Mechanics, Heat, Oscillations and Waves
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Lecture – 24
Non – Inertial Forces ("Pseudo - Forces")

Today, we will take up this topic of non inertial forces also known as pseudo forces sometimes called fictitious forces. The concept in Newtonian dynamics that has actually quite simple, a quite simple explanation, but it is occasionally very confusing especially because of a lack of clarity as to where these forces come from and what exactly they mean. It will turn out that, what we mean by a non inertial force should really be called a non inertial acceleration.

And then, once you multiplied by a suitable mass of a body which you are tracking, it becomes a force apparent force acting on the body and this is the whole origin of it. Extremely simple, it has to do with the fact that Newton's laws of motion are valid in inertial frames, Newton second law of motion which relates mass times the acceleration for constant mass to the force acting on the body is valid in an inertial frame. And an inertial frame is one in which the first law of motion is valid.

So, the question arise is as to, what happens if you have an accelerated frame, a frame that is in that is undergoing an acceleration with respect to the family of inertial frames. What happens to Newton's law? What does it look like in that instants and this is what we are going to address today. So, to make matters simple, let us take a very simple specific example.

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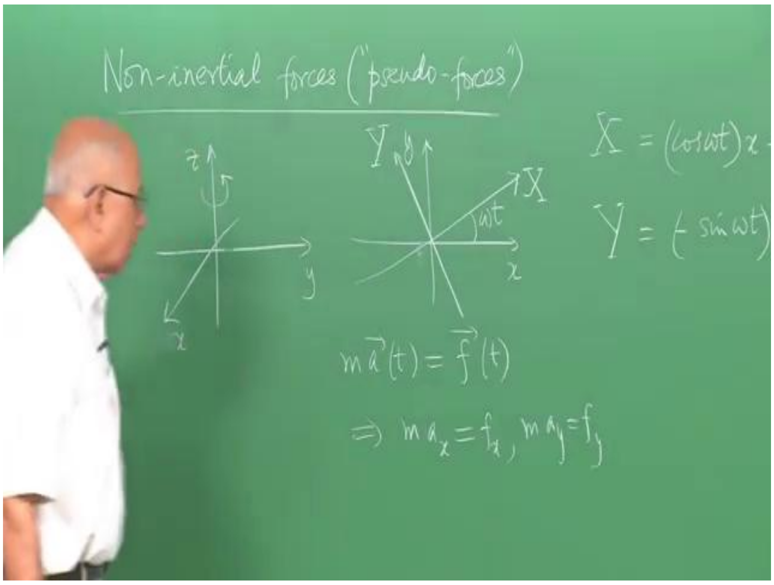

I have in mind a coordinate system in an inertial frame like this in x, y and z and I have in mind transforming from this frame of reference to a frame of reference that is rotating about the z axis at a uniform angular speed omega. So, we have a new frame that is rotating with respect to this frame, the x and y axis become some other variable x prime and y prime or capital X, capital Y, while the z axis remains unchanged. So, if you look at it from above, it looks like this, the original frame and I am going to use small letters for the original frame.

So, here is x and here is y that is the origin and in the new frame of reference, the x and y coordinates rotate at a uniform angular speed. So, at some instant of time the coordinates, the new coordinates look like this, let us call that capital X and this capital Y and the rotation is a steady rotation, so at time t this angle is omega t. And then, the question is if I give you a mass m and it is undergoing some dynamics in the x y plane, what does it looks like. If looked at from the coordinate system, the capital X and capital Y which are changing with time.

So, the way to transform from little x to capital X and capital Y is the familiar one. So, I have X of t and I am not going to write the t dependence here, it is explicit, it is implicit this is equal to cos omega t times little x plus sin omega t times little y, whereas capital Y is equal to minus sin omega t times little x plus cos omega t times little y. Now, we have dynamics going on in the little x little y plane.

So, we have a particle of mass m which then under goes an acceleration a, which is a function of t and that is equal to the force acting on it and I will denote that by a small f here, that too in generally changing with time. Now of course, I can resolve this into x and y components and it is clear that this will imply that m times a x equal to f sub x m times a y equal to f sub y in Cartesian coordinates. What happens here in this case?

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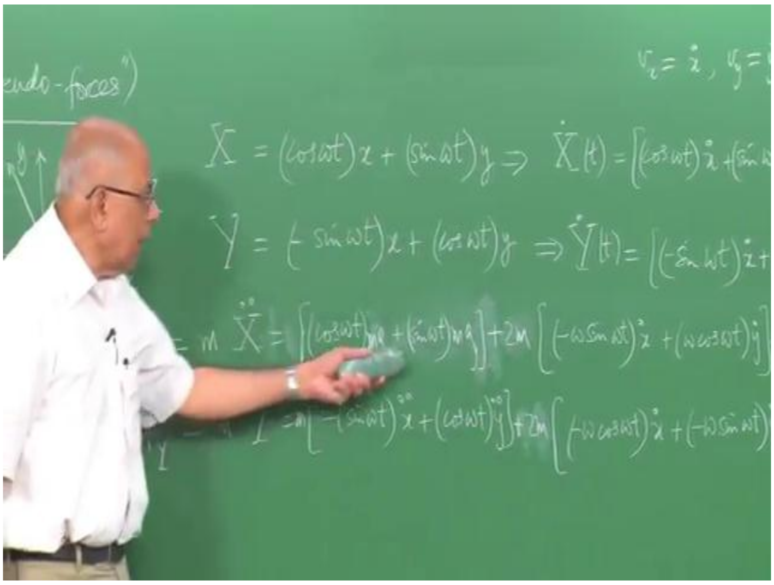

Well the first thing we have to do is to differentiate these with respect to time and this will immediately imply that x dot of t and let us, saves space by putting overhead dots for time derivatives x dot of t is equal to there is one term in which x and y get differentiated to give me little x dot and little y dot. So, this is equal to and let us put brackets to combine terms together suitably. So, this is cos omega times little x dot plus sin omega t times little y dot and then, there are two other terms which are plus.

So, we have an omega coming out, so plus let us put omega insides. So, this is minus omega sin omega t times little x plus omega cos omega t times little y and that is it, once I differentiated the first time not only does do these get differentiated, because the particle is moving has some velocity components. But, in addition these two quantities, which are the coefficients relating the whole coordinates to the new coordinates also get differentiated with time and this is what happens, those terms are sitting here.

Similarly, this will imply that y dot of t is equal to again the same thing minus sin omega t times little x plus cos omega t times little y plus extra terms y dot in this side plus extra terms, which correspond to minus omega cos omega t times little x minus omega sin omega t. So, let us put a plus sign and I want to keep all the minus signs right, so let me put them in brackets in the sin omega t little 1 in this fashion.

So, we have two extra terms, which come from the differentiation of the manner in which the coordinates, the new moving coordinates depend on the old coordinates here. Now, these of course, x dot and y dot are components, so v x equal to x dot v y equal to y dot and we will deal with these in a minute. What happens if I differentiate one second, I need to differentiate one second, because I want to relate the acceleration in the new frame to the force in the new frame and find out if Newton's law is valid.

Or, if it is not valid as I suspected it would not be because of these extra terms, what are the extra terms that get in, so this is the target to find out, to what extend can I write Newton's laws after identify the new force in the new frame of reference and the new acceleration. So, the second derivative x double dot of t is equal to again I differentiate this term and this term first and I get cos omega t x double dot plus sin omega t times y double dot that is the first path.

And then, there are several other terms coming from the fact that I have to differentiate this and of course, I get twice this bracket now. Because, if I differentiate this I get minus omega sin omega t plus omega cos omega t and this time I am going to get, if I differentiate these two the dots on top. So, there is a term which is equal to twice and then whatever inside the bracket omega sin omega t x dot this time plus omega cos omega t y dot this time twice.

Because, once from this term and once by differentiating these two the x and y there. And finally, there is a third kind of term which comes from differentiating this quantity that is minus omega square cos omega t minus omega square sin omega t. So, let us put a plus and in minus this guy here I should be careful times little x of course, plus. So, let us put the minus sign out minus omega square minus omega squared cos omega t sin omega t y expression.

So, we are comes from differentiating X a second time and let us differentiate y quickly a second time. And then, we see the structure of the whole thing this is equal to again I differentiate this part and this part, so it is exactly as before minus sin omega t x double dot plus cos omega t y double dot plus twice. And now, I am going to get these derivatives, so it is going to be minus omega cos omega t x dot, this time plus minus omega sin omega t y dot this time.

And finally, whatever I get by differentiating this and this second time. So, plus omega square sin omega t the minus sign goes away here, x and then the derivative of this, which is minus omega square cos omega t y and that is it. We finish differentiating and we got six terms on either side, because I started with two and I differentiate twice. Now, it is immediately clear that if I multiply this by mass, I get here the component of the acceleration along the capital X direction in the moving frame.

So, this is equal to A let us call it A and this is the X component of the A, A sub X that is m X double dot the A X component of the, let us know let us be careful here, this quantity here is the mass times the acceleration as it stands. So, let me call this m times A subscript x, by definition A subscript x is x double dot just as little a subscript x is little x double dot. So, I am going to use a capital letters for the rotating frame, small letters for the original frame the inertial frame of reference.

And similarly, I multiplied by m here, this is m times A y and now I need to multiply this whole thing by m. So, there is a m here there is a m here plus twice m here, plus twice m here plus m times this plus m times this, that is it. Now, what is m x double dot? It is equal to and I am going to write on this board itself to say myself, space in right time and writing. This is equal to a subscript little x m times x double dot and m times y double dot is a subscript little y times and m, so let us put m a little x plus sin omega t m a little y plus.

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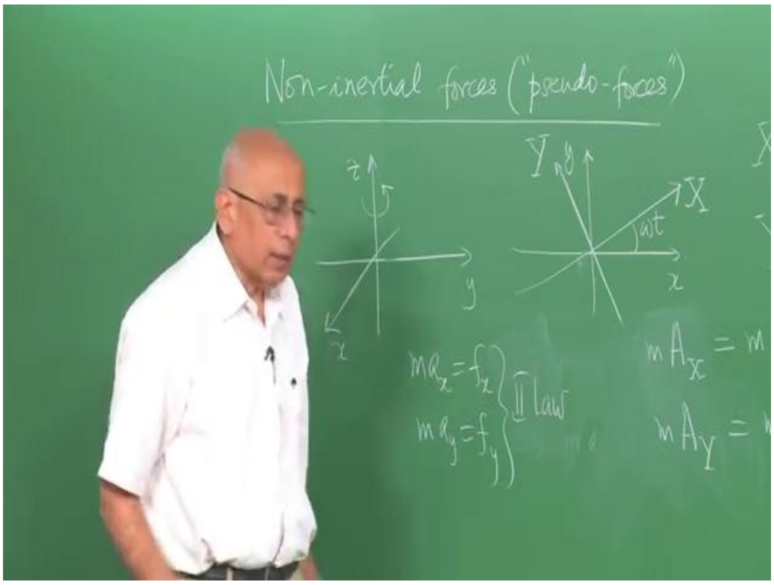

But, what is m a x? I know that in the inertial frame of reference ((Refer Time: 12:57)) m times m a x equal to f sub x, m times a y equal to f sub y, this is Newton second law, where f sub x and f sub y are the Cartesian components in the inertial frame of reference of whatever forces acting on this body of mass m. ((Refer Time: 13:21)) So, I replace those here and I get f subscript x and f subscript y out here, in exactly the same way on this equation I get this is equal to once I multiplied by m, I get f subscript x f subscript y plus this the term whatever is here. But, what is this quantity here? This is the transformation of the force, the new x component when you rotate the frame of reference through an angle omega t. Remember, that the force is a vector.

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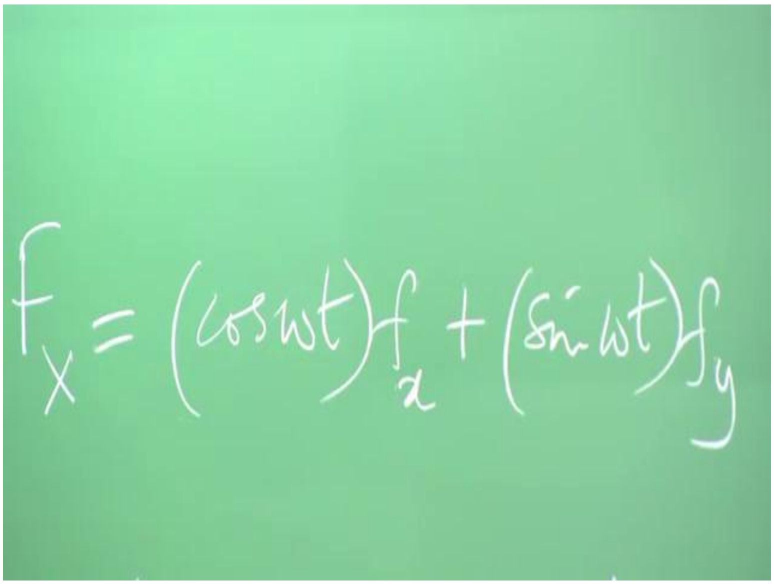

So, it is immediately clear that F subscript X must be equal to cos omega t f x plus sin omega t f y by the very meaning of the statement that the force is a vector. So, this combination here, by it is very definition is the force the new x component of the force and let us use a capital F for that.

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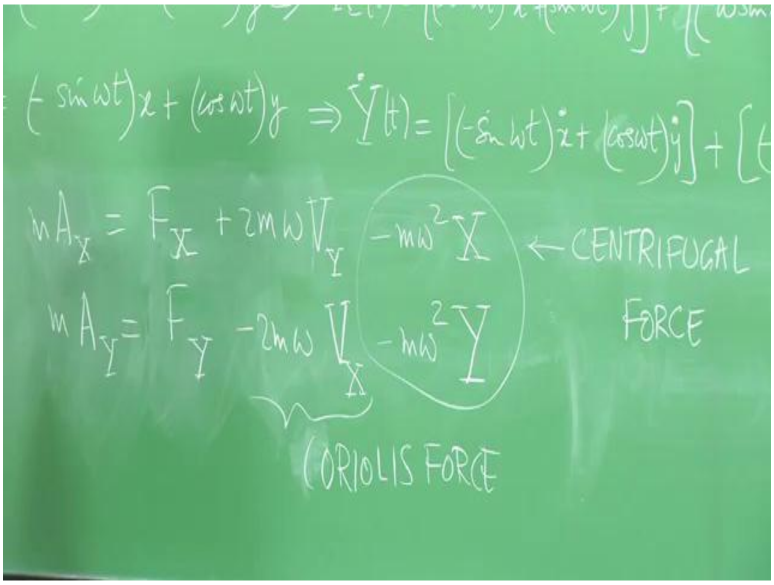

So, this whole thing goes way and I get F subscript X and in exactly plus all this terms and in exactly the same way I get here F subscript Y plus a whole lot of other terms. So, it is immediately clear that Newton second law is not valid in the original form in the rotated frame of reference, so the accelerated frame of reference. ((Refer Time: 15:05)) Remember when you have rotational motion is always an acceleration you cannot kill the acceleration. So, this immediately implies that new ton second law is not valid.

So, write in Newton second law in this form tells ((Refer Time: 15:21)) you that m A X is equal to F subscript X plus some other piece. And similarly, m A sub Y is equal to F sun Y plus some other piece and these are the dimensions of force components, which is why these are called pseudo forces or non inertial forces the arise entirely from the fact that the co efficient in the expansion a time depend and we have to differentiate them.

So, although there is no physical agency that produces these forces for instants sub F x is some kind of force, which arises to some electrical charges or electrical force there is no such agency produces these terms these arise purely from the kinematic some accelerated motion. So, since we multiplied this by m to produce forces, what really arise initially was differentiating this coordinate we discover extra terms in the acceleration. So, I should really call these in inertial a effects are acceleration pseudo acceleration if you like.

But, then once I multiplied by m they become forces are some kind, they are forces in the sense that in the new coordinate system the system does see this forces. Because, this acceleration has a this transformed force the physical force and then there are these extra terms here, which are sitting here. But, what do this terms look like well immediately see that this term for pull out and omega and let us, do that ((Refer Time: 16:55)) 2 m omega and, so this omega goes away that omega goes away this term alone remember that this is v sub x and this is v sub y.

And, what is it look like it is clear that it is look like the velocities also vector this looks like the y component the transform the y component of the velocity. So, the immediate identification remember again are original rules of transformation for the x and y this tells you, what is going to happen this think here ((Refer Time: 17:51)) is v capital V sub y here. So, this is capital V sub y, so that part is identify and this term here, with the minus sin and then omega is capital V sub x.

So, the some structure imagine here, this tells you that in the new coordinate system, if you measure velocities, in the velocity of the object multiplied by omega times m. Whatever, then this, this extra terms sitting here, and the dimensionally these things are dimensions of force and they are these term and what are these terms like well again you can see that this is minus m omega square. So, pull out minus some omega square I get cos omega t x plus sin omega t y that is capital X

So, this is minus then omega square capital X and that is gone and in exactly the same way pull out sin this gives me minus sin omega t plus cos omega t, so this is minus m omega square capital Y. So, these are final expressions for the x and y components of the acceleration the mass time acceleration is actual physical force transform and looked at a new coordinates plus these extra terms or the extra forces. If, you like non inertial forces are rather forces arising in a non inertial system due to the kinematics due to the fact that your accelerated frame of reference.

This term these two term this or this are the components of the, so call coriolis force, while these two terms are the components are the, so call centrifugal force. Can we write this in a more elegant form, then this in the answer is here, what is needed is the followed give rotated about the z access.

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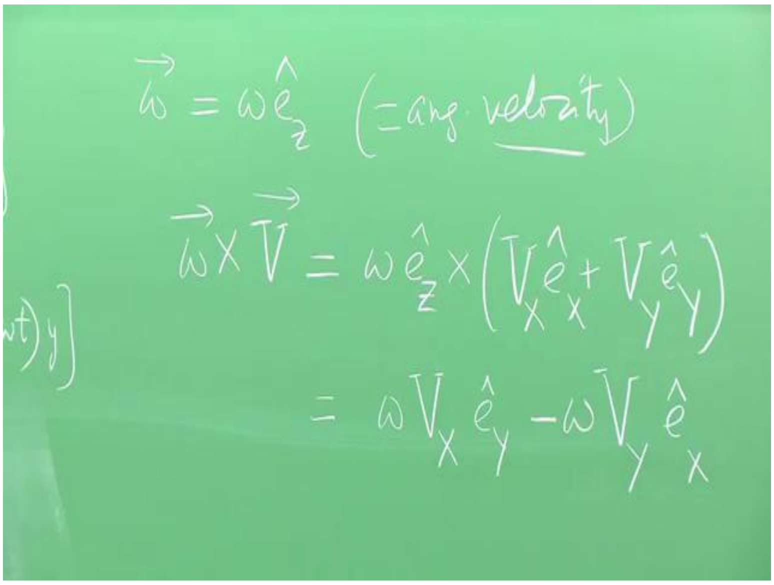

So, let us define an angular velocity to be equal to omega times is sub set that equal to the angular velocity and what is this term here ((Refer Time: 20:27)), then the omega it part of is a component of the angular velocity z components. So, if you consider, what omega cross V is this is equal to omega e subscript that small z or capital Z does not matter does not change at all cross and here. You have V x e x plus V y e y here, and this quantity is equal to z e x is minus is plus e y.

So, it is omega is a cross product e y and then, z cross to y is minus y cross z, which is x, so minus omega e y e x in this fashion this a vector ((Refer Time: 21:33)) we have a 2 m omega V Y seeing there. So, it is minus the X component of omega cross V, so I could remove this and write it as minus 2 m omega cross V the X component of it because we had a Y and that was minus omega V Y and this is equal to this term here is minus V X e y and sitting V X sitting there. So, it is minus 2 m omega cross V the y component of there.

So, you see we are beginning to get a structure here m times a the X component is F the X component does not two m omega cross V the X component and ditto here, on this side here. It remains to identify this and it is clear here that this X and Y or the components of the capital R. The new, if I call capital R has X and Y components given by this quantity and this respectively well little more intricate that put the structure down.

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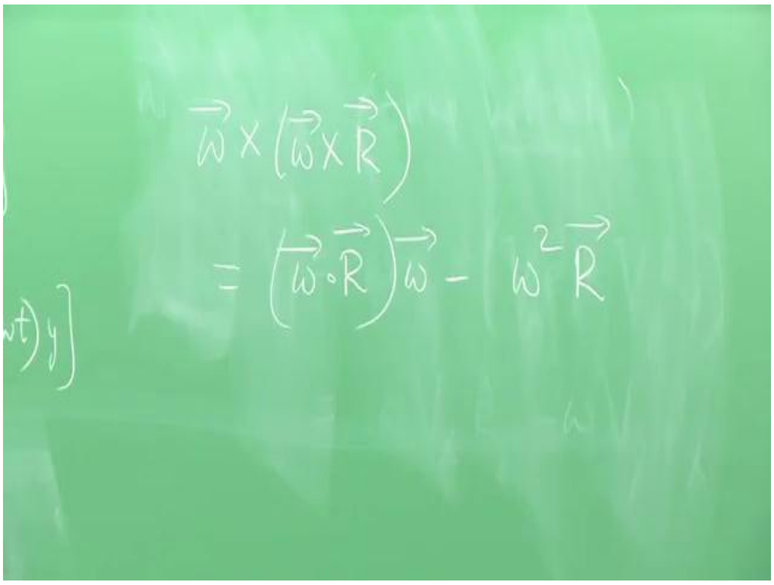

If, you look at what omega cross omega cross R is and use the vector identity, then it is equal to omega dot R times omega minus omega dot omega, which is omega square times R. And that is precisely, what we have here, apart from the fact that m omega dot R is 0, if R has only x and y components has we assume we will set this object moving in the x y plane that does not change never get to the z axis. ((Refer Time: 23:34)) So, these two components of omega cross omega cross R the x and y components is respectively with the minus sign.

So, let us write this has minus m omega cross omega cross R the x component and minus plus it was a minus sign is already there this minus sign, which is already sitting there, so plus m times omega cross omega cross R the y component.

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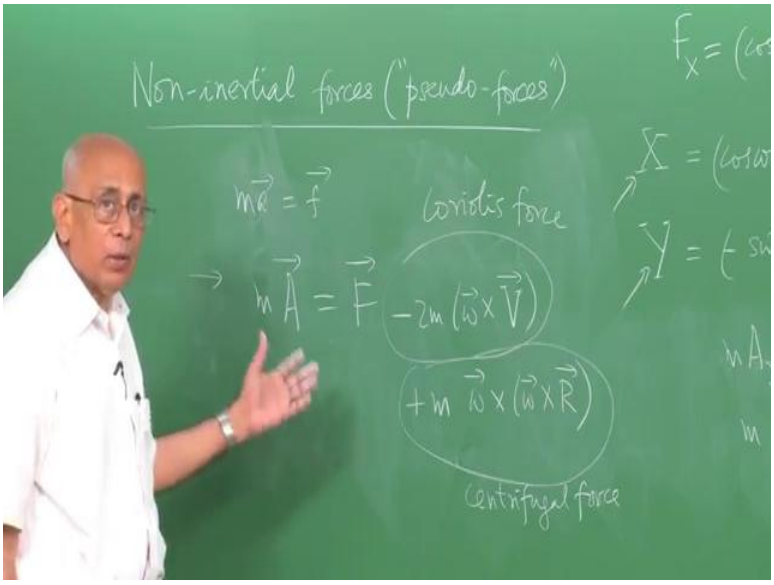

So, we have result now, it says that in vector of form the dynamics will look like this, what originally look like m a equal to f. Now, looks like that goes over into m times A the acceleration equal to the force F out here minus 2 m omega cross V, then velocity new frame. So, everything is equation refers to the new frame of the reference rotated frame of reference plus m times omega cross omega cross R that term is the coriolis force and that term is the centrifugal force and that is it.

So, we have this extra force is a pair in to forces, if you like pseudo forces, non inertial forces, what are you call it. Appearing entirely, because of the fact that you have move the over to an accelerated frame of reference in which Newton's law is not m times a equal to the force applied are the external force, but there are these kinematic contribution here. When I multiply those acceleration by m I get forces effective forces these forces do exist and they do act on the body as seen in the retaining frame of references.

One point you note immediately is that, if in the rotated frame of reference the object does not move in other words V is 0 the coriolis force is 0 identically and you have the centrifugal force that always present no matter, what? Even, if you have a static object it does not move in the new frame of reference it still experience this force this term is still present. So, the ways it is coordinate related to acceleration that is this extra contribution here.

And of course, in the simple instance, we have looked at of a constant rotating uniformly rotating x y plane this thing reduces somewhat minus m omega square with corresponding component ((refer Time: 26:38)) of R that part, but this is the general form in general. For arbitrary omega no matter, what this is the general form like here, we need not now restrict ourselves to the x y plane omega rotation. There is one assumption we made namely we rotated this frame at a constant rate angular speed.

So, the question is, what happens, if I also change the rate of rotation or change even the axis of a rotation as a function of time. There is one more pseudo force or inertial force, so it is called the oil of force and it is depended on omega dot as you would expect I am not going to write that down since that beyond into scope of what I want to say here. These are the most common forces here and we can actually compute these quantities.

Basically, what we need to compute is the correction to is the acceleration in the centrifugal acceleration we can estimate, what this centrifugal acceleration is in various cases the most common example of course, is what happens of the center of the earth from, let us estimate that immediately.

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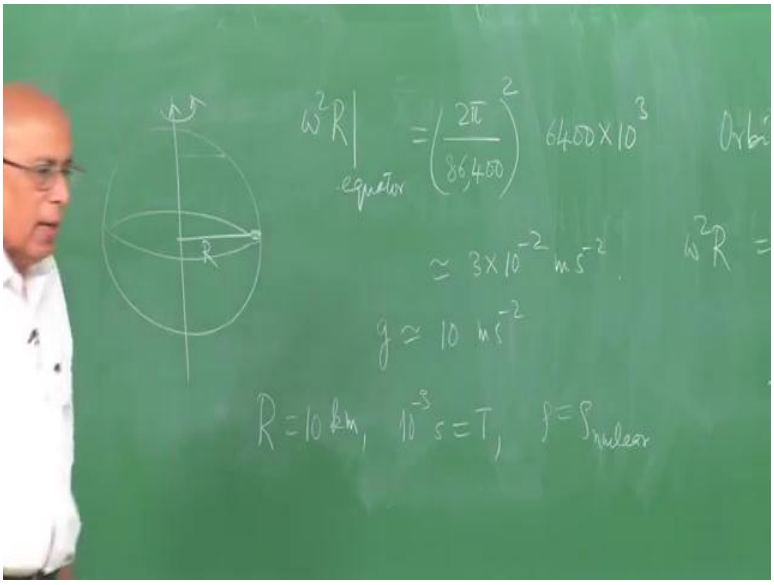

So, we have the earth rotating about it is axis are an angle that is the equator and let us look at, what happens to a point on the equator. Because then, we have to worry about the total radius of this earth on the other hand, if you are in this latitude is only this action distance that effectively contributes to the rotation that is the one that appears in this formula the R. So, what we need to compute is, what is omega square R in the case of point on the equator of the earth.

So, omega square R on the equator is equal to 2 pi divided by the time period just 24 hours and 24 hours as you know has 86400 seconds. So, we need to square this and multiplied by the radius of the earth and the radius of the earth 6400 kilometers. So, it is 6400 multiplied by 10 to the power of 3 in meters and it is not have to say this number is well known this is approximately 3 times 10 to the minus 2 meter second is minus 2 that is the correction this units.

So, given that in little g is of order of 10 meter second this is like three parts in a 1000 meter second this is like three parts in a 1000 the correction more precise number is between 980 and 983 at the pole this correction would not be there, but at this equator it is maximum. So, it is a small correction to the force of the gravity here and it is cost that is out word filing of the due to the rotation of the earth not in a inertial frame. But, you practically in inertial frame, because this correction is pretty small compare to little g, which is of the order of 10.

So, the ratio is the order of 10 to the minus 3 and therefore, what happens, now when the sun go the earth goes on the sun that two is rotating frame of reference. So, we can compute, what happens the quickly.

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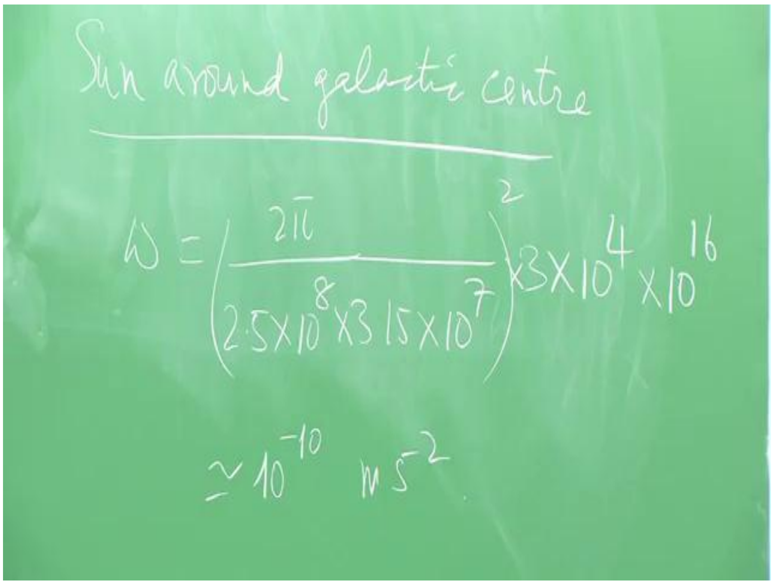

So, orbit around the sun orbit around the sun and we need to compute omega square R. So, omega in this case is 2 pi divided by a year and year is got 365 days multiplied by 8600, whatever this is a 3.15 into 10 to the 7 seconds that is useful figure to remember all ways, because 3 point 15 is pretty close to pi and we can cancel the pi's on top to get an order of magnitude estimate. So, that omega and R so let us put omega square R less then square times R is the rate less the distance from sun to the earth and this is a 150 million kilometers.

So, it is 1.5 into 10 to the 8 into 10 to the 3, so that is into 10 to the 11 and this is, what we have to compute and this is of the order of 10 to the minus 3. Again a very small correction this acceleration is very, very small compare to little g, well that is not the end of the story, because a sun goes around the center of the galaxy. So, one could ask is that going to make a difference that is going to be hopefully and much smaller contribution.

So, sun around galactic center omega in this case is 2 pi divided by the period around it takes to go, that is 21 and 50 million years approximately. So, it is a form a 2.5 in to 10 to the power 8 times 3.15 times 10 to the power 7 that is in seconds this whole thing is square multiplied by the distance from here to the center of the galaxy. Well this galaxy is of the order of 100 thousands kilo a light years in terms of it is diameter and we are about two thirds in the way center.

So, perhaps about 30 thousand light years also, so this is 3 times 10 to the 4 light years and the light year 10 to the 7. So, this is of the order of 10 to the 16 meters and we need to compute this and this of course, immediately show it is 16 small this of the order of 10 to the minus 10, 10 to the minus 9, 10 to the minus 10 etc meters square. So, again very small correction very, very small correction due to this non inertial force this centrifugal force, you could ask what happens, if you has much more indecent gravity and much more fastest spinning.

So, as a little exercise calculate what happens when this is a pulsar 10 kilo meters and radius R equal to 10 kilometers and it is a millisecond pulsar. So, the thing takes 10 to the minus 3 seconds is a time period and you need to, now find out what is omega square R in this case and I leave you to figure out what happens on the surface of this. And compare it to the gravity there assuming that the row density equal to nuclear density. So, say 100 protons in a volume, which is linear dimension extend to the minus 15 meters.

So, compute that and find out the correction is in other words find out, what omega square R divided by little on the surface of a pulsar a millisecond pulsar that is a 10 kilometers all that object rotating of nuclear density rotating at the rate of a millisecond that some interesting exercise look at.