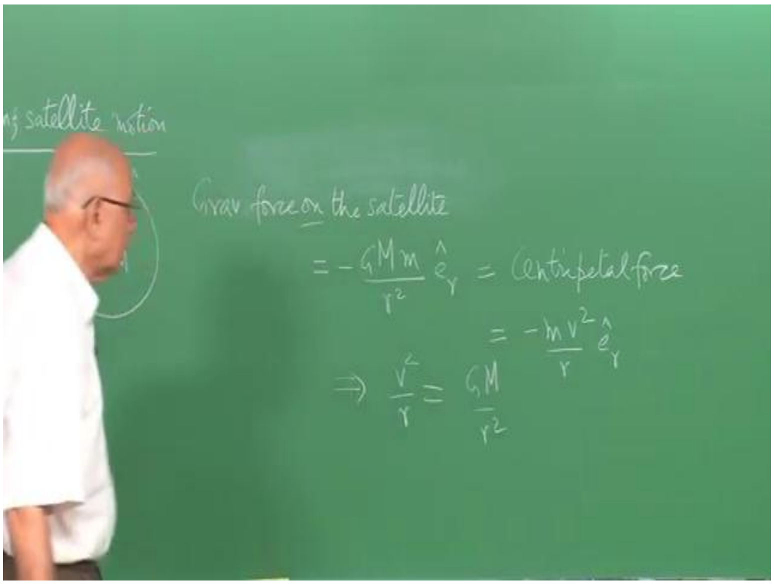

So, the gravitational force on the satellite due to the earth, due to the attracting body this is equal to minus G, the mass of the earth or the attracting body times the mass of the satellite divided by the square of the distance between them radially inverts and that is a physical force, the gravitational force. And in this force, this object moves in a path which is dictated not only by the force, but also by the initial conditions on this body, namely what to the initial velocity is in positions coordinates like.

One should equate this force, one should equated to the radially invert force of an object moving for instance in a circular orbit, this should be equal to the inverse centripetal force, which from kinematics we know is equal to minus m times v squared over r e sub r and that part is kinematics. So, this part is dynamics, define that tells you that gravity exists the mass exerts a force of gravity on all other masses that is dynamics and this portion comes from kinematics. We have already seen that when an object moves in a circular path for instance, then it is got a centripetal acceleration and if you multiply by m, it gives of the centrifugal force on the object.

And together these two equations will imply m cancels out as you can see, it implies that v squared over r is equal to G times capital M over r squared. So, again to be very careful this portion comes from kinematics and it is the centripetal acceleration multiplied by the mass and this portion comes from dynamics from Newton's law of gravitation and together for consistency this condition has to be satisfied.

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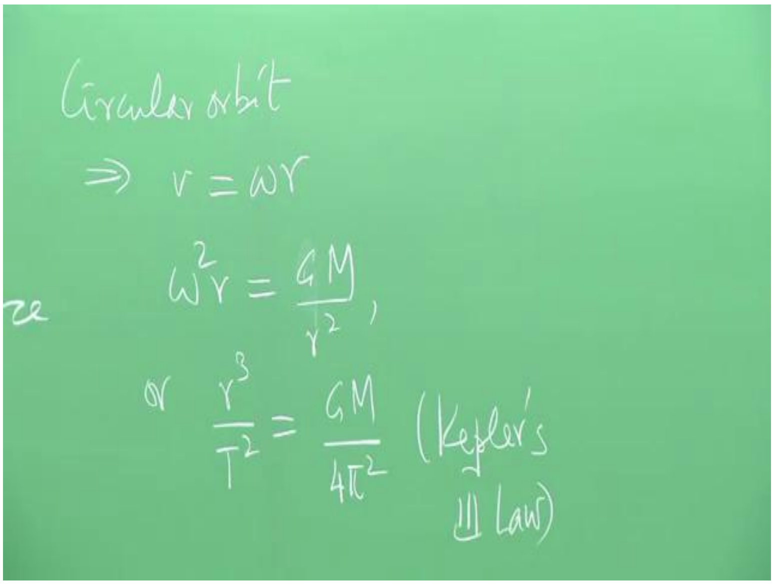

And if it is in circular orbit for instance, would imply that the linear velocity is the angular speed multiplied by the radial distance r and if you put that in, you get omega squared over times r is equal to G m over r squared or since omega is 2 phi over the time period, it says r cubed over time periods squared equal to G m over 4 phi squared, this is Kepler's third law for a circular orbit. Kepler's third law in the special case of a circular orbit of radius r. We know that Kepler's third law is applicable even to elliptic orbits and we say last time, that a simple scaling argument for the coulomb potential or coulomb force tells you why this is so.

But, this law here you know arises very, very simply in this frame work for a circular orbit, it is a simple derivation here of Kepler's third law that argue to this proportional to T squared, which is equivalent to saying that all possible orbits of all possible radii are possible, such that the radius in the time period are related by this common ratio by this ratio is invariant, it is a constant for a given attracting body.

Now, what does it look like from the pointer? Well, what is the actual state of motion of this object; it is in free fall not only is the satellite, but every point in it, every object in it is in free fall with exactly the same centripetal acceleration. And therefore, there is no weight, there is no reaction on a scale for instance you can also not predominant scale that is the reason there is also not is weight less when an artificial satellites in free fall.

Incidentally this is exactly the reason why we do not on this earth feel the centrifugal force due to our revolutionary motion around the sun. Because, it is exactly balanced by the gravitational attraction of the sun from our point of view, but from the sun point of view from a frame fixed at the center of the sun, the earth and everything on this earth is in free fall in the gravitational field of the sun. And therefore, objects here do not feel that force we will see this in a different frame work in a minute and that is you look at it from the point of view of some body is sitting in the satellite ((Refer Time: 08:02)).

So, let say satellite frame in the satellite frames something very different happens, here it is a rotating frame of reference and therefore that is centrifugal force and a inertial force in this frame of reference, which is experienced by an, as soon I am not sitting in there in addition to the gravitational force here. So, in this case the two forces are F gravitational inverts into towards the center of the earth plus F centrifugal and this is equal to 0, because they balanced each other exactly.

This force if you recall has magnitude G m over r squared times little m and it is directed inverts and if you recall what we derived for the centrifugal force is was minus m omega squared r, where omega is the angular speed of this rotating frame as seen from the inertial frame. In other words, it is the angular frequency of the orbit and this is equal to 0 and you will be recognized that it is exactly the same equation as in this case.

So, from the point of view of a person sitting in the satellite there are two equal and opposite forces acting on him, not an action reaction pair by any means, the two equal and opposite forces acting on in. The invert force due to the earth gravitation and the outward force due to the fact that this is a non inertial frame and this is a centrifugal force which is running to flinging them out and these two cancel each other and you get exactly the same equation as before.

This of course, is the same thing as happens to us when we are on the surface of the earth we do not feel the gravitational force of the sun, while we are around the earth, because we have also an equal and opposite centrifugal force can into fling as outputs, these two cancels and therefore as for as we are concerned there is no net force as due to the sun. So, once that is clear let us go on to the next statement and this have to do with the scaling relationship and I need to explain something here.

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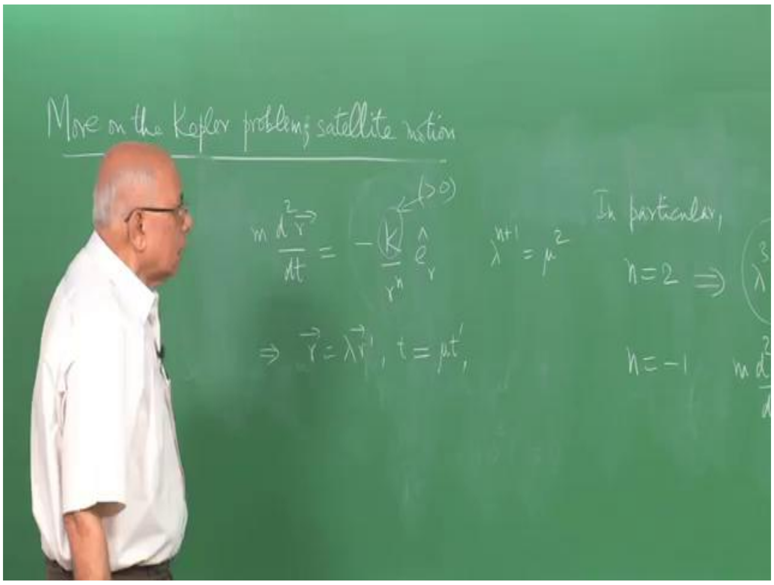

We saw that if you have a central force in which the equation of motion an attractive central force a, which is of the form minus k over r to the power n e sub r, where n is some index, some number. Then, we shall that this will imply that if you change the length scale by a lambda, so I put r equal to lambda r prime and the time scale by some factor mu t prime, then the physics of the situation is unchanged, if lambda to the power n plus 1 is equal to mu squared. This came by substituting for r and t in terms of r prime and t prime on both sides and recognizing that the factor lambda is going to come out from here and lambda to the n here. So, the two add up to give this and mu squared comes in the denominator here and ((Refer Time: 11:25)) this condition here.

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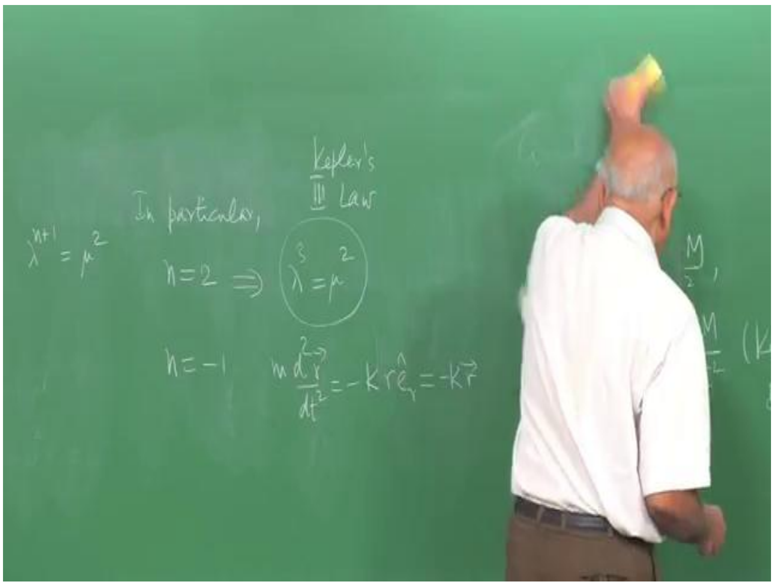

In particular, and these are the two important cases n equal to 2 that is the Kepler problem inverse square law of force will imply that the lambda cubed equal to mu squared, this is Kepler third law in the following sense, it says that if this force 1 over r squared force supports and orbit with the mean radius r and this orbit has a time scale or a time period t then it also supports the same potential of force also supports any other orbit of radius r prime and time period t prime such that r cubed over t squared is equal to r prime cubed over t prime squared.

So, that is the content of this law, this scaling argument does not prove that the inverse square law potential supposed closed orbits in the form of ellipses it does in do that, it only says if it does. So, if it is supports close motion with these parameters mean radius r and time period t it also supports any other r prime and t prime such that r over r prime cubed is equal to t over t primes squared, so that is all it says.

In the other case, the other special case was n equal to minus 1 this corresponded to r going up here and would give you and equation of motion which is d 2 r over d t 2 equal to minus k times r times e r, but that is the same as minus k times r. In other words, you have a force which is the restoring force trying to push back into the origin proportional to the instantaneous displacement and this is the simple harmonic oscillated, this force is the harmonic oscillated in three dimensions, not a linear oscillator.

But, in three dimensions and in this case there is no relation here, this becomes lambda to the power 0 which says the time scale or time period is independent of the length scale of the motion. In other words it is not dependent on the amplitude of the motion this is the property of isochronicity of the simple harmonic motion, namely the time period is the independent of the amplitude are the total energy. So, that well known property emerges from this scaling and here. Now, one might get the impression from this argument that somehow for in arbitrary value of n positive or negative you have closed orbits not true not true at all.

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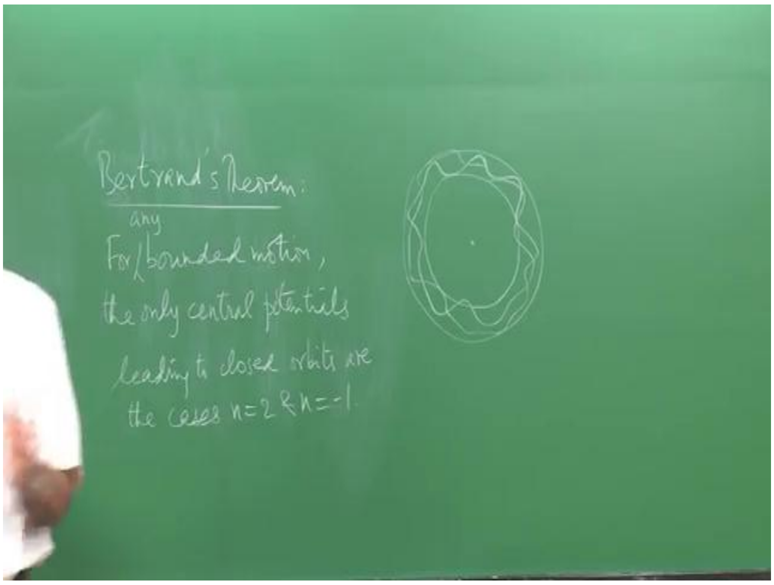

In fact, it is much more riggers then that it is says there is a theorem which says it is got a names. So, let me write this down it is an important theorem it is called Bertrand's theorem and it says for bounded motion namely motion such that the radial distance from the center of attraction does in go of two infinity. But, looking at only that class of motion which is bounded, for bounded motion the only central potentials leading to closed for any bounded motion closed orbits are the cases n equal to 2 and n equal to minus 1.

In other words, saying it is every bounded motion and one of these forces either the harmonic oscillated force or the inverse square law force leads to a trajectory for the particle in physical space which is the closed orbits whether it is an ellipse are not is the separate question and the answer is yes, where ellipses in general. But, these are the only two cases in which you have closed orbits, in general you do not have closed orbits for other values of n, even if the central potential.

So, central forces we do not have bounded motion need not have closed orbits. For instance, you might have bounded motion in some of these forces for different values of n which look like this. So, there is a minimum distance or minimum and a maximum distance or maximum and the particle on the body does motion like this, in some complicated curve which may go in and out orbit change never believe this annual region, but it does not close on itself not a periodic orbit.

So, that is the very, very crucial point with the Bertrand's theorem which tells you that only these two potentials for deep technical reasons, these potential have some deep symmetric properties which we will going to here, which leave to this beautiful property that all bounded motion in these special forces lead to closed orbits. The difference between this potential and this potential is that here Kepler's third law says that the time period squared is proportional to the cube of the main distance from this attraction center.

But, in this case it is says the same kind of arguments says that the time period is independent of the amplitude of the motion or the linear dimensions of the orbit. In both cases you have elliptic orbits, in this case Kepler's first law tells you that the center of attraction is at one of the force higher the ellipse, but in this case Kepler's law tells you this is no first at a Kepler's law here, it tells you that the ellipses at the center of attraction is at the center of the ellipse, rather then one of the four side.

That said one can ask what the energy looks like for different classes of orbits in the case of the of tragedies in the case of that inverse square law potential. Well, here you got to go to this is an attractive inverse square law force, because is the minus sign here and case a positive constant here as you know if you have a repulsive coulomb potential or 1 over r squared force then you have a only open orbits there is no question of a closed orbits, no capture just scatter in a hyperbolic orbit.

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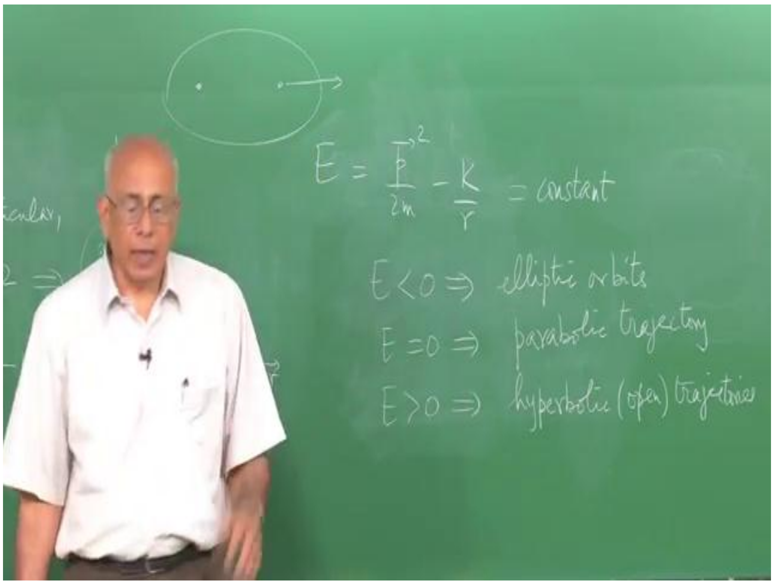

In this case what happens is that the total energy of the system is the some of the kinetic energy minus the gravitational potential minus K over r in this notation that have chosen K is capital G capital M etcetera, etcetera not bother about it some constant over r squared here for this. This quantity is a constant in the sense that for any given motion for every orbit this is some value, this is some constant here then what happens is that if E is less than this implies elliptic orbits.

But, if E is greater than 0 this implies hyperbolic open trajectories different values of E different lead to different hyperbolic orbits. But, there not ellipses there not it is not captured it is not bound motion and you could now ask what is E equal to 0 corresponding to exactly equal to 0 to correspond to. Everywhere on this trajectory E remains 0 it is a constant of the motion, what is a correspond to not surprisingly it will correspond to an elliptic orbit in which if you recall here is an elliptic orbit with the sun vector the center of attraction at one of the force side it will correspond to a case where the other focus goes of to infinite this moves of to infinite and you get a parabolic orbit of trajectory.

So, this value of the total energy provides you with the way of classifying the kind of motion you have, either you have un bounded motion in a hyperbola or you have bounded motion in an ellipse or the case between the two the case at separates these two classes of motion corresponds to E being exactly 0 at all instances of time and that correspond to parabolic trajectory here. Now, what tells as special about this up inverse square law force and here is the fact which is very, very significant which is what makes this potential so this is a force shows special.

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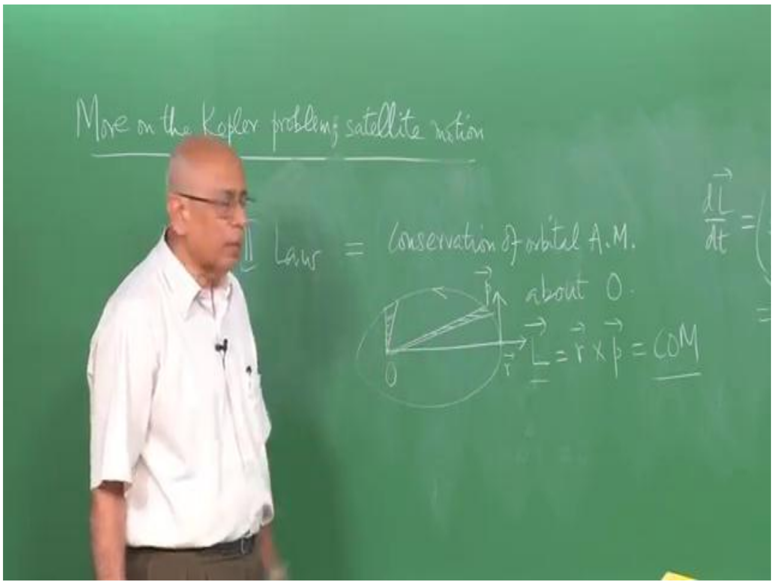

We know that the second law of Kepler, Kepler second law which says that the radius vector sweeps equal areas in equal times. So, here is radius vector it is sweeps in some given interval of time it is sweep that is area later on it is sweeps in the same interval of time it is sweeps an equal area and so on, we already saw that this is nothing but, the conservation of angular momentum. So, this is equal to conservation of orbital angular momentum about the center of attraction.

In other words, this quantity L equal to r cross p is a constant of the motion and therefore, you can evaluated at any instant of time it is going to be exactly the same thing at all points on the orbit. In particular, if you evaluated at this point when it is here r is in this direction that is the direction of r from o outwards and if the orbit is in take describe in that sense p is sitting here. So, r cross p a vector comes out of the plane of the board.

So, in this case the orbital angular momentum comes out of the plane of the board and this constant in magnitude as well as direction. So, everywhere on the path L has exactly the same direction and the same magnitude mo matter where you are and that is easy to see.

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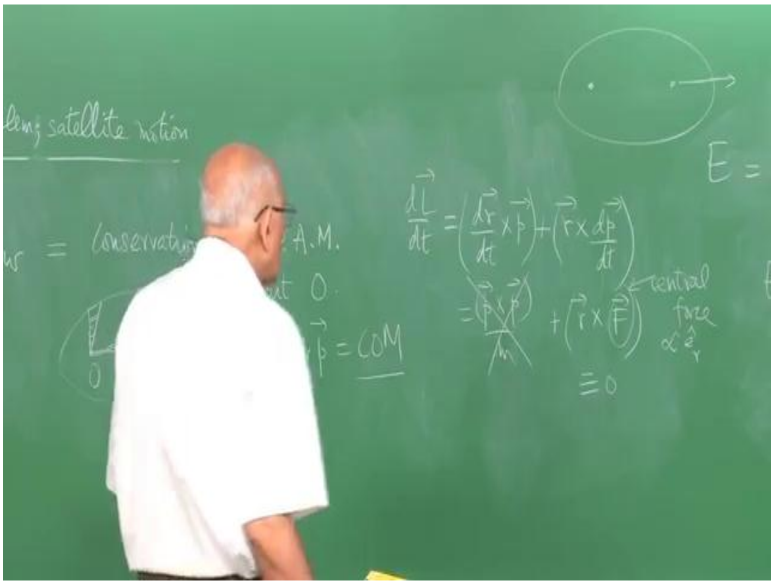

Because, what we have to do it should take d l over d t this is equal to d r over d t cross p plus r cross d p over d t, but d r over d t is the momentum divided by m. So, this is equal to p cross p divided by m and that is 0, because the angle between p and itself is 0 in the sign of that angle is 0 plus r cross the force. Because, d p over d t by Newton second law is the force on the particle, but this force is inverse square, so this is equal to r cross minus k over r squared e sub r and that is identically 0, because again it is a cross product of r with the radius unit vector in the radial direction and that 0 indent.

So, this implies that L is the constant of the motion, no where have you used that this is an inverse square law really. Because, this force as long as this force is the central force put immediately imply that this is equal to 0, because this force will be proportional to e sun r does not have to be inverse square law, but it is still true. So, this is crucial point that angular momentum is conserved about the center of attraction is conserved, center of force is conserved for all central forces, orbital angular momentum is conserved, but now a little piece of magic appears and that is the following.

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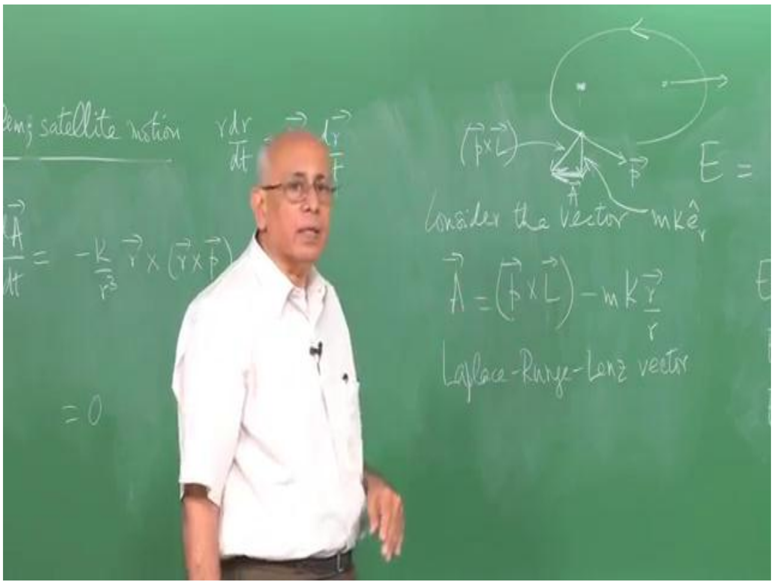

Consider the vector and let me call it A, this is equal to p cross l minus m times k times the unit vector in the radial direction. So, m k r over r consider this vector and it is got a special name, it is called a Laplace runge in a lenz vector and the claim is it too is the constant of the motion, it to reminds and changed has the particles moves in it is orbits and I am going to the give this an assignment to you, but you can see how this is going to work.

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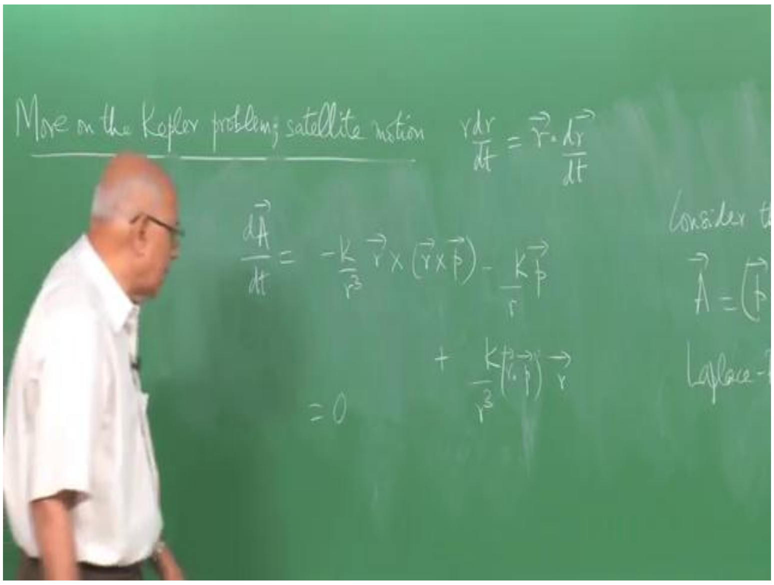

Because, what we need to do and let me go through couple of steps d over d t is the time derivative of this times L. So, d p over d t remember is the force, so the first term is going to be the force which is minus k over r cubed r that is the unit vector k over r squared times unit vector here cross l is r cross p. So, that is the first term and the second term is p cross d l over d t, but d l over d t is 0, we know that l is the constant of the motion.

So, that portion this hole derivative of this hole term is just this minus m k and now you got to very careful first you differentiate this r. So, it is m k over r d r over d t, but d r vector over d t is just the momentum p plus the derivative of this 1 over r. So, it is equal to m k over r squared in this function times d r over d t and then this vector sits there in between, but this is a mesh we do not know what it is. So, the trick is to multiply by r and divide by r ((Refer Time: 26:58)) r cubed there is an r cubed here you see what is go happen and then use the fact that r d r over d t equal to r dot d r over d t that is the vector identity be prove little earlier in general for any time dependent vector.

So, if you road it like that this is r dot d r over d t, but now you combine this m and the d r over d t and write it has r dot p. Now, use the identity for the vector triple product here and a home this vector has a 0 is identically 0. So, this vector is a constant of the motion and now it reminds to figure out where it is and you can do it what it looks like wins direction it is in and that can be done where ever you are on the orbit, any convenient point.

For instance, it will here on the orbit at this point for instance, the p vector is in this direction, the r vector is in this direction r cross p the angular momentum goes in and then you got to do a p cross l whatever as gone in and the result is going to be a vector which one added to this follow here is going to be view. So, that vector is an this direction this is going to be p cross l and this vector minus this is this vector here that is A this is e r minus this guy here which minus is m k e r and we want to negative of that.

So, you want this plus this which is this, so it is in the direction along the semi major axis pointing from the center of attraction towards the shorter end, towards the point of nearest approach of this planet or this orbiting body and this vector is a constant I am a at as will mention that it is related to the eccentricities it is magnitude is related to the eccentricities of the orbit and the fact that this vector is a constant of the motion is in some sense responsible for all the magic properties that you have in the case of the Kepler problem.

I thought at mention this because it is an interesting little exercise to show that in addition to l and the energy itself you also have this constant of the motion and a little bit of vector algebra vector calculus as it tells you how this is the constant of the motion and what it is physical significances has. I mention it is geometrical interpretation is that it relates directly to the eccentricity of this elliptic orbit.