Mechanics, Heat, Oscillations and Waves
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Lecture – 23
Kepler's Law of Planetary Motion

Now, let us turn to the problem of a planetary motion as encapsulated in Kepler's laws of planetary motion. As you were aware, there are three such laws and what I am going to try to do here is not to drive these laws or anything like that. But, merely to take these laws and put them in a broader perspective of understanding the two bodies central force problem itself. And point out, what is common between the general problem and the particular problem of Kepler's of planetary motion and what is not.

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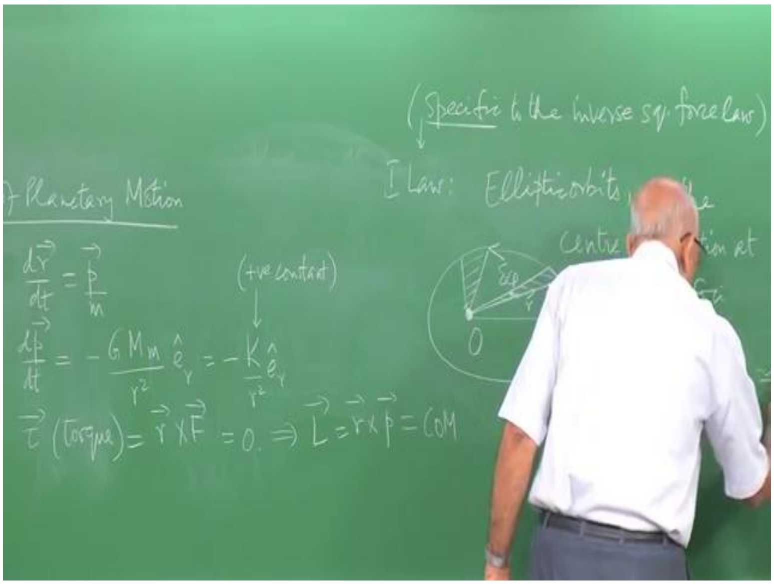

So, let us start by saying that we have a particle of a mass little m, instead of mu, I might as well as use little m. And we write down, it is equation of motion, which is m times d r over d t equal to it is momentum p and a rate of change of momentum is given by the force on the particle by Newton second law of motion, which in this case is capital G. The mass of the sun, the center, whatever is attracting it, times the mass of the planet divided by r squared times e sub r.

Now, let me simplify this a little bit by writing this as equal to minus some constant, capital K over r squared e sub r. So, this here is some positive constant, whose value is going to depend on the particular units that you use, the particular masses and so on and so forth. So, this is the problem that one has to solve, these are two coupled equations for the vectors r and p and the statement is, given r of 0 and p of 0, one should be able to deduce, what r of t and p of t is by solving these two couple differential equations.

But, because there is r dependence sitting here this problem is not trivial at all and this is what Newton addressed. After seeing a lot of data from Kepler, who then enunciated his laws empirically, Newton derived those laws completely from the fact that he could write down the equations of motion and solved this problem in some sense. Today, we would not solve these equations, the way Newton try to solve them by the very specific geometrical tricks that he used and so on. We just use the calculus on our knowledge of differential equations to try to solve, to solve these particular equations.

But, the interesting thing is, the interpretation of the solution more than anything else and we know that, the solution to this is that, when you have an attractive potential 1 over r potential, there are bounds states possible, which are orbits, which are ellipses and so on and so forth.

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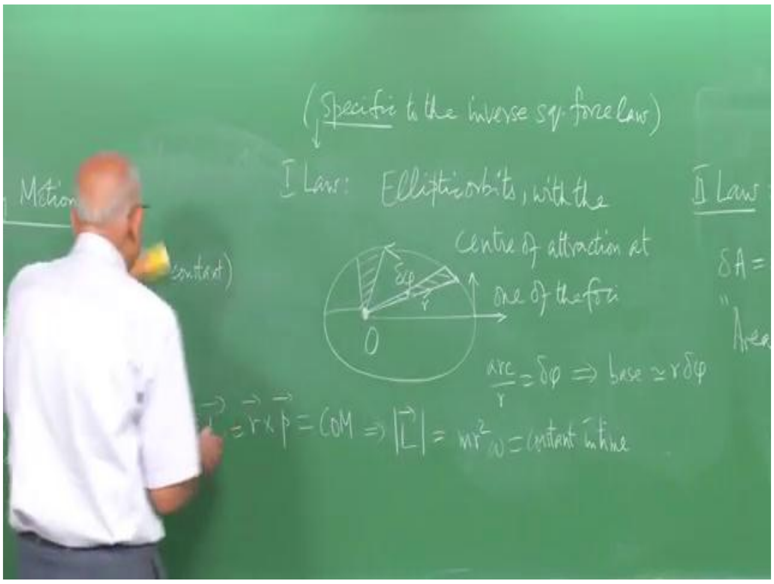

Now, where does this take us, what does it tell us, well the first law says that this mass m, now let me do it in an abstract form moves in an ellipse for any initial condition. It moves in an initial, in an ellipse with the sun at one of the foci. So, there are two foci and the center of attraction is here, so that is the origin this. So, already you can see that even though, this problem looks completely spherically symmetrical, the solution does not have that symmetry, the solution has an eccentricity, it is an ellipse, it is not a circle.

Circular orbits are a special case of elliptic orbits; however, in this particular instance, it turns out the planets have elliptic orbits with undoubtedly small eccentricity as that happens, but nevertheless there ellipse is. And in principle, the elliptic curve, the eccentricity could be arbitrary, after all we do know of comics, which have highly eccentric elliptic orbits. We know of artificial satellites which have extremely eccentric orbits, a very large semi major axis or a very small semi minor axis and yet, it is in stable orbit around the earth.

So, in exactly the same way, we have ellipses, elliptic orbits with the sun or rather the center of attraction or origin of coordinates in this case at one of the foci, so that is the first statement here. Now, what I like to ask is, is there any other potential, any other form of this potential, instead of this we have something else, some other force law would that also lead to elliptic orbits.

Even, if it says it is center of potential also, would it lead to elliptic orbits, then the answers turns out to be yes, I will come to back this and point out, there is one other central force in three dimensions for which you have elliptic orbits possible. But, with the slight difference from what this law says, we will come to that subsequently, so that is what the first law says.

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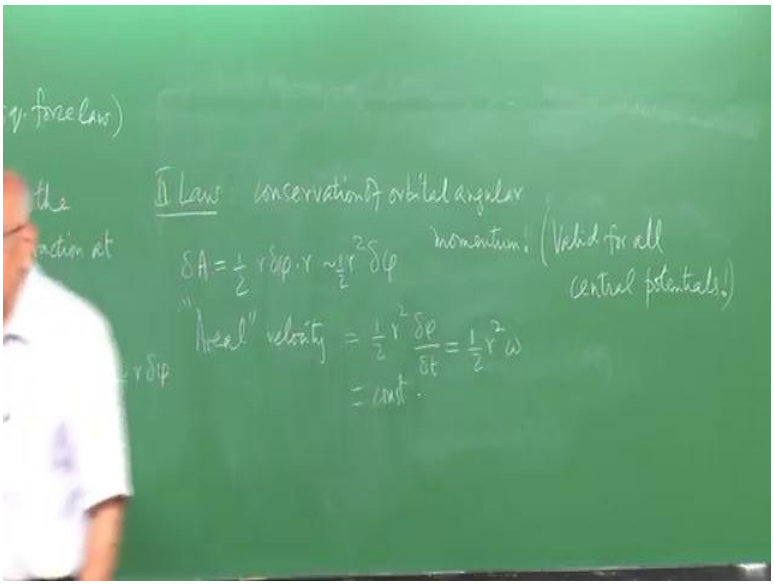

The second law say something more general, but before that ((Refer Time: 05:32)) let me point out here that this particular statement elliptic orbits with the centre of attraction at one of the foci for the bound orbits is specific to the 1 over r potential or the inverse square law force. So, let me write that down, that this first law is specific to the inverse square force law, if it is not the inverse force square law, this is not true. You do not have elliptic orbits with the center of attraction at one of the foci.

To cut a long story short, there is another potential which we will talk about towards the end, where you again have elliptic orbits possible, but in that case the center of attraction is at the center of the ellipse and not at the one of the foci. So, this is what distinguishes motion and that potential from motion under 1 over r squared a force or 1 over r potential, the Kepler problem or the Coulomb potential.

So, the second law says the radius vector from the planet from the center of attraction to the instantaneous position of the planet traces out equal areas in equal times. Let us look at this little carefully, it says that if in an infinite decimal interval of time d t, this is the area traced out. Then, at a later instance of time, while the radius vector changes, the angular speed is changes, everything changes, but this area in the same interval of time d t has got to be equal to that; that is what this says.

Now, what is this actually telling us, while first of all the moment I say that this orbit is an ellipse, it is clear that is the orbit is planar, it is not a three dimensional curve, but a planar curve. Why is that? Because, it is a central force and there is no net torque on this particle, ((Refer Time: 07:38) because notice that the torque on the particle tau, torque is equal to r cross F, but F is in the radial direction inverse. And therefore, r cross F, the sin of the angle between the direction of the force and the radius vector is 0 and therefore, this is equal to 0.

And if the torque on the particle is 0 as it is in any central force, it immediately implies, this implies the orbital angle of momentum L equal to r cross p equal to the constant of the motion. Now, when the orbital angle on momentum is a constant of the motion, this is the vector and therefore, when you say vector is a constant of motion not only is it is direction is a constant, but it is magnitude is also a constant.

And what is it mean to say, it is direction is constant, if the particle is moving in an ellipse in the plane of the black board. If instantaneously, this is the radius vector in this direction and the momentum at this instance of the time is this way, r cross p goes in to the plane of the black board at this point of time. And at a later point of time, it should again go in to the plane of the black board and at any other later instance, it should again go into the plane of the black board normal to it, which means that, all these little area elements are in the same plane, which is equivalent to saying that the orbit is planar.

So, the planarity of the orbit is really arising as the consequence of the fact the angular, orbital angular momentum is the constant of the motion, both in magnitude as well as direction here. Now, the second law says something further, it says equal aerial velocity, it is says that this little triangle and this triangle have exactly the same area. But, what is the actual area of this triangle, if you write out of any instant of time, if this angle here is delta phi and this distance is r. Then, this arc as you can see the base of this triangle essentially is arc divided by r equal to delta phi.

So, it is says that this arc or the base is r delta phi, the base of this infinite decimal triangle is r times delta phi. What is the area of the triangle, it half time the base times the height, so this little triangle has an area delta A, which is equal to 1 half of the base of the triangle, which is r delta phi times the height of the triangle which is essentially r. Apart from the factor of 2, says this, where put that factor in ((Refer Time: 10:53)).

The rate of change of this area this area velocity, therefore aerial I should say aerial speed actually, because it is not a vector. The area element is a vector though we can associated direction with it, this is equal to 1 half r squared delta phi over delta t, which is equal to 1 half r squared the angular speed omega, which is delta phi over delta t. So, it is really saying that the angular speed multiplied by the squared of the instantaneous radius vector is a constant is time; that is what it is saying. ((Refer Time: 11:38))

On the other hand, what is the orbital angular momentum in magnitude, the magnitude of L is equal to the moment of inertia multiplied by the angular speed and the moment of inertia is n times r squared respective the origin multiply by omega and this is the constant in time. So, you have omega r squared is constant in time times m, which is the constant by the conservation of angular momentum and this second law says half r squared omega is a constant in time, saying essentially the same thing.

So, it says while omega may change and while r may change, the combination r squared omega does not change, the product r squared omega does not change. Out here, r is large omega is small, at this point r is small and omega is large, is it through much faster here in much in here to change over rate of change of the angle azimuthal angle. So, essentially it is says the second law the recall the fact that that this is equal to constant time is the time as the conservation of orbital angular momentum. So, the point I want to make is that, Kepler second law is not specific to the inverse square law of force.

It is in fact general statement; it says that, no torque on the particle in the center of force, therefore it is arbitral angular momentum is a constant of the motion, both in direction as well as magnitude and that all it is saying. So, this law while it is specific as I said to, the first law is specific to the inverse square law of force the second law is true in general valid for all central potentials.

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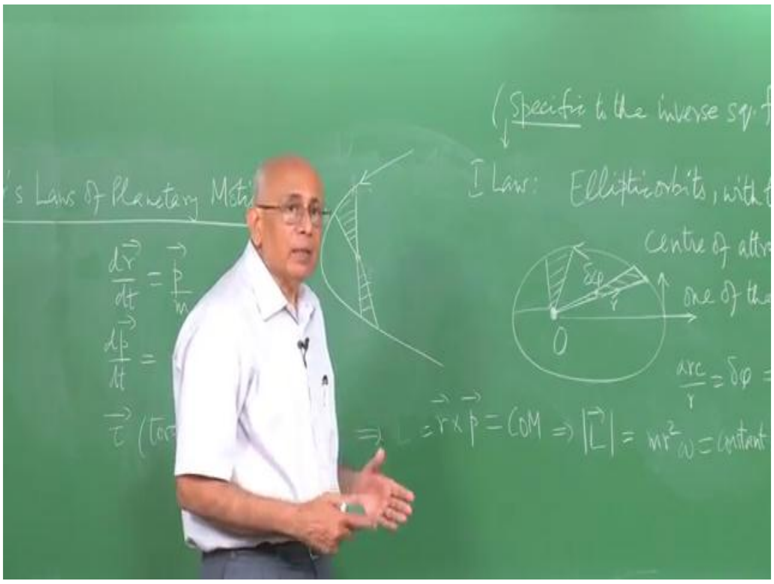

In the Kepler problem itself, it is in fact valid even if the orbit is open. So, it is certainly true that if you have a common hyperbolic orbit. So, here is the center of attraction and here is the orbit of this comet, even if it is not in elliptic orbit in equal instance of time, this area would equal to the area traced out in the same interval of time. So, the second law Kepler's second law is not specific to planetary motion, it is true for all motion in any center potential whatsoever.

And in the case of the inverse square law of force, it is valid whether the orbit is an ellipse or a hyperbola or parabola, it is still true, even for non-periodic motion, such as that of common, it is still true, this law is still true. So, it is worth appreciating in the fact;

that is law is really a general statement conservation of angular momentum and not the specifically restricted to the elliptic orbits in the inverse square law force.

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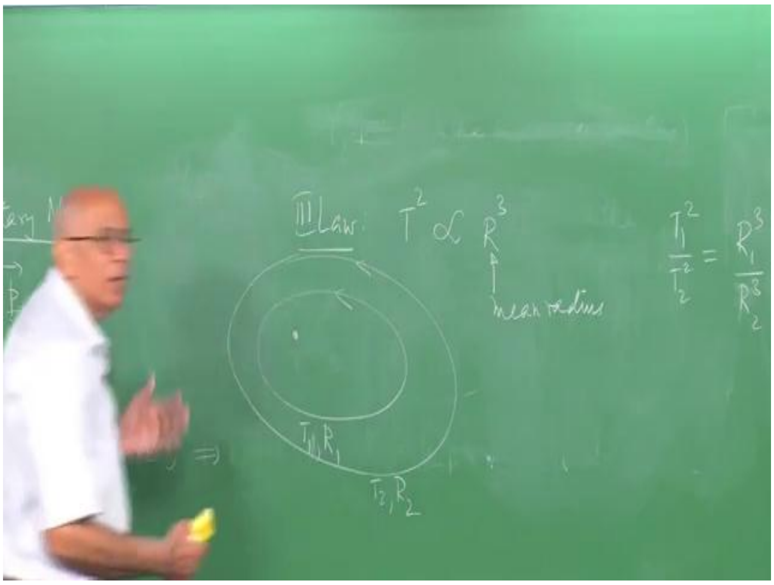

What about the third law, the third law once again is specific to the inverse square law force and it says as we know. The square of the time period T squared is propositional to the cube of the mean distance from the sun, which in this case would be some kind of average over the elliptic orbit. So, this is propositional to the R cube and let say this is the mean radius is not important, where it is a semi major axis or the semi minor axis or whatever, it some mean value; it is a length scale which prescribe the size of the orbit and cube of that is propositional to t squared.

So, what it is really saying is that, the same potential, the same force can now support different orbits. So, there is an orbit of this kind with the mean radius with say time period T 1 and mean radius R 1. In the statement is that, the same potential with different condition initial condition could also support an orbit of that kind with T 2 and R 2. And what is being said by this statement, what is meant is that T 1 squared over T 2 squared is guarantee to be R 1 cubed over R 2 cubed.

So, this is the content of the third law planetary of planetary motion and where does this come from, well elementary derivation for a circle orbit or possible, then not very difficult. Well, easy to say that, assume this special case of circular orbit, you can easily prove that with 1 over r squared force, you going to have property precisely this kind for a circular orbit. For an elliptic orbit, it is little more complicated to prove, but I am not going to do that.

Instant proving that I going to show you that this law is really a property of 1 over r squared force and follows from something called scaling, a scaling relationship. It does not prove that the orbit are ellipses, it does not prove the first law anything like that. But, it still tells you, why we have these particular powers, why 2 here and why 3 there, why not something else and y this relation at all to start with.

So, the way to understand that law is in this sense namely, if the potential, if the force excreted by this center of attraction, this inverse square law force can support a closed orbit an ellipse with mean radius R 1 with time periods T 1. It can also support the same potential also support another orbit of mean radius R 2 and time T 2, where the ratio of T 1 and T 2 is related to ratio of R 1 and R 2 in this particular manner.

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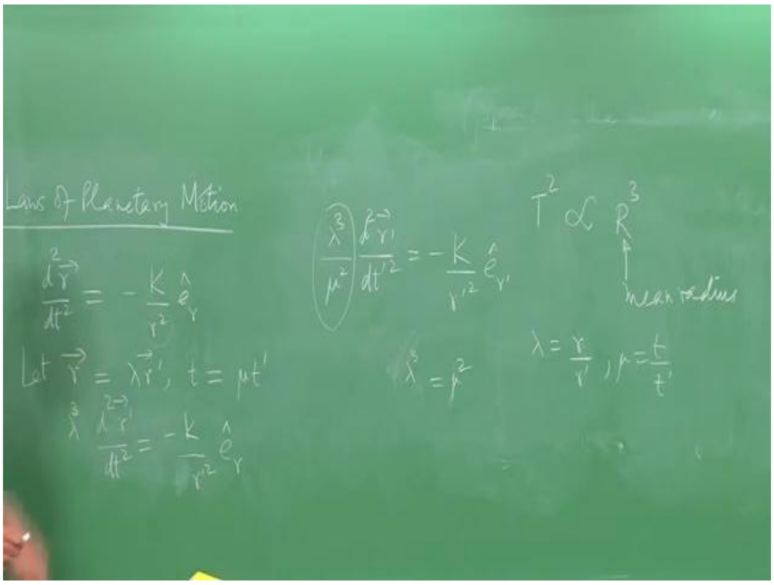

So, that is what this law is really saying and where does it come from, well it comes from a little bit of glorified dimension analysis, nothing more than that let me show you how, let us write the equation of motion in Newton's from for this situation. So, you have m times d 2 r over d t 2, this is equal to minus the force or some constant minus g m on m 2 etcetera, whatever it is finally, cancel out all this write this as minus k over r squared e subscript r expansion.

So, that is really what the force is and this is the acceleration that some positive constant, once you give me the masses of the sun and the capital G and little m and do on, so this is giving to me here. Now, I stare at this and I say well the dimensions, physical dimensions of the constant is the important, this is got dimension length is as dimension is time here, this is unit vector. It has no dimension at all, this is 1 over length squared and this takes care of the balance with dimensions in this case here.

Now, I come along and say that let me change the length scale in the problem by going from r to some r prime, so let say lambda times and r times. So, I do not like the unit that I am using, instead of choosing the particular units I have, I choose different unit. So, that what I have the length r magnitude r becomes lambda times r prime in this case, no physics would really change once this is done.

And let see what the consequences of this is, so this tells me that I have lambda time d two r prime over d T 2 we could minus K over and here I have r squared. That is lambda squared r prime squared e r, e r or e r prime does not matter because parallel to each other. So, it is a same unit vector and it dimension less magnitude is 1 and it direction is that of radius vector, which is the same whether you use r of r prime. So, this is the new equation and I bring the lambda that to left hand side I get lambda cube times this.

Moment, I have this, I say well if I change this scale of length, why not change the scale of time as well and so measuring time in seconds may be measure in minutes or hours or whatever, it should not change any physics. So, let me also change in variables from t to r t prime, which is equal to some mu times t prime, mu some constant, some number, pure number.

Then, the equation becomes in the new variable, I have lambda cube d 2 r prime over d t prime 2 and there is a mu squared; that is it here equal to minus K over r prime squared e prime or r prime, but that look exactly like, because I pull out the new squared, because if I differentiate with respective t, like differentiating with respective t prime with the mu pulled out.

So, this says the equation of motion is identical to the old equation; same physics is going to be describe, except that I have this extra multiplicative factor here. So, this factor is set equal to 1. then the physics is unchanged completely as it should be under this change the definition of units or scales of length and time. If you do that, it says lambda cube over mu squared equal to one would imply that, now what is lambda, it is r over r prime and mu is t over t prime in this case.

And what we are saying here is that lambda cube is equal to mu squared will ensure that this problem the physics of it is independent of the choices of scale in length and time. And that implies that, if you have two length scales in the problem R 1 and R 2 and the ratio is lambda and you 2 times scale problem, the ratio is mu is t over t time, then this square of this ratio is equal to the cube of this ratio, which is precisely the law. This does not prove that the planets move in ellipses, it does not prove that one the ellipse is the sun is set to the center of attraction is one of the forces; it does not do that any of that.

It says, if all this argument says is, if there is motion supported by this potential the 1 over r squared potential with the closed orbit supported by it with the time scale T 1 time period T 1 and mean radius R 1. It also supports the same potential is also going to support another orbit with time period T 2 and mean radius R 2, but you guaranteed that T 1 over T 2 squared is R 1 of R 2 cube here and this is Kepler's third law.

So, Kepler's third law is very specific to this force here, because we have to put a lambda square here to bring that appear and that is how we got this lambda cubed out here propositional and then, the set it equal to mu squared here. So, Kepler's third law, if we like a consequence of the scaling properties of this potential of this force, well the moment you get the natural question is to ask is, can I generalize this, can I have some other potential, can I have any other potential if so can I find similar rule, yes indeed you can.

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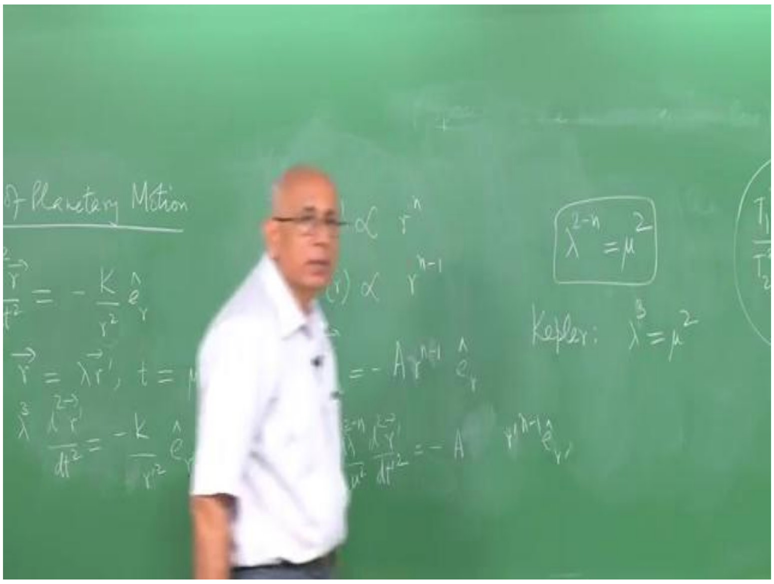

Let us assume without proving, what kind of orbits you have for such potential, let suppose that you have a V of r, which is propositional to r to the power n. well, since we talked about inverse square potential, let us leave it as a r to the power n, when the force F of r is propositional to the derivative this potential. So, it is propositional to r to the power n minus 1.

So, what you have in this formula here would be d to r over d t 2 is equal to some constant lets some other constant call it A, it does not matter, if it is attractive, it go to this way in the minus sign times r to the n plus 1, n minus 1 times e some r. And I do the same trick as before and then, turns out that lambda times d to r prime over d t prime to and again I get a mu squared here, because I change from t to mu times t prime I said t equal to mu times t prime.

That is equal to minus a lambda to the n minus 1 these r prime to the m minus 1 e subscribe r, r prime, it does not matter. So, this is lambda to the n minus 1. So, write it as 1 over lambda to the 1 minus n, I multiplied through here, so you get a 2 minus n and this says immediately that lambda to the 2 minus n should be equal to the mu 2. Therefore, any motion with the characteristics length scale r and a time scale t, if r and t take on values R 1 and T 1, then that same potential or force will also support a similar motion will open, close, we do not cares similar motion with time scale T 2 and R 2.

Such that, the ratio of length scale to the ratio of time scale satisfies this condition here and in the case of the Kepler problem the potential was 1 over r, so n was minus 1 in the force was 1 over r squared. So, you set minus 1 here and you get lambda cubed equal to mu squared leading to this is equation here in the Kepler problem, but I mention in the beginning that one more potential for which this happen.

So, you still get ellipse is etcetera and that is the harmonic capsulated potential, which corresponds to putting V of r equal to r squared. So, what we have is a partial attach to the origin with the spring and it is moving in a line, it moves in a three dimensional space it is call the isotopic oscillator.

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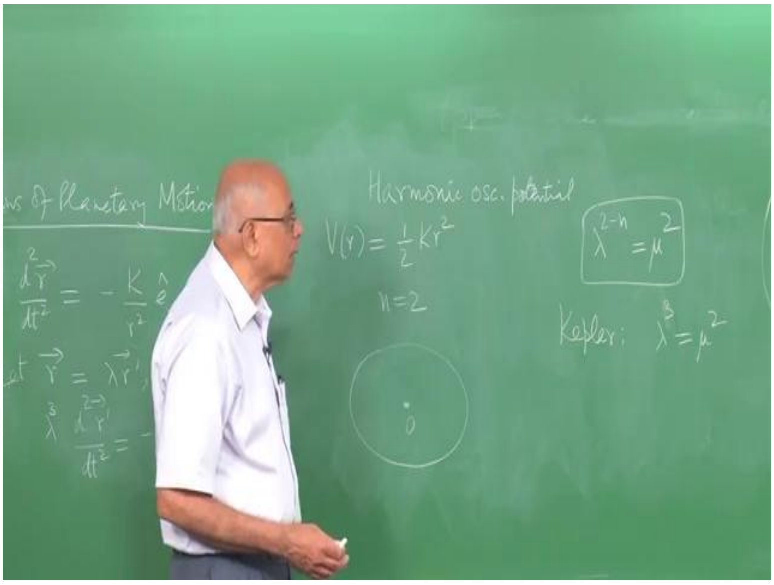

And the V of r in that case equal to 1 half and spring constant times r squared. So, it is like three harmonic consulted right angles to each other the x, y, z, directions and the result in motion. This problem the potential n equal to 2, in this case and in indeed discover that in this case that is no depends on lambda at all of the time scale here and what is that saying, it telling us this way property which mean to and to later.

And when you have simple harmonic motion the time period is independent of the amplitude, in turn it means the time period, this independent of the energy. This property is very, very specific to the harmonic oscillator problem; it is actually one of the defining property of the oscillator. All through it is no mean is really unique to the assaulted, there are other cases which are not so simple, where you have exactly the same property.

But, as for as central potential the concern the only central potential to the power law of this kind r to the power n for the potential for which the time period of the motion is independent of the amplitude of the size of the aptitude is the harmonic capsulated potential n equal to 2. And it also follows as the corresponding equivalent of Kepler's third law, because this thing here becomes lambda to the 0 is the case n equal to 2; that is the only case in which it has happened.

Let me mention finally, that the first law, which is I said was specific to first Kepler's first law, which has said was specific to 1 over r squared potential; the elliptic orbit part of it is not unique. There is one other potential for which the orbit are ellipses the bound orbit the ellipses and that is this potential. The different says in this case is the harmonic oscillator potential.

And the difference in this case is that, the orbits are ellipse is with the center of attraction at the center of the ellipse rather than the 2, any of either of the four psi of the ellipse. So, in that sense r squared potential in the 1 over r potential different from each other, but the conservation of angular momentum is still true. The still elliptic orbit with the center at different point, center at different case, center and one case focus in the other case and third law is different in the two cases. In one case, you have t squared propositional to r cube and the other case you have t squared you have constant independent of r. So, that is the different between these two potential and I thought it should be placed in these perspective.