Mechanics, Heat, Oscillations and Waves
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Lecture – 20
Conservative Forces: The Concept of a Potential

We will now go on to a very important basic concept in the study of dynamics of motion in general. And it is what to do with, what are called conservative forces, and we will introduce the very, very important concept of a potential associated with such forces. So, as always let us begin with an extremely simple observation and that is the following.

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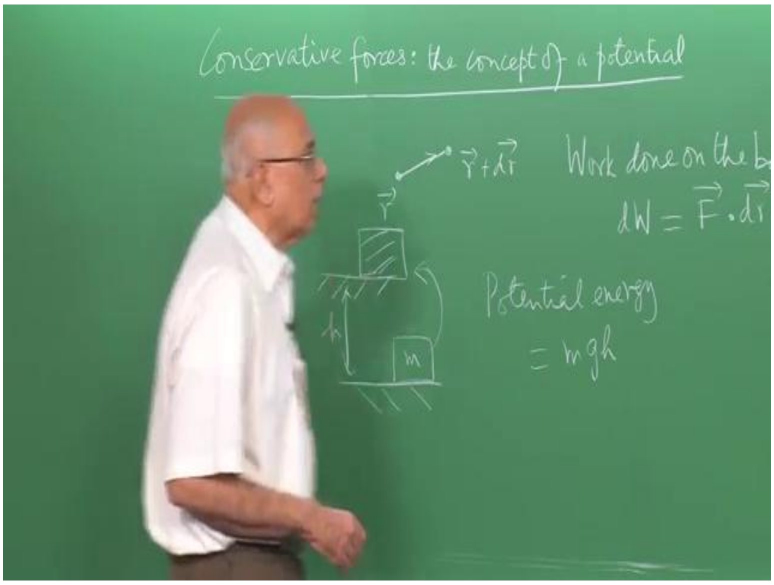

We know that if you have a particle or a body of some mass m at some position vector r and a force acts on it and displaces this object to enabling point, let say r plus d r. And some kind of force acts on it and displaces it by this infinite decimal amount. Then, the work done on the particle or the body, let us call it d W, this is equal to F dotted with d r. In other words, that component of the force which is along the direction of the displacement does the work and it is equal to d W and this amount is given by the scalar product of F with d r.

So, this immediately suggest, well for one thing, it says that in Cartesian coordinates for instance, you could write this as F x, d x plus F y, d y plus F z, d z. Remembering that d r in general can be decomposed along E x, E y and E z and similarly for F. So, this suggests immediately that each component F x, F y, F z is some kind of derivative of this infinite decimal amount of this work with respect to the corresponding displacement.

For instants, suppose you did not have any displacement in the y and z directions, then d W over d x would be equal to F sub x. So, we have this idea that somehow of the other, the force and the work it does on a body are related to each other by this rule of differentiation. You differentiate this quantity and you sort and you end up with the component of the force.

More generally, where does this force go, where does this work go; it goes into the stored energy of this system. To give an example, a very simple example, suppose here is a ground level and I have a mass m sitting here and we have a shelf at some height h about the ground and I lift this mass up and put it up here. I have done work against the gravitational field of the earth in lifting this mass up and putting it up at a height h, certain amount of height; that is certain height h.

And if I started at rest here and it is at rest here at this point, then it is clear that some kind of stored energy of this particle has increased by certain fixed amount. In this case, we know the answer, it is m g h and this means that, that m g h is stored as energy in the system. And we say that the potential energy of this object in this case is equal to m g h and the reason for the word potential is that, if I go to drop this object, when it attains the ground level. This energy, this amount of energy gets converted into kinetic energy in this object hits the ground, because certain impact.

So, that is the reason for this word potential energy, but already you can see that the idea that an object can store energy in a form other than it is kinetics energy is implicit in this very, very simple example. Now, what concept of a potential and conservative force is going to do is to make this much more general as we will see and we will do this by a sequence of examples of various kinds.

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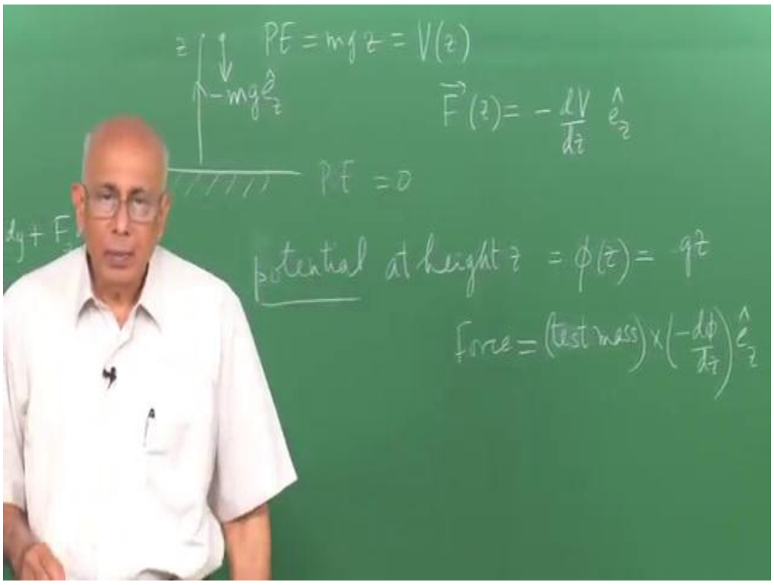

Taking this example itself to start with, what it implies is that, if you have the ground level and I say that the potential energy at this point is 0. On the other hand, at a height z, let us put the z coordinate in a vertical direction about the ground level, the potential energy P E equal to m g z. Then, there is a relation between the force acting on the body here and this potential energy.

In the following sense, you can see that the force acting on the body is downwards, like this and it is equal to minus g times the unit vector and this z direction minus, because it is acting downwards here. It immediately suggests that the relation between these two is that, if I call this V of z, the potential energy at a height z above the ground, taking the ground level potential energy to be 0. Then, it immediately follows that the force on this body F, when it is at a height z above the ground, this is equal to minus d V over d z times e z. So, V itself is m g z.

So, d V over d z is m g and we have put a minus e z and you have a force, which is minus m g z in this fashion. So, this is the fundamental relation it says the force is the minus the derivative of the potential or potential energy in this case, you can appropriate unit vector to get the direction. Now, of course, you could put different masses at the same height. So, you would like to have a concept, which is related to the gravitation force alone without bringing in the specific mass.

If I put in some other mass capital M, then again this would be minus capital M g z. So, I could define not a potential energy, but a potential gravitational potential, because it is the gravitational force, potential at height z equal to say phi of z and this is equal to, in this problem this is equal to g time z itself. So, you could look at the potential in this sense as the potential energy per unit mass in this case. So, you multiply by the mass and you get the actual potential energy of the object and the force of gravity on that object would be minus d V over d z.

The force of gravity on any object would be m times the derivative of phi of z. So, force equal to mass let me call it test mass to see, whatever be the mass that you put at this point multiplied by minus d phi over d z times e z, this fashion. Now, what is the use of this concept of a potential, we will see this in a minute after we look at a few more examples, then we can see that in each of these cases, it will turn out.

That the force, corresponding force will always be a suitable derivative of some kind of a certain potential or if you give a general object with some specific mass or charge or something like that, it will be that quantity multiplied by the potential. And it will turn out to have the physical meaning of a potential energy of this object. Let us look at a second example here immediately.

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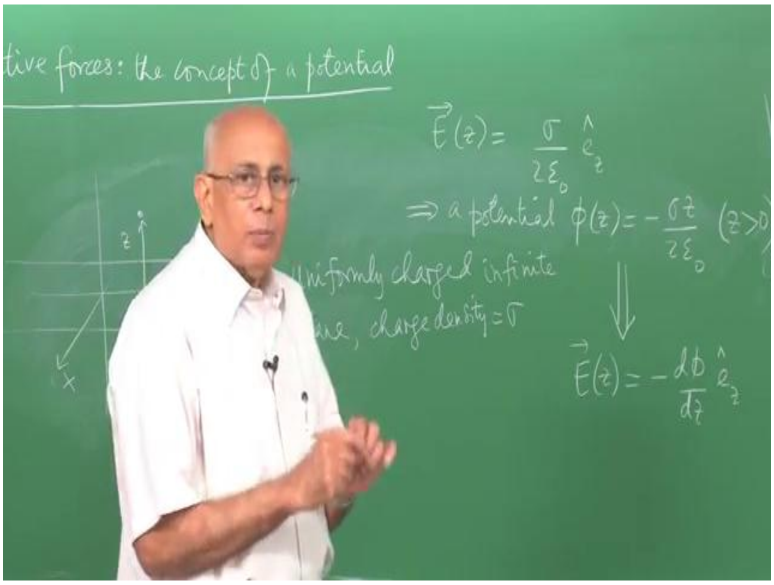

Suppose, you have and this is very familiar example in an electrostatics, suppose you have an infinite uniformly charged plane. So, the x y plane is uniformly charged plane with a surface density equal to sigma say. So, sigma is the amount of charge per unit area of this infinite x y plane and then at a height… So, let us draw some coordinate system; that is the x y plane; that is x; that is y infinite plane.

So, the plane is of infinite extent in the x y directions and now, at any arbitrary height, anywhere for that matter, it does not matter where we choose the origin on this plane, at any height z above the plane at this point z, height z above the plane, we can find out, what the electric field looks like. The electric field as we know, we put a test charge here, a positive test charge and sigma is positive will be perpendicular would be normal to this x y plane.

It will be directly upwards and it will have a magnitude E, well the vector itself E at a height z will be equal to sigma over 2 Epsilon naught, where Epsilon naught is the permittivity of the vacuum, multiplied by the unit vector e z in this fashion; that is e of z. The question is, can I relate this e of z to some potential, such that the derivative of that potential with an appropriate minus sign is going to give me this force.

And the answer is yes immediately, because this will correspond to a potential phi, which will be a function of z of course, equal to minus sigma z over 2 Epsilon naught and this probably that e of z equal to minus d phi over d z times e z. The minus signs will cancel out and give me this force here. So, this means in this particular problem where the force is actually a constant, independent of the height above the x y plane, because it is an infinite plane in this direction.

In this problem, it so happens once again like in the problem of gravitational field there, the potential is linear in the coordinate. Depends on a single coordinate, there is no dependence on x and y, because this thing is uniform and infinitely extend uniformly charged. And therefore, this potential does not depend on where you are as far as the x and y coordinates concerned, only depends on how far you are above this plane or below the plane, we come to that in a minute.

So, such a potential a linear potential, if you differentiated it once will give you constant force exactly as in that case here. This is slight difference here, because you could potentially go down, you could also go down below the plane, and then if you put a unit test charge, a positive test charge here and sigma is positive the force will go in the opposite direction. So, what you think the potential is going to be for z less than 0, namely below the x y plane that the answer is easy once again to guess.

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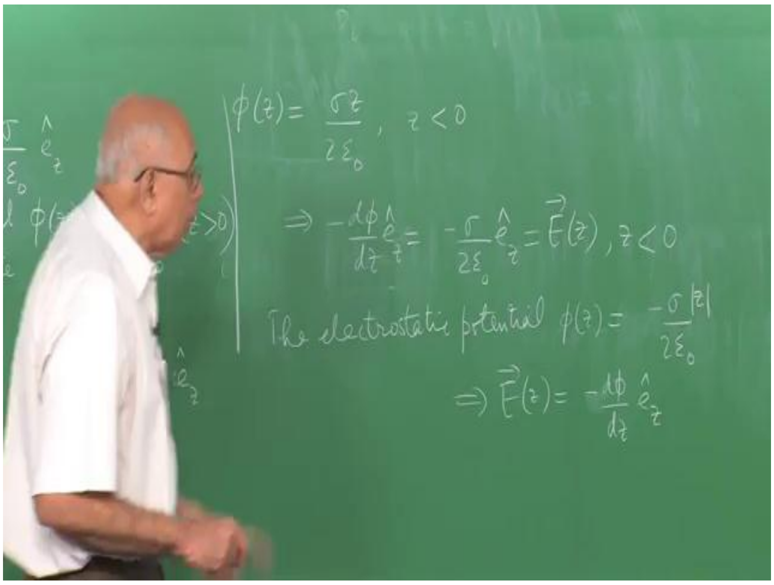

In this case, phi of z would be equal to sigma z over 2 Epsilon naught. So, this is for z greater than 0, z less than 0. Because, now if you differentiate it and write down minus d phi over d z, this would imply therefore that minus d phi over d z is equal to minus sigma over 2 Epsilon naught. And if you multiply this by e z, those here which is indeed the direction in which the force acts on unit positive test charge here.

So, this again tells you that you can combine both these quantities and this is equal to the electric field at a point z, because that is less than 0. So, you could combine both these things and say the potential, the electrostatic potential due to an infinite uniformly charged plane with charge density sigma sitting on the x y plane. This electrostatic potential phi of z must be equal to that we have here minus sigma over 2 Epsilon naught; the modulus of z, mod z is minus z in the case of z less than 0. So, it gives you plus sign here and you have a minus sign here.

From this, it follows this implies that the electric field E of z equal to, we need to differentiate with a minus sign, so this is equal to whatever derivative we had minus d phi over d z, e z. Remembering that because the field or the force points upwards above the z axis, above the x y plane and below downwards, below the x y plane, it is clear that the force changes direction as you go from a above to below, the plane. And therefore, the potential is discontinues as far as it is derivative is concerned, because the derivative gives the field.

The and indeed you see that discontinuity coming from the fact that this potential is a function of modulus of z, which has you know the V shape curve plot mod z verses z. It is 45 degree line for z positive, 135 degree line for z negative with a sharp point at z equal to 0. So, as far as that is concerned, we have a simple expression for the potential and we can derive the field from this potential.

Now, the potential energy itself would depend on or the force on the particle itself rather would depend on the test charge that you put. So, if you put a charge q, you know that the force on the charge is q times e. So, once you have the field finding the force itself only is a multiplication a question of multiplication by a certain constant. Now, let us look at a less trivial example than this.

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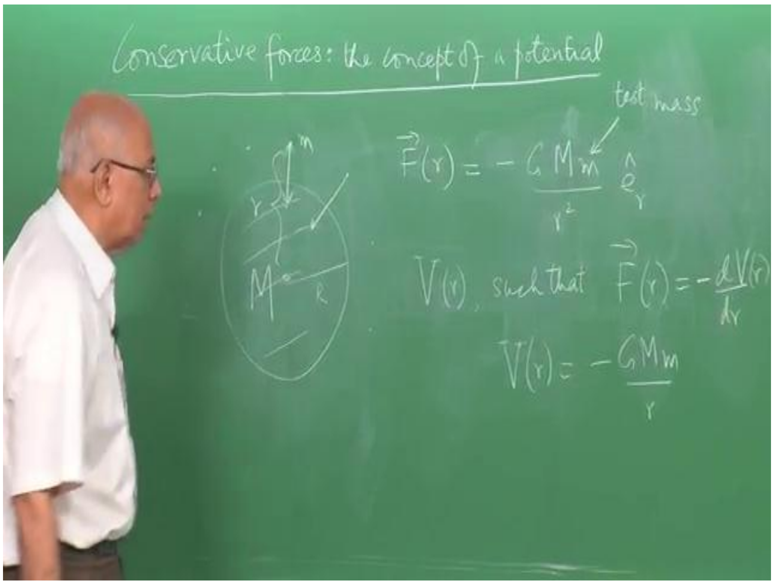

Let us look at what happens, if you have a gravitation force due to us very spherical bodies, such as the earth and we ask, given this earth with some radius capital R. This is the earth and I ask what is the gravitational field at a point here? Now, this is a spherically symmetrical mass distribution, we assume the earth to be a perfect sphere.

For such a situation, Newton's laws of gravitation terms out to yield a gravitational force here, which is directed towards the centre of the earth from the point concerned. It does not depend on where you are as for as the latitude or longitude this concerned.

So, it does not matter what the angular coordinates are, at all points at a given distance a little r from the centre of the earth the gravitational force is the same magnitude and acts always towards the centre of the earth, towards the centre of this mass distribution. So, the force at this point acts inwards in this way, the force at that point acts inwards in this way. And therefore, we could write this force at a distance r from the origin.

So, this distance is r, this is equal to minus g times m, let us call this the mass of the earth times little m; that is my test mass, whatever mass I put at this point, like the test charge divided by r squared times the unit vector and the radial direction with a minus sign to show that the force is directed radically inwards as suppose to outwards. So, that is the force and this we know from Newton's law of gravitation.

Together with the fact that this is the gravitational force on this point due to every mass element of the earth and the remarkable saying which was discover by Newton first is that, when you have a spherically symmetrical mass distribution. Then, the field at any point out side does not depend on the angular coordinates of that point with respect to this origin, but only on the radial distance from this point.

So, this magic property of spherically symmetric mass distribution is common also to the case of Coulomb potential or the Coulomb electrostatic between charges. There to if you have a spherically symmetrical charge distribution, we will see the potential at any external points is dependent only on the radial distance from the centre of this charge distribution to the point concern and not on the angular coordinates.

That the reason, I have return r here rather than write r comma theta comma phi, I have put those in, no angular coordinates only the radial coordinate decimal put here without a vector symbol to show that, it is only dependent on the radial distance. So, this is the magnitude of the force without the minus sign and that is the direction minus e r is the direction here. So, the question is, can I write down, what the potential energy of this object is, in other words, is there of function V of r, such that, F of r is equal to minus d V over d r, d of r times e sub r. Can be find such a way of r and inspection shows you that you can, because you can immediately see that if I choose V of r equal to minus g m m over r and nothing else.

Then if I differentiate this quantity, because I have a 1 over r the derivative is minus r squared. So, this minus sign from there and there is already minus sign here. So, that is gives me a plus sign, but I have to find minus d V over d r that puts the minus sign back once again and I multiplied by e r and I get the gravitation force here. So, that is the potential and it is in fact, the potential energy of a test mass equal to little m at a distance r from the centre of the earth, this is the capital M, the mass M here.

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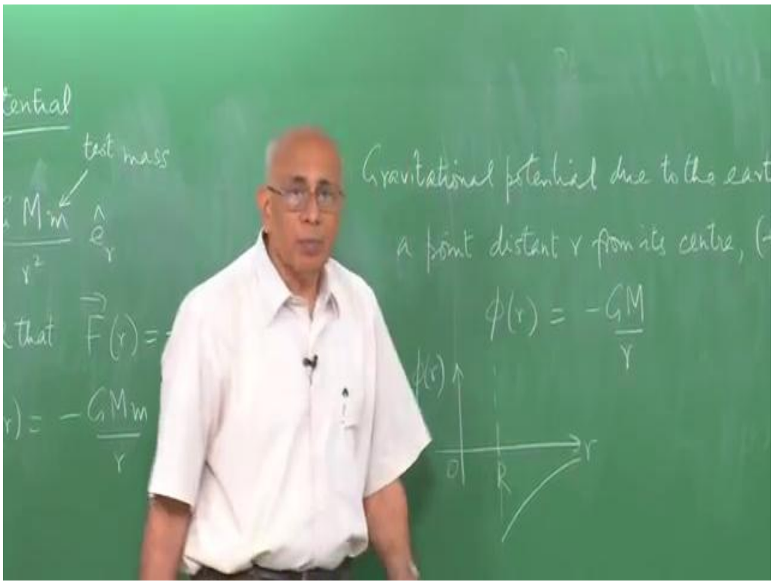

So, exactly acts in the electrostatic case I can define gravitational potential. So, gravitational potential due to the earth at a point at a point distance r from it centre, for r greater than or equal to r. We need to have the force due to the entire distribution, the full mass of the earth, this quantity is phi of r, it is a functional only of r not of theta and the azimuthal angle phi, the latitude or longitude and this quantity is equal to minus g m over r.

So, it is function only of a radial distance, once again I differentiate this quantity, I get plus g m over r squared take minus the derivative and I get minus g m over r square. And I multiplied by the mass to give me the force on a given mass little m and multiply by e r and I get this gravitation field here. Now, what is this thing look like, if I plot this functions, it says if I plot little r here is 0 and this is phi of r this looks like, we do not know, what is going on till you hit capital R.

This formula is only true for little r greater than capital R, greater than equal to capital R, namely from the surface of the earth upwards into this space. Something else happen inside here, we will come to that sequentially later on, but the graph as for as this is concern, start at some finite point minus g m over r and goes like that. So, this quantity here G capital M over capital R is what use to calling as little g as you know and with an appropriate sign, it is says things for downwards towards a ground.

So, this again tells you that the effect of this force, which is a vector after all, can actually the force itself can be expressed as a suitable derivative of a scalar function, call the potential in this case. And this is an extremely useful idea, the fact that you have a scalar function, which you can read with which gives you all the information; that is actually contain in the force itself.

Now, you might have one very important question, which is the following, after all, if you have the derivative of this function, equal to this force and this is what a physically measurable. We could always say, then that if I add a constant to this potential, it does not get differentiated at all, the constant derivative is 0, and therefore the force, the physical force will not change.

Does that not mean that the potential in every case is determine only up to an additive constant, which is completely arbitrary. Because, it will not show up, when you differentiate to get the force and the answer is yes. You always when you specify a potential here, have to specify a reference level with respect to which you are measuring in this potential are you computing this potential.

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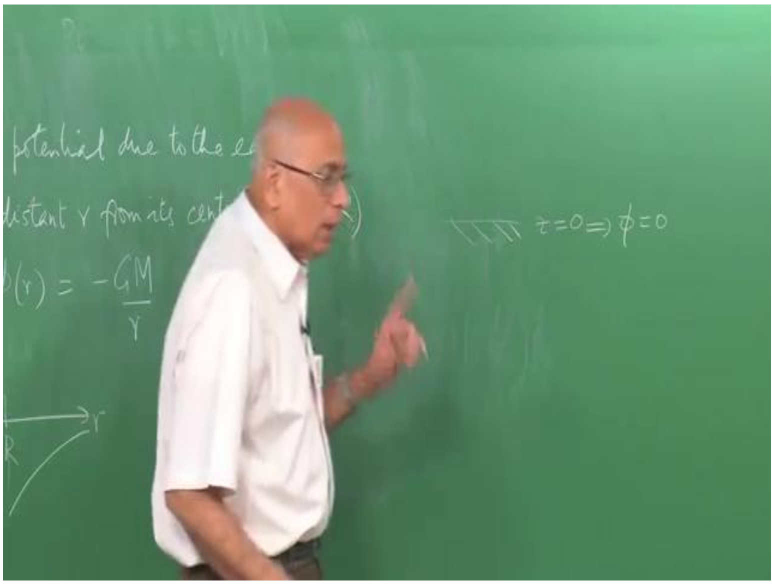

In the case of the example, the normal example of the ground being 0 potential, this you say z equal to 0 implies phi equal to 0 with respect to the ground level taken to be 0, the gravitational potential at any point at height z above the ground is g time z with respect to that 0. In this problem, the reference level is taken to be 0 at infinitely. In other words, when an object is infinitely for away from the earth centre, the gravitational potential at that point is taken to be 0 and with respect to that 0, the gravitational at any infinite distance r is minus G capital M over r.

So, this is important to remember that in every case, we will always implicitly or otherwise have a reference level for the potential or the potential energy with respect to which you compute or write down expressions for these quantities. This is an important point and this reference level is chosen at your convenient. So, as you can see even from this simple examples in some problems, it may convenient choose it at some finite point, such as the ground, the 0 potential.

On the other hand in other case, it might we convenient to choose it at infinitely on physical ground saying that infinitely for away, there is no force, gravitational force is vanishing, the small and I choose the potential there to be 0. Now, can we generalize these things a little more and the answer is again yes.

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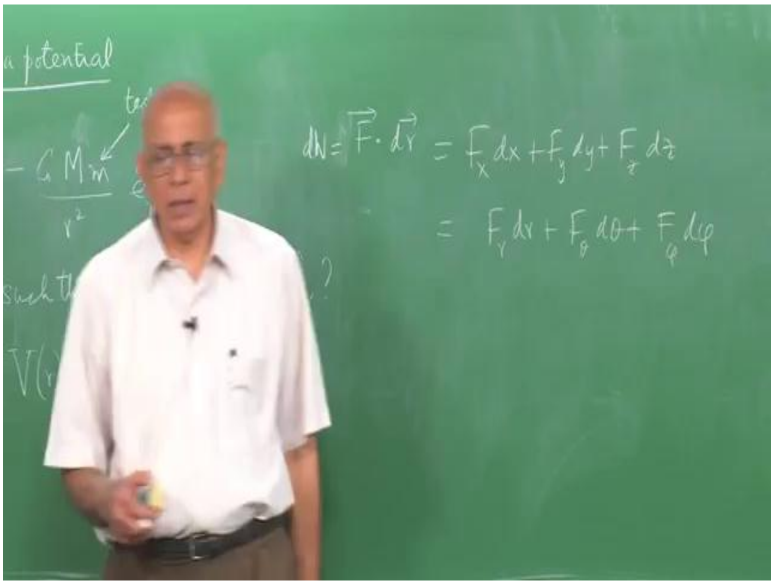

How was z that able to do this calculation in terms of r, without bringing in the other coordinates, after all, when we started with the statement that F; that the work done d W was equal to F dot d r. And if I write this Cartesians, it is F x, d x plus F y, d y plus d z. In general there are three independent coordinates x, y, z or if go to write this in terms of spherical pole coordinates for instants. This would be equal to F r, d r plus F theta d theta plus F phi d phi with appropriate dimensions for F r, F theta and F phi remembering that.

This quantity here does not have dimensions, this thing here dimensions of length and this therefore, dimensions really of a force, but these quantities, these are dimensions quantity. So, these components already have force times length as physical dimensions in them and that is got we remember. So, how is it that I was able to get away with the fact that only r, only this dependence came in and the answer a rows in these examples, the answer come from the fact, that I know already, that the force is radially inwards or radially outwards and such forces are called central forces.

Force is which in spherical polar coordinate do not depend on the angular coordinates, but only the radial coordinate they are call central force. The force is directed only radially, it is called a central force. We will come a will elaborate on this concept a little more as we go along this concept a little more as we go along. But, that what enable to show to speak inverts this expression, because the other two terms in some sense, where not present at all. When the present, you need a little more general way of writing this derivative and it involves, what is called the gradient operator, which I will not going to at this stage. But, we will come across examples where you need that, but we will see why the central force is a very important concept, why it appears so often.