Mechanics, Heat, Oscillations and Waves
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Lecture - 21
Central Potential and Central Force

Ready now to take up the idea of what a central potential is and what a central force is. But, before, we go on to the formal way of deriving or defining these quantities, one more example will not be out of place here. We have already saying the example of a gravitational potential due to spherically symmetric mass distribution.

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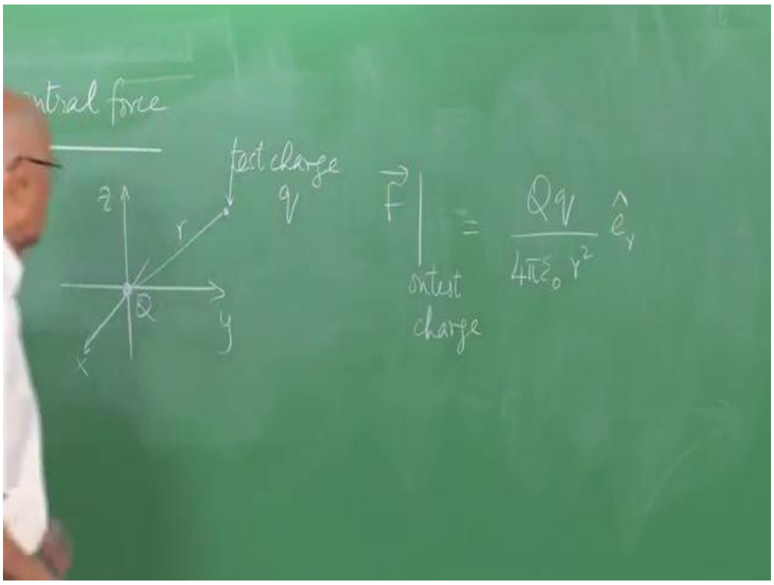

The another example, the famous example is an electro statics were coulomb's law tells you that electro static force between two charges, point charges Q 1 and Q 2 or little q and capital Q say is of the following kind. So, if you put one charge here, capital Q at the origin of coordinates without loss of generality, here x, y and z. And you put a test charge here, little q there at some distance little r from the origin of coordinates.

Then, the force on this test charge F on test charge, the electro static force as you know is capital Q multiplied by little Q divided by 4 pi Epsilon naught times r squared multiplied by the radius vector of the unit radius vector, unit radial direction vector from the origin to this point here.

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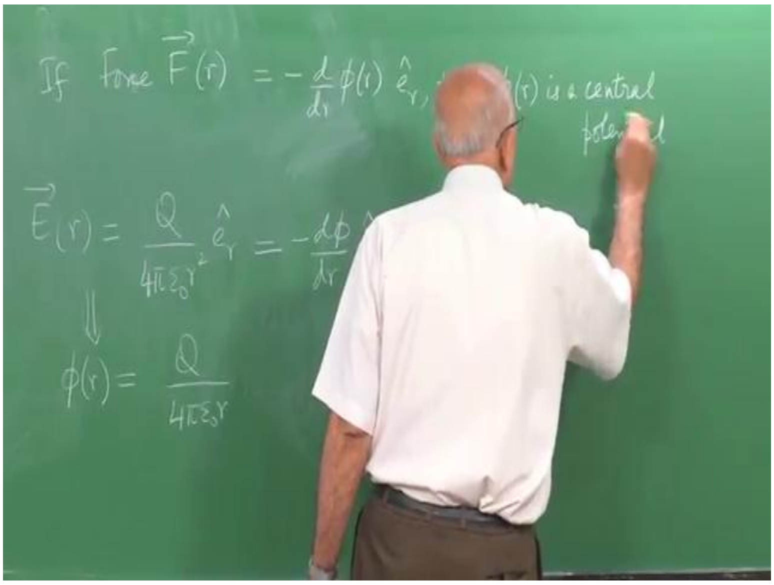

So, exactly as in the earlier case, we could ask, what is the potential? Well, actually this is the force, but the electro static field itself E at any point distant r from the origin, is not depend once again, because it is the point charge at the origin. So, spherically symmetric, it is not depend on this spherical polar coordinates theta and phi, which depend only on r. So, to indicate that, I write it as a function of r without putting in a theta or phi there, this quantity is equal to Q of 4 pi Epsilon naught r squared e sub r.

When you multiplied by the charge, the test charge, you get the force on the test charge, because we know the force is charge times the field, the electric field. Now, this thing here, will implied that it is derivable from a potential phi of r, again function of little r this is the electro static potential at any point distant little r from the center, from the origin, this is equal to Q over 4 pi Epsilon naught r. And there is no vector, it is scalar, this quantity is a scalar.

Because, now you see that if I differentiate d phi over d r and put a minus sin, this quantity is Q over 4 pi Epsilon naught r squared and multiplied by e sub r. Therefore, this quantity can be written as minus d phi over d r, e sub r. So, once again I have a potential, which goes like the gravitation in potential, it goes like 1 over r. The fact, that you have a plus sin here and you had a minus sin in the gravitational case.

Had to do with the fact, that when you have two masses, both are which are positive, the force of gravitation between them is always attractive. Whereas, if you have two charges, which are like charges, both little and capital Q are the same sin, the force between them is repulsive, whereas, in the case of unlike charges, charges of opposite sin, the force is attractive.

So, to take that in to a count this will to minus sin that appears or does not appear in the potential. But, the functional dependence, the 1 over r dependence is exactly the same as before. Now, this is what the electro static potential is due to a single charge and it again, now the statement is that any time, you have in general, you have a force, which is derivable from a potential in this sense.

In other words, whenever you have a force F, which depends only on the radial distance at any point, if this is equal to minus d over d r phi of r times e sub r, if this force is equal to this, then phi of r is a central potential. Central in the sense that the corresponding field or force, it is derivative is directed in the radial direction inwards or outwards does not matter, no angular dependence. And the corresponding force is called central force, the potential is a central potential.

So, the two famous examples of central potential or the gravitational potential due to a spherically symmetric mass distribution or a point mass at the origin and a charge in electro charge. In each of these cases, if the distribution deviates from the spherically symmetric distribution, immediately you have much more general kind of potential, it is no longer as central potential necessarily, because there is no spherical symmetry left in the problem. In fact, you can see immediately, that if I have two charges, this goes away at once, this properties immediately lost.

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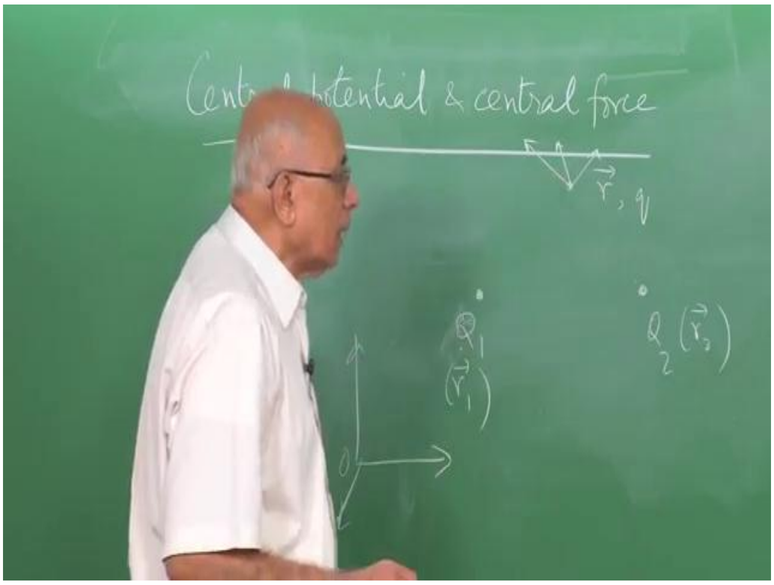

If I have the charge Q 1 here, because the capital Q at position r 1 and a charge Q 2 at position r 2 and I ask, what is the potential and the electro static field at some point arbitrary point p, whose position vector is r. How do I do that? I put a test charge here, unit positive charge or it will charge little q there and ask, what is the force on this charge due to these two points, which are kept fixed, these two charges are assumed be kept fixed at these locations.

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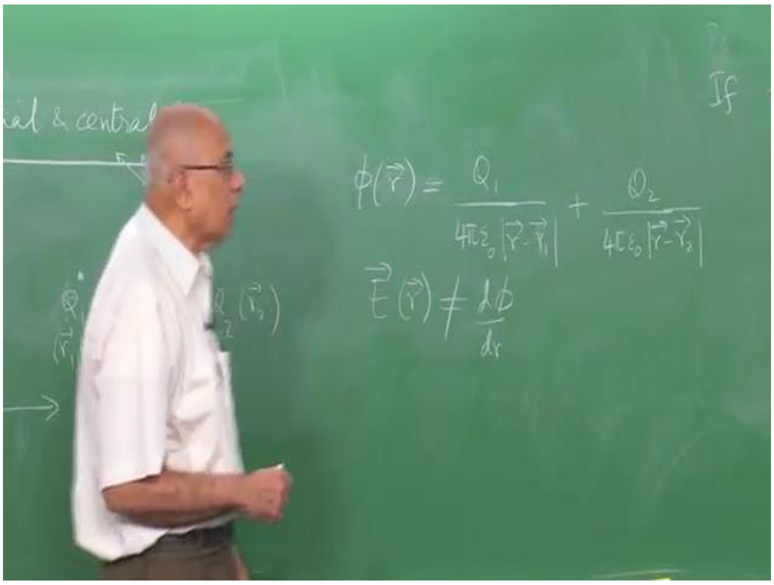

Well the electro static potential phi; now is a function of r theta as well as phi. So, let me call it phi of r at this point and just to show that, it is at this point, I put in a little r here with the vector symbol to show that, it changes from the point to point. And I do not care, where the origin is, here is an axis and that had a distance little r from it and these are at r 1 and r 2. This is equal to Q 1 over 4 pi Epsilon naught times the distance between these two points, which is of course equal to modulus of r minus r 1 plus Q 2 over 4 phi Epsilon naught modulus of r minus r 2.

That is the electro static potential at this point, to find the field at that point, I need to differentiate with respect to the coordinate r of that point. But, this r and this r here, now all the coordinate all the various three coordinates r theta and phi will all appear here, when I write this expression down is not a function of little r alone. So, in this case, the potential is not the central potential, it arrows from the central potentials, but the two potentials have got superposed and this is crucial.

The forces or the field at this point due to charge 1 and charge 2, vectorially add up to give the net field at this point. For instance, if this is positive; that is positive and this test charge here Q is also positive, this exacts a force away. That exacts a force away in this direction and there is some result in the field between the two results and force, which is the sum of these two, the vector some of these forces here.

But, this involves adding vectors and when you have more and more sources this addition becomes more and more cumbersome. On the other hand, in the potential adds up it is simply a scalar. So, this just algebraic addition of these numbers and there is no vector here involved. The vectors comes, when you start differentiating with respect to the different coordinates here.

In this case, it is not possible for make to write down the electric field at the point r, this is not equal to d phi over d r alone not true. Because, there is also dependence on theta and phi on the polar coordinates of all the points concerned as including that as the point where the point is that field is being measured here. There is a more general formula for it, which we will write down in suitable form later on, but right now, the important thing to remember is that, whatever be the force, it is still derivable from a potential.

There exists a suitable potential from which you can find with suitable differentiations with respect to the coordinates, you can find the field at this point. And the great advantage of using potentials as suppose to using the forces or fields themselves is that potentials or scalar quantities and you add this scalar quantities from different sources corresponding potentials.

And then, calculate the field once and for all by differentiations rather than finding the fields you to each individual charge and then, trying to add them up pictorially; that is much more intricate. So, that is one great advantage of using the idea of a potential. So, we said something about the central potential and the movement you have a central potential does not have to go like 1 over r and anything like that any function of little r alone is a central potential. And then, it you guaranteed with the appropriate force a field is given by this formula here exact settle down here.

And the important cases are spherically symmetric charge distributions and spherically symmetric mass distributions they will lead correspondingly. Because, of Newton’s law gravitation and coulomb’s law respectively, they lead actually to forces, which as central forces. This another physical example of a central force is very important, so we will go to that I will mentioned that then, we will come back and look at the other advantage of using is the other important property of a central force.

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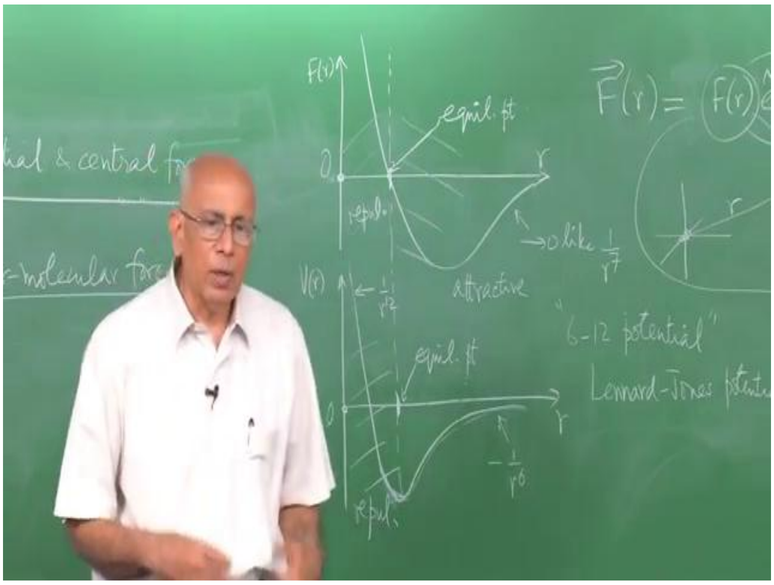

And this has to do with the inter molecular force that is an intricate subject the force between molecules in matter it is actually quite complicated, because the molecules themselves quite complicated. And in generally involves calculation of these forces from first principles involves fairly intricate quantum mechanics and really very difficult in most cases except in the simplest instances. But, we do know something, which is that in general there is a very, very a there is standard qualitative way in which, this force generally behaves and the suitable conditions.

So, we look at the case of neutral molecules is such as neutrally electric molecules and which, some degree of spherically symmetric. So, once this spherical symmetry is present, then the intermolecular force looks like this it is a very typical graph it is a central force. So, what we have in mind is at this origin of coordinates putting one molecule at the origin of coordinates always, so one molecule is sitting at the origin of coordinates and the other molecule is a distance r away from it.

So, we have in mind is situation is molecule at the origin of coordinates here and some other molecule here at a distance r from it. And the question is, what is the potentials? And that point a, what is the force, of on this molecule due to this molecule at a given distance r from it. So, that is the typical question we like to answer and it goes us follows it turns out that this force looks very generally like this and this force F of r at the central force one of the input away origin the central force goes like some quantity F of r times is of r.

Because, it is central it must be directed either along the line it is always directed along the line joining these two points molecules either inwards or outwards depending on the other forces are attractive or repulsive . So, if F of r is positive greater than 0 implies repulsive force and if it is negative it implies attractive force. And it turns out in most cases this force becomes extremely large and repulsive and then, it will becomes this kind of think.

So, there is a point here where it changes sin and to the left of that point a plot F of r here, as a function of r to the left of that point this force is repulsive these two molecules repulse each other. But, to the right of that point in this region here, the force is attractive and takes off to 0 as you would physically expect when the distance between the two molecules, becomes unbounded. Now, this can be derived from a potential and the answer is indeed yes there is a minus d v or d r here.

And the corresponding, so this thing here can be written in the form minus d v d r and the v of r itself has the following shape at the point, where this force is 0. Since that, force must be come from minus d v over d r it means v must have an extreme. And out here you can see that d v over d r, which is F extending towards infinity this point and the minus sin the slope here is negative. So, this v of r must also 10 to infinity at this point here is 0.

So, this source must look like this extending this point here it must look like this. So, this kind of minimum here, where the slope is flat and that is why, F is 0, because this F is minus d v over d r this quantity is equal to that in. So, where it crosses where, the force cross is 0, you have equilibrium there is no force net force on the molecule the attraction and repulsion kind of cancel each other out and this is a point of equilibrium, not stable equilibrium.

Because, if you displace at a little bit move out unless this thing here is a deep enough well on the other hand, if you give it a large displacement you may dissociate these two molecules. But, otherwise they could be oscillations about this minimum here, which will talk about, when we talk about oscillations. On the other hand, what happens at this point the d v over d r in this entire region this repulsive, because the slope is negative minus this slope is positive and that is, what F is doing up here very steep.

On this side the slope is positive after the minimum, therefore, minus the slope is negative and the force is indeed negative here with the certain equilibrium point., so this is the point of equilibrium, equilibrium point So, if no other effects and you manage to put these two molecule exactly this distance apart this system will be in equilibrium, in stable equilibrium, in this point. Of course, in a real substance there are thermal agitations and so on.

So, the particle actually they are they will move above perhaps oscillate above this point of equilibrium leading to write you interesting affects. Now, what this due to, what is this generic thing due to and it is work understanding a little bit, where this force comes from in this simplest instance. If you have to electrically neutral molecules this means that there is are atoms for that matters there is a negatively charged electron cloud and there is a positive charge at the center.

And similarly, you have negatively charge electron cloud and you have a positive charge the center and strictly speaking, if these are static object the electro static field outside this distribution at any point here is 0. Because, it is spherically symmetrical and then, the electric field as you know is as, if the entire charge distribution was concentrate at this center. And, if the object is electrically neutral the total negative and positive charge is cancel each other and give you no field here at all.

On the other hand, these are dynamic objects and what happens is fluctuations this spherical symmetry of this distribution induce. So, called electric dipoles, which in turn polarize this distribution and, so you have an interaction between a dipole and a dipole, which is not this does not satisfy. This property that the electric field outside is 0 or anything like that is it is not spherically symmetric anymore and there is a net force between these two objects here.

This force turns out to be an attractive force in most instances and is in fact, responsible for this part of the potential namely the attractive force here. And this force typically dies down like a very large power of the distance between them this force F of r tends to 0 typically like 1 over r to the power 7, 7 appears quite naturally and therefore, the potential it goes like 1 over r to the power 6 typically. Later on when we do electricity and magnetism I will give an explanation to very simple one of, why this should power particular power 6 should appear in the potential of all powers.

And this side; however, when you bring them sufficiently close together, then the Powderly exclusion principles prevents. So, you from having electrons two electrons in the same state simultaneously and this leads to a repulsion that is largely responsible for this repulsive part of the force. And it in fact, goes to infinity extremely rapidly, typically this would go like 1 over r to the power 12 with the plus sin and that with the minus sin, because it is from below and this extending infinity with the 1 over r to the 12.

So, this kind of potential is often called the 6 12 potential it is also called the Lennard Jone potential between neutral atoms and the neutral molecules, which are roughly spherically symmetric. And very typically of while the details in different cases they differ the shape of the potential and therefore, the shape of this central force is more or less standard in this form. Now, what this power is will depend on the details of the molecules as well whatever happens here that is easier to model it, then this fairly complicated quantum effects are involved in trying to discover, what the repulsive force is at very short distances.

But, this thing here is the generic shape of this potential and it is a central potential. Therefore, we can write downward the force is central force in the first approximation then this potential is valid. One actually has a way of modeling in the molecular sources and then, preceding to see when, what happens you put a lot of them together to contents in some form of contents patriotically or a solid.

So, I thought I should mentioned this here it has a crucial and very important application of the idea of the central potential many, many other applications of the central potentials will compare to this as will go along. Meanwhile for a conservative force, that is very one important property, which is crucial and that is as follows.

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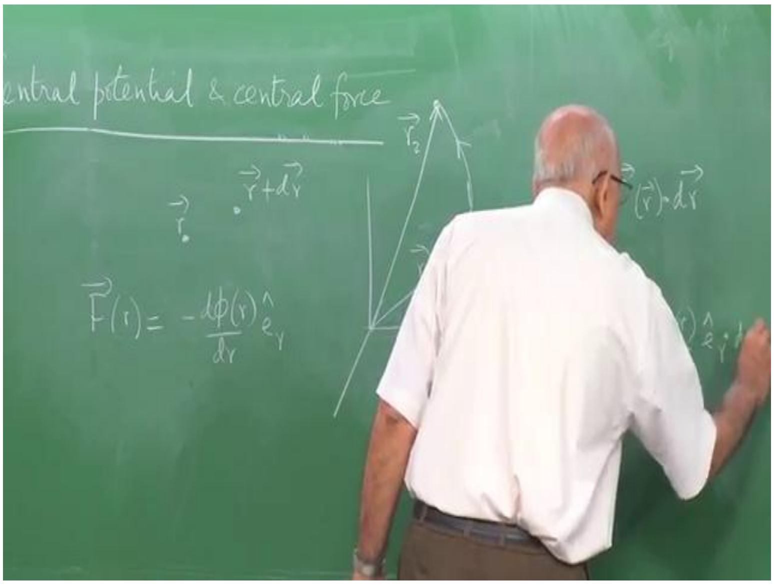

I already mentioned that, if you have the potential at every point, so here is r here is r plus d r say and the force that is acting on your particle or object is F of r this is equal to a derived from a potentials. So, this equal to minus d phi r over d r e r for a central force, then the work done in taking this particle from here to some point like this with respect to some origin. So, let us draw this origin here, here is the point r vector, and let us suppose to move to the point, some other point, some finite distance away let us move that point, along some describe path.

So, move in this expansion this point here, which let is start with r 1 and I move to r 2 expansion. So, under the action of this force this particle moves from r 1 and r 2 and the work done is equal to F at each point F of r dot d r and you must intricate or add up all these infinities decimal places. So, here is the first displacement the next displacement the next the next etc these are all little infinities decimal r vectors and the vector some of them will move you from one point to another the total is integration from the point r 1 to the point r 2 in general that is not a trivial integral to evaluate in general.

On the other hand, when the force is the central force, then you can see that the force at this point is radially, let say outwards in this passion and the object moves here to this place here. The actual work done is the component of this displacement along this radial direction here. So, it is only the radial portion that contributes and that is not hot to write down, because once I write this is this form here. Then, you can see this is equal to integral from r 1 r 2; whatever be F of r we do not care minus sin d phi only of the distance r over d r e r dot d r, but look at, what happens d r.

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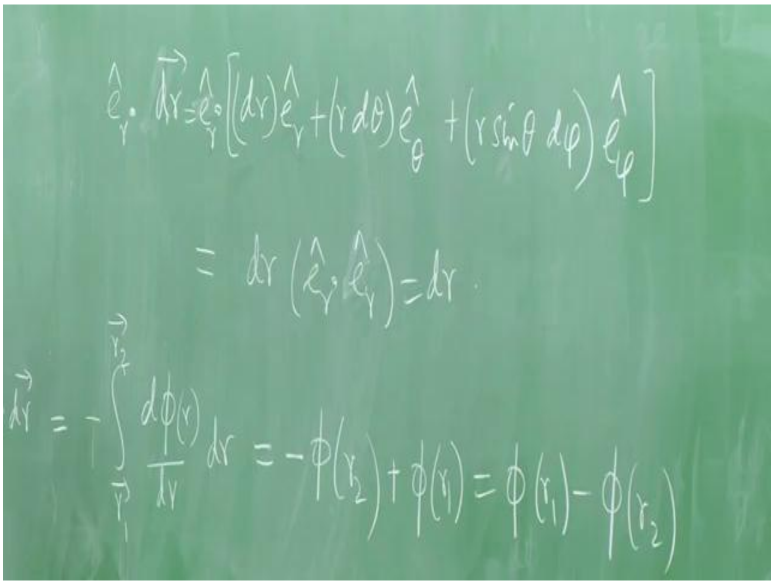

We know in general that d r is equal to d r times the unit vector in the radial direction plus r d theta times the unit vector along the polar increasing polar angle direction plus r sin theta d azimuthal angle e pi unit vector along the direction of increasing azimuthal angle. Now, you have to take that and dot it with e r dot is equal to e r dot this quantity, but e r e theta and e phi or right angle to each other they form a right hand tried. So, e r dot e theta 0 e r dot e phi is 0 and therefore, this reduces to d r times e r dotted with e r, which is equal to 1.

So, this is just d r and therefore, this reduces to minus the integral from the point r 1 to the point r 2 d phi r over d r times d r without any vector sins and that is a simple integral to do, because this is minus phi at the point r 2. But, now this is the function only of the distance of any given point and therefore, it is only r 2 without the vector symbol plus phi of r 1, which is equal to phi of r 1 minus phi r 2. So, we have this beautiful property that the work done on their system by this force in moving from a point from r 1 to r 2 is not dependent on the details of the path at all.

But, this only at difference equal to the difference depends only on the potential and is the difference between the potential the initial point minus the potential at the final point here. And in the case of central potential this is a function of only this distance little r 1 and the distance r 2 here. Therefore, if I took any other part all that would the relevant would be in each infinite decimals displacement, what is the component of this force in the radial direction, let us all that relevant and immediately leads to this answer here phi of minus r 1 and phi of minus r 2.

So, it immediately suggested that you could as well replace this path by straight line path and you get exactly the same answer no matter, what you do? What kind of path you do work done you going from here is a independent of a path and depends only on the end points which also says that you come back in a close loop the total work done must be 0. Because, it is the difference between the potential at a given point and itself, which is of course identically 0.

So, we had this very important property that when you have a conservative force well with this central force in this case derive from a potential. But, when you have a more general conservative force the same property turns out to be true that involves suitable writing down suitably at derivative here, which is not dependent only on e r, but also e theta and e phi. But, we will see we will show later on that even in, that case it will be only at difference of the potential at initial and static point.

So, the second important property of a conservative force of, which a central force from a potential energy is a special case is that the work done in a close path is 0. The work done by this force on a system a close path is 0 or equivalently the work done in going from any point when the other point is independent of the actual part taken between the two points and dependence only on the potential difference between the two points. And that is tacitly implicitly put in by us in elementary calculation when over the compute the change in energy and so on.

By simply computing the potential difference, but it is based on the fact that in the case of a conservative force the work done in a close path is 0 or the work done in going from any point at any point is a function only at the potential difference between the two points. And in the case of central potential is particularly easy to demonstrate that this is, so we will see more example of this property as we go along.