Mechanics, Heat, Oscillations and Waves
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Lecture - 22
The 2- Body Central Force Problem

A very important problem in dynamics, one which has got a long history and which is absolutely crucial to the study of dynamics is the so called 2 Body Central Force Problem. If you recall, we had mention the last time that we could regard the gravitational force between two particles as arising from a certain gravitational potential, which depended on the product of the masses, multiplied by a gravitational constant, divided by the distance between these two particles, that led to the inverse square law of force between these two particles, the gravitational force. Now, in general 2 body central force problem is as follows.

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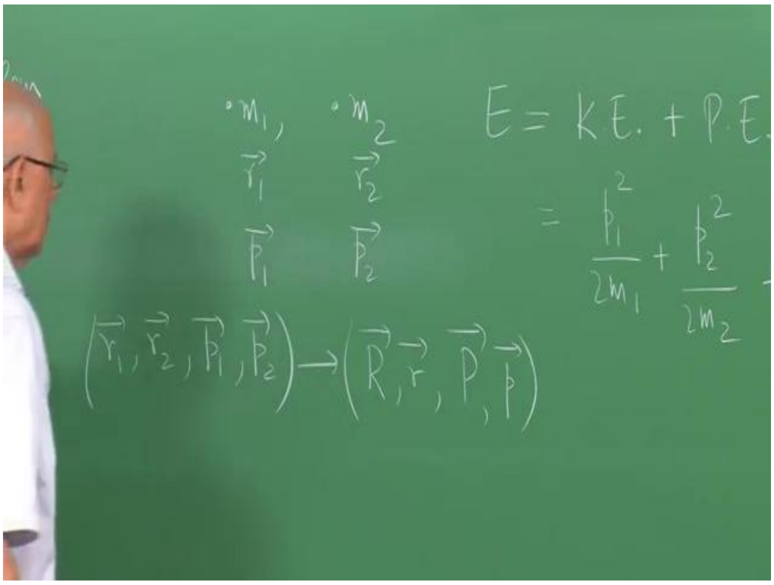

You have two particles of masses m 1 and m 2 with some prescribed momentum to start with and then, the question is if they interact with each other through a central force, namely a force which is directed along the line joining these two particles and which is got, which is derived from some kind of potential. Then, can we solve for all the dynamical variables in this problem, namely can we find the coordinates, instantaneous coordinates r 1 and r 2 of these particles, can we find the instantaneous momentum p 1 and p 2 of these particles.

So, given the fact that some initial conditions and initial coordinates are given, can we find all these dynamical variables as functions of time. In particular, if they should go in to some kind of periodic motion around the common center of mass, can we deduce this or not. So, that is the general 2 body central force problem and what I shall do is to show you the steps by which this problem is approached, not necessarily solve it in the most general case, but then we specialize to the case of the Kepler problem.

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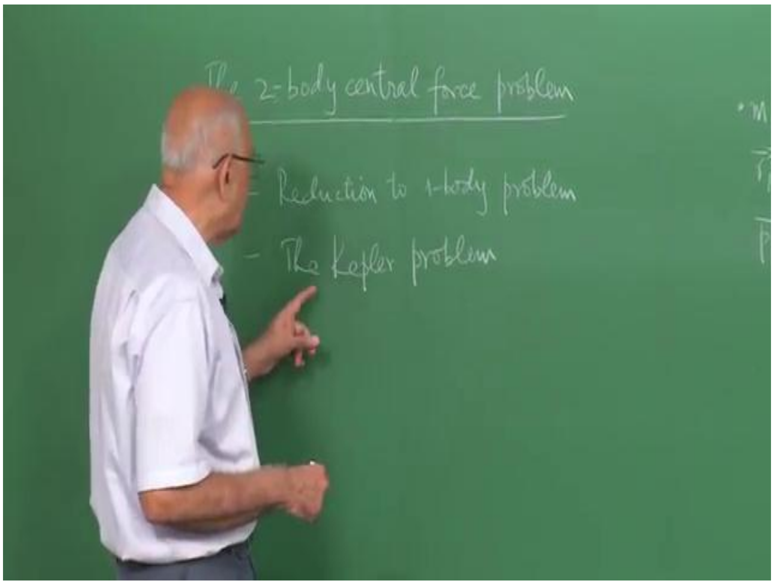

So, after showing that this 2 body problem is essentially a 1 body problem in a certain specific sense, I will show that the problem in the case of the Kepler problem namely the gravitational force between two particles, this leads to orbits the very familiar Kepler's laws of motion and so on. So, this would be our goal to try to understand, how this happens starting from first principles.

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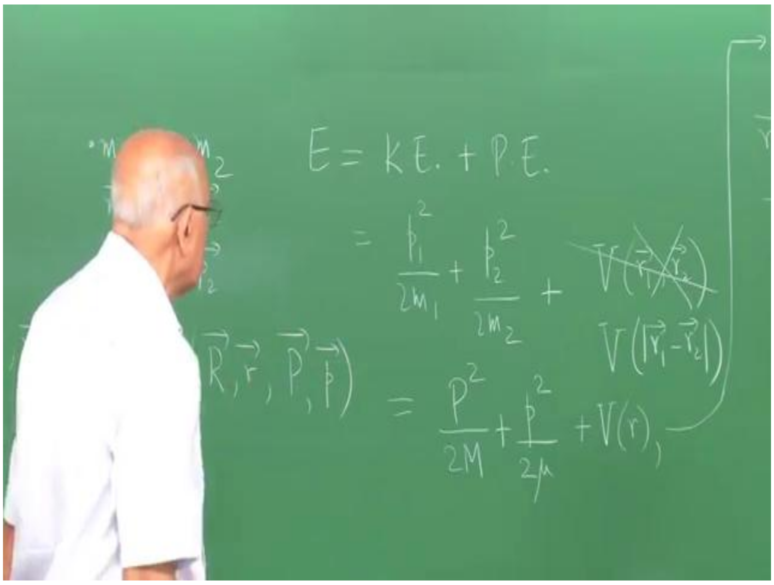

Now, the first thing that we have is that the total energy of the system, of this system of two particles and there is nothing else in the universe, just these two particles moving slow compared to the speed of light in vacuum, so it is non relativistic. The total energy is the kinetic energy of the two particles plus the potential energy of the two particles and we believe that, because time is homogeneous, there is nothing to specify what the origin of time is one way or another.

We believe that we know, we can prove that the total energy is a constant of the motion and this is given by the sum of kinetic energies. So, it is p 1 squared over twice m 1 plus p 2 squared over twice m 2, these are the squares of the magnitudes of these two vectors plus the potential energy, which we will now assumed to be the following. We will assume this to be a function V not of the general coordinates r 1 and r 2, no; but rather a function V of the distance between the two of them.

So, instead of V as a general function of r 1 and r 2, we have a potential which a potential energy in this case which depends only on the distance between the two particles, the actual orientation of this vector r 1 minus r 2 is relevant. So, not only have you saying that the absolute origin of space coordinates does not a matter, it is only a difference between the two vectors r 1 and r 2 that matters. We going further and saying that the energy of interaction, the potential energy of these two particles is a function only of the distance between the two particles and nothing else.

So, under such a situation, given this total energy we would like to deduce from the equations of motions which we can write down, Newton's second law of motion. For each of the particles, we can try to write down what the equations of motions are and try to solve them given some specific conditions. Now, that is a very general problem, in general it is not easy to write down explicit solutions to these equations of motions, but a first a very great simplifying fact arise is, as soon as we say that the potential energy is the function only of the distance between the two particles.

It immediately suggest that we should not use the coordinates r 1, r 2, p 1 and p 2, but rather a coordinate by specifically is a linear combination of r 1 and r 2 in the following sense. So, from r 1, r 2 ((Refer Time: 05:22)) p 1, p 2 from this set of variables one goes to a new set of variables, also the same in number finally, because you cannot take any variables which takes into account the fact, that what appears here is only a function of the difference, only a function of the magnitude of the relative co ordinate r 1 minus r 2 between these two particles and these new coordinates are the following. By uses capital R, little r, a capital P corresponding to capital R and a little p which have the following properties.

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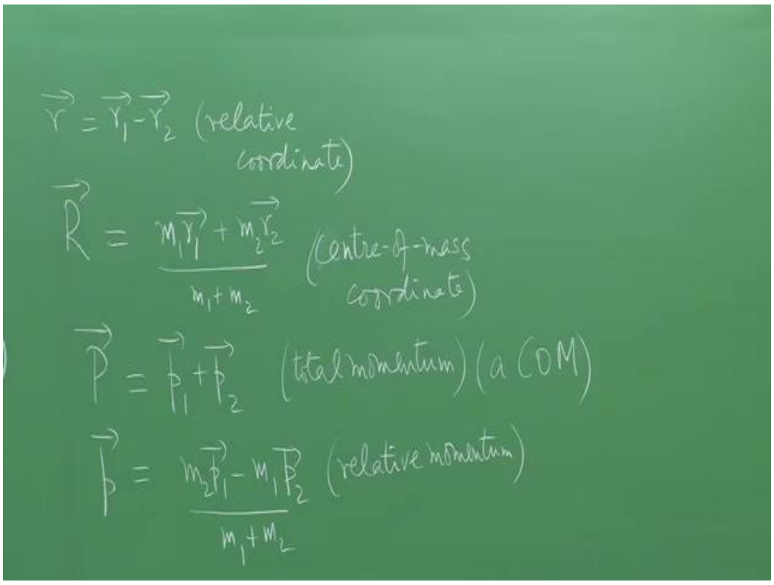

This coordinate capital R, little r is by definition r 1 minus r 2 that is the relative coordinate and capital R is the coordinate, not surprisingly of the center of mass of the two particles. So, with respect to any origin what so ever, you define a center of mass coordinate which is m 1, r 1. So, it is the coordinate of the first particles weighted with it is mass plus m 2 r 2 divided by m 1 plus m 2. If the two masses are equal, m 1 equal to m 2, then this is the half way point r 1 plus r 2 divided by 2, otherwise this is the general expression for the coordinate of the center of mass, so center of coordinate.

So, instead of the vectors little r 1 and little r 2, one uses two new linear commutations of these vectors, one of which is the relative coordinate and the other is the center of mass coordinate. Now, it is immediately clear that I could have defined the relative coordinate as r 2 minus r 1, it will not make a difference and basically that is because, this is only a function of the magnitude of this vector, it is absolute sign does not matter at all.

Corresponding to those two coordinate vectors, there are two momentum vectors and in this case it turns out and I justify why this should be so. The total momentum p 1 plus p 2, this is the total momentum, linear momentum and little p is a relative momentum, little p it is not equal to p 1 minus p 2 or p 2 minus p 1, it is rather equal to m 2 p 1 minus m 1 p 2 and it is called the relative momentum divided by it. It should have the dimensions of a linear momentum's, so there is a division here of by the total mass.

Now, there is a technical reason and rather there is a systematic procedure by which one can arrive at the conclusion that the relative momentum is this combination and not any other combination, the total momentum is fairly straight forward. Now, the point is that when you write the total energy down in this way, there is a method by which given the total energy is called the Hamiltonian in this case, thus a way of writing down Newton's equations, the equations of motion from this total energy.

Exploit in the fact that if this ((Refer Time: 09:29)), this thing here is a function only of the magnitude of the relative coordinate here, then it is straight forward to show that these are the combinations. The right combinations that may be chosen and as would have guessed by, now since this inter system is not being acted upon by any external force when anywhere or any external torque. It is immediately clear from what we have said earlier, that one expects this momentum to be a constant of the motion.

Because, by Newton's third law the force exacted on particle 1 by particle 2 will be equal and opposite to the force exacted by particle 2 on particle 1 and as we saw, that immediately implies that the sum of the two momentum in the 2 body problem is the constant of the motion. So, this one here is the constant of the motion, just as the energy itself is a constant of the motion, because the whole thing is homogenous in time. There is no reference to any origin when t equal to zero instant, any special instant therefore energy is also conserved.

So, we have at our hands, the statement that the momentum the total momentum is conserved, the energy is conserved and then given this fact, given these changes of variables what we have to do is to go back and rewrite the total energy in terms of these new variables. So, all we done is to change from ((Refer Time: 10:58)) little r 1, little r 2, little p 1, little p 2 to capital R, little r, capital P, little p, put linear combinations; one can invert those relations and solve for the original variables in terms of the new one and substitute in here and then what happens is the following.

This becomes equal to ((Refer Time: 11:16)) and this is the piece of algebra which is easily carried out and becomes equal to p square over twice m plus little p square over twice mu. I will explain what mu is in the minute plus V of little r, because now this vector is little r and it is magnitude is little r without an arrow up there.

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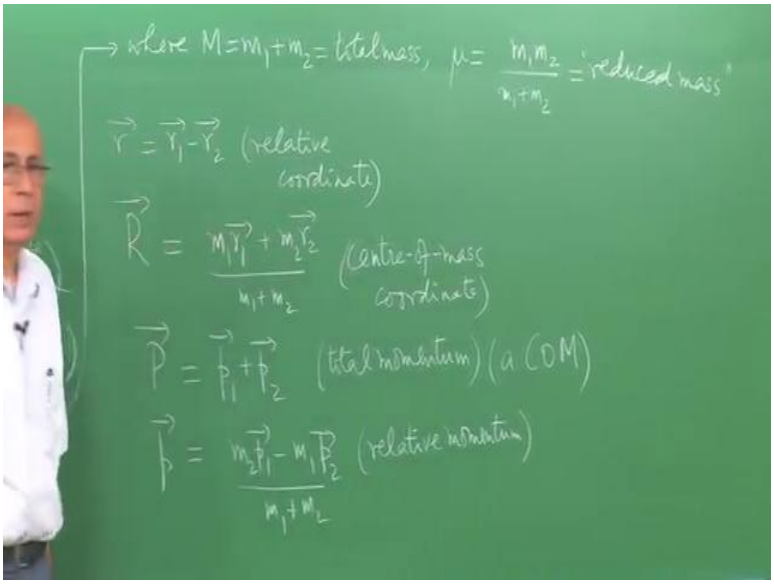

So, this total energy reduces to this expression here, where capital M is the sum of the masses and that is the total mass and little mu is equal m 1 m 2 over m 1 pus m 2 equal to the so called reduced mass, this function. So we have explicitly, explicitly here an expression for the total energy in terms of the total momentum, the relative momentum and the potential function, the potential energy which depends only on the magnitude of the relative coordinate.

It is immediately obvious for inspections, that this coordinate, the center of mass coordinate does not appear anywhere here, it does not appear in the expression of the total energy at all and then, of course it follows that just as the total momentum is the constant. It follows that this coordinate which could be regarded as defining the coordinate of a particle with the total mass with momentum capital P ((Refer Time: 13:12)) which is the constant of the motion is like a free particle.

So, it is as if you have a different problem now, the original problem of mass is m 1 and m 2 as being reduced to the problem of a capital M ((Refer Time: 13:26)), particle of mass capital M moving at a constant momentum capital P, like a free particle in a straight line together. With the problem of another particle of effective mass little mu with momentum little p moving in a central potential V of r. So, these two problems are got decoupled and once that is done, this problem reduces and simplifies.

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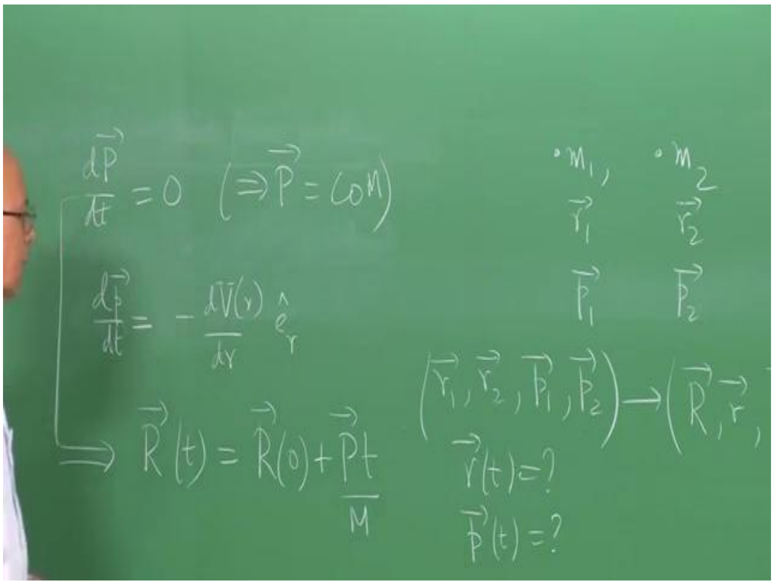

As a consequence, if I wrote this equations down for change rate of change of these coordinates and so on, it turns out then that d p over d t equal to 0 and I shall write down exactly how this is derived from the total energy, but it is fairly straight forward to do this. So, this will imply that p equal to the constant of the motion and d p over d t, the rate of change of this thing here is equal to by Newton's third law, by Newton second law ((Refer Time: 14:38)) it is equal to the force seen by this particle would be derived from this potential and that is the central potential.

So, this becomes equal to minus d V r over d r times e sub r. So, the problem involving little r and little p is isolated from the problem involving capital P and this will also implied immediately, that this code unit r of t by integrating this equation. Now, R of t is equal to R of 0 plus whatever a free particle does with a given initial momentum and a mass. So, this is P times t over M. So, we have a complete solution for what the center of mass does, it moves linearly, so this whole system it is center of mass with the effective total mass equal to m 1 plus m 2 moves in the straight line depending on what the initial conditions were.

So, if you specify for me the initial momentum of these two particles, I know this and I know the mass and if you tell me, where the center of mass was located initially, I know where it is located at any further instant of time. So, this part of the problem is like a free particle and is solved trivially by the laws of motion being explicitly integrated out. It leaves us with this problem here that is the central force problem. The problem of a particle of effective mass mu in the central force or central potential is V of r.

And now of course you have to tell me, what this e of r is, in order for me to make progress, to make further progress and say what kind of motion we expect of this. Now, we need initial conditions for this, so you need to know what is the initial coordinates of these two and you need to know the initial values here. The some of them remain constants and then, it is a question of how these quantities evolve as a function of time.

Specifically, we would like to know, what is r of t; given r of 0 what is this r of t and of course what is this p of t equal to. So, this is the central force problem, the one body central force problem, because we now have a particle whose effective mass is mu, the reduced mass and we would like to know what it does here, now notice something about this reduced mass.

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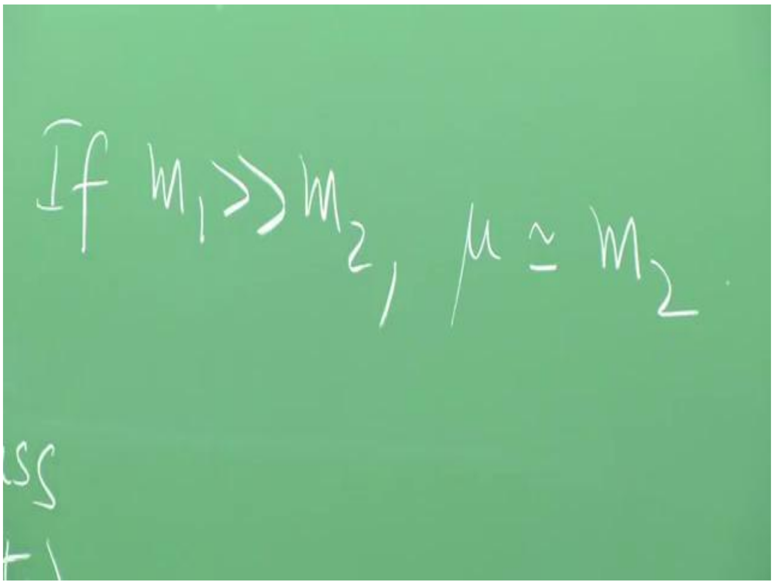

Notice that if m 1 is much, much greater than m 2, so one of the particles is very much larger, very much massive more massive than the other one, then mu is approximately equal to… You can neglect this m 2 correspond, compare to this m 1, m 1 cancels out, then this is approximately equal to m 2. This is what happens, for instance if you consider any of the planets moving around the sun, so the mass of the sun is m 1 and the mass of the planet is m 2, the effective motion of this planet is like that of body of mass effectively that of the planet itself.

The mass of the sun cancels out top and bottom and you have effectively a reduced mass which is essentially the mass of the planet itself. And in this case, it is also clear that if m 1 is much, much bigger than m 2, then as just as you can neglect this m 2 compared to this m 1, you can neglect this compared to that and the center of mass coordinate is essentially the coordinate of the very much heavier particle or very much more massive object.

So, this is the reason why we can regard to a first approximation, the sun as a massive infinitely massive objects sitting at the center of coordinates as center of attraction and the planets moving around the sun in orbits like a 1 body problem. Because, the center of mass of the earth sun system is essentially the coordinate of the sun itself. So, it is essentially at the center of the sun, a little bit displacement at the center of the sun.

Just as even in the earth moon system, the case of the earth moon system the center of mass of the earth moon system is actually inside the surface of the earth at any instant of time, because the earth is so much more heavier than the moon is. And in the case of the sun and the planets, the comparison is much, much it is even more so; the sun is very much heavier than the earth is. So, this is how a general 2 body problem, a central force 2 body problem gets reduced to an effective 1 body problem moving in a potential, the certain potential V of r is specified potential, from which we can find the equations of motion, this equation of motion and attempt to solve this equation of motion.

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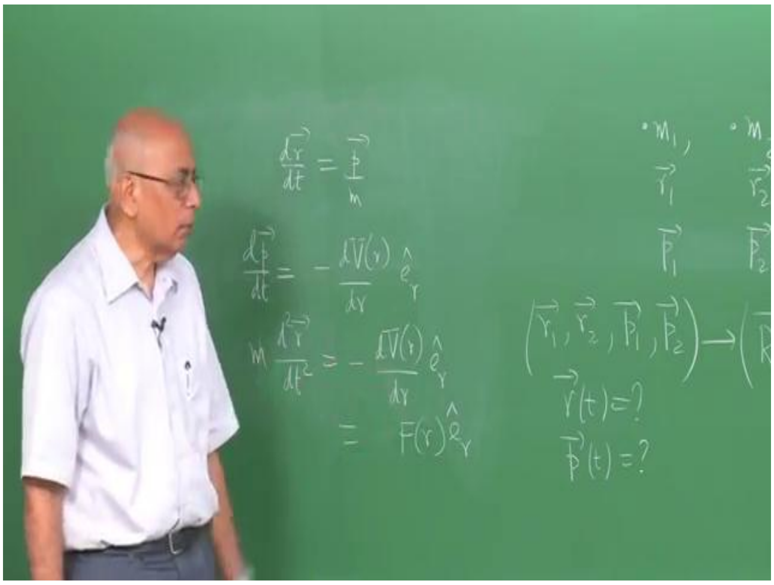

By the way, this equation here should be accompanied by another equations which is fairly straight forward here, which is m d r over d t equal to p, for rather I should write this as p over m. So, you see you have really a pair of equations for the unknowns r and p and they are couple to each other and when I eliminate this, then I end up within a equations for the second derivative of r which is the acceleration.

So, just to write that down, this immediately implies if I differentiate p with respect to this, it says second derivative here. So, it says 1 over m d 2 r sorry, so let us differentiate d 2 r over d t 2, this is equal to and multiplied by m here, that is d p over d t this is equal to minus d V r over d r e sub r and this is equal to some forces. So, let us write this as F of r e sub r and this gives us the familiar form of Newton's equations with which you are familiar, namely mass times the acceleration, the vector acceleration is equal to the force and it is in this case, it is the radial force. In the case of the sun, it is an attractive force and that is the next thing we have to do, this is the Kepler problem, so let us do that.

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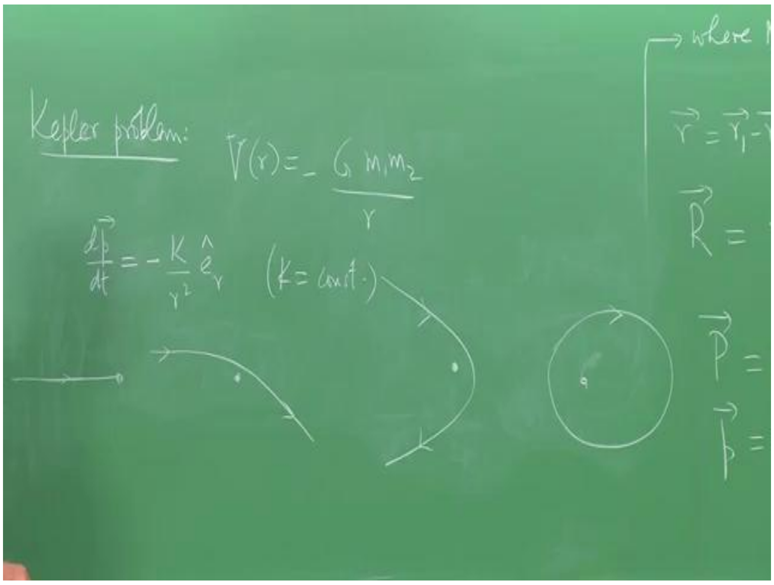

In the case of the Kepler problem which is planetary motion, we have a… This is equal to G times M mass of the sun times the mass of the planet. So, I should write in my notation, we should write this as m 1 m 2 divided by the distance between the sun and the planet, little r here with the minus sign, it is some constant divided by 1 over r that is all we are interested in. So, let me rewrite that in the form, let us rewrite this thing in a simpler form, which is d p over d t equal to well the gravitational force.

So, it is some constant divided by r square e sub r with a minus sign, some constant which is the product of the masses capital G and so on and so forth. So, this is the problem that one has to solve in the case of the gravitational problem, where you have a planet orbiting around the earth and what was shown by Newton was that, the solutions of this equation are such that this vector r, it is locus is a conic section.

So, under all consumable circumstances the locus is an iconic section always, for any given initial conditions for instance. Suppose, you have the center of attraction here and you have a planet, the initial momentum is in the direction which falls in to this center of mass, while the planet will fall in to this center of attraction and that is it, so it is a straight line. On the other hand, if the center of the attraction is here and the initial momentum is like this, it is possible that this gets scattered in the form and trace a hyperbola in a space.

There could be a case where here is the center of attraction and the initial condition is such that in the limiting case, it traces a parabola of this kind. This is a specific value of the total energy for which this happens by the way and then, there is a case where we have the center here and the planet traces out, an elliptic path that the sun or the center of attraction at one of the foci. So, we have all these possibilities and this is, this was a Newton's great achievement to essentially solve this analytically for all possible initial conditions and to show that the orbit in real space is always a conic section of some kind.

Either, you have periodic motion, in which case the particle executes an ellipse, an elliptic orbit in the general case or a parabolic orbit or a hyperbolic orbit, these two are open orbits in the particle essentially escapes. So, these are the kinds of orbits the comic would traces, typically they would execute. If the comic never returns, it would typically execute an ellipse, an orbit which is the hyperbola of this kind.

Now, it is interesting to ask under what conditions you have this intermediate case of a parabola, because if you think about it for a minute, you realize that an ellipse has two foci and a parabola is a limiting case of an ellipse when one of the foci moves away to infinite. So, it turns out that in the case of this orbit, the energy of this particle whose relative, whose c ordinate is little r and momentum is p is positive, in this case it is exactly zero and in this case the energy is negative, because it is a bound orbits.

So, these three cases are distinguished by the value of the energy of this effective particle of mass move here and that is what determines whether it is an ellipse for, whether it is a hyperbola or a parabola or an ellipse. The parabola being the limiting conditions, that separate this, separate tricks as it is called, that separates open motion namely the hyperbolic motion with the closed orbits, periodic motion which is the elliptic orbit.

The next thing to do is to write, proving this by the way was non trivial and that is what Newton's achievement. Today, we know how to prove it fairly straight forwardly, but I am not going to go and solve this differential equations here, but rather we are going to assume that the Kepler's laws in my next task is to explain what the significance of this law is, why they come from and what they tell us in some sense. So, that will be the next thing we take up.