Mechanics, Heat, Oscillations and Waves
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Lecture – 02
Fundamental Constants and Dimensional Analysis

Let us begin today with where we left off last time. I started taking about the Fundamental Constants of nature and we listed three of them to start with.

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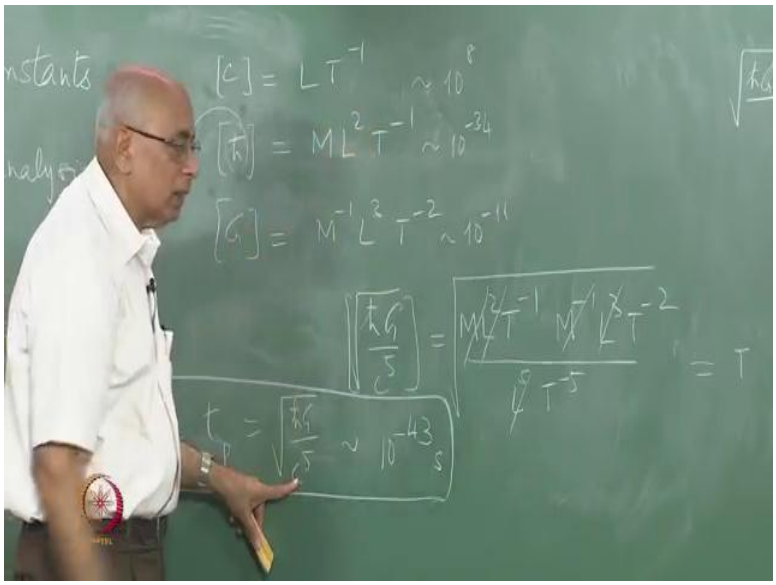

The first was the speed of light and vacuum denoted by c and recall that the physical dimensions of c was that of the speed L T inverse. Then, Planks quantum mechanical constant Plank's constants and this had physical dimensions of M L squared T to the minus 1, energy multiplied by time and then there was Newton gravitational constant which had the dimensions of M inverse L cube T to the minus 2.

The numerical values of these constants in standard international units, this is 3 times 10 to the 8 meters per second, this is of the order of 10 to the minus 34 Joule seconds and this is of the order of 10 to the minus 11 in standard international units. I mention that these work truly fundamental constants in some sense of course, you can immediately think of many other constants of varying kinds.

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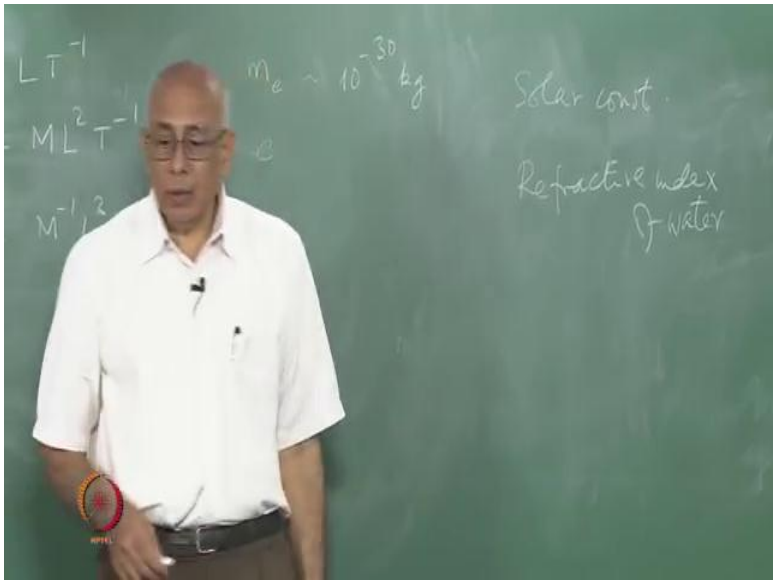

For instance, there is the mass of the electron which is of the order of 10 to the minus 30 kilo grams. Similarly, there is the mass of the proton, the mass of the neutron and so on. Then various masses of nuclear, there all constants in that sense constants of nature and then there is a charge on the electron 10 to the minus 19 Coulombs. And then, there are various other kinds of constants one can associate with these quantities and other derived quantities.

For instance, you might say how about the solar constant, how about the constant like the specific heat of a particular material like water for instance or how about the reflective index of a liquid or a solid and so on, they are all constants of nature. But, the point is if the fundamental difference between this sort of constant and this sort of constant and this sort of constant for instance, we say refractive index of what for instance that too is a constant.

But, there is a big difference between this kind of constant, that and that, this kind of constant refers to specific materials, specific objects, specific conditions such as the amount of radiation received at a distance from the sun equal to the average distance of the earth is from the sun per unit area and per unit time and so on that is the solar constant. This will of course, change if you go to another planet or some other distance, so another star.

The refractive index of water depends on the temperature of the water, it depends on the pressure it is under and other physical conditions, so in that sense it is specific to a particular substance and a particular conditions. The mass of the electron and the charge of the electron on the other hand are not of that kind at all, the electron has a same mass no matter, where it is in the universe it has the same charge, no matter where it is in the universe.

So, these are truly constants of nature in that sense; however, they again refer to very specific objects, such as the electron in this case or the proton or the neutron. On the other hand these are fundamental in a different sense, they describe various properties of the universe itself in some very, very fundamental sense. They are independent of the objects which this universe contains, their intrinsic constants in the following sense. This c according to the special relativity theory puts a fundamental limit on this speed that material objects can have.

So, this is in fact a limiting velocity in the universe the way it is, this on the other hand tells you what happens when you go to very small objects, like atoms or molecules or electrons, individual particles, nuclei and so on. It is a quantum mechanical constant and it is start playing a rule as soon as quantum mechanics takes over from classical physics and you need to include the effects of quantum mechanics in whatever you are describing.

So, it is a fundamental property of the universe as we believe, this G tells you another fundamental property in this universe, namely masses attract each other according to in the first approximation Newton's law of gravitation. So, in that sense this tells you something about the large scale structure of the universe and this tells you something about the very small and the very large here and this puts a fundamental limitation on velocity itself or deep freeze and sub symmetric.

So, in that sense these constants have a slightly different role to play then these are the constants and that is why we call these truly fundamental constants of nature. Now having done that, having said that one can ask now, after all my basic quantities I talk about are M, L and T. So, is it possible that I can construct from these three constants of nature which we claim a truly fundamental constants? Is it possible that we can construct a quantity which has physical dimensions of a mass or a length or a time?

Very conveniently for us we can do that, because there are three of these quantities here which have physical dimensions as written here and from them we can construct quantities which have individual dimensions mass, length and time. All we have to do is to put these quantities together in such a way that the other two dimensions cancel out and you left with either an M or an L or a T. For instance, suppose you construct as I said last time it is more useful to talk about h cross which is h over 2 pi, in Physics we always take about h cross because that is a quantity that appears naturally.

When considering the following, we would like to have for instance a mass. So, I would like to have a quantity of dimensions mass and this must be equal to some combination of h cross c and G and it is obvious from here that if you multiply h cross with G this mass cancels out immediately. So, if you write h cross G here note there, then this cancels out and you are L to the power 5 and T to the minus 3. So, this quantity has dimensions L to the 5, T to the minus 3 and you want to cancel out both these fellows.

So, what we do is for instance take a c to the power 5 and look at what the dimensions of this quantity R. So, physical dimensions of this R M L squared T to the minus 1 times G which is M inverse L cube T to the minus 2 divided by c to the power 5 and what c to the power 5 does is L to the 5, T to the minus 5. So, this is equal to the M cancels out here, the L cancels out completely, this 5 goes up on top and you left with T squared.

Therefore, if you took the square root of this quantity we have a quantity square root of this which is equal to T. So, this says that the combination h cross G over c to the power 5 has the physical dimensions of the time and nothing more all the L's cancel out. So, what would you call this time, it is call the plank time T plank, T plank is equal to square root of h cross G over c to the power 5. So, these three fundamental constants give us a natural quantity which has a natural combination which has the physical dimensions of a time and it is a fundamental time in nature in some sense is called the plank time denoted by T sub p and it is h cross G over c to the power 5.

Now, it is interesting to ask question, what is the numerical value of this quantity, well if you put in the figures we already have for this, this was of the order of 10 to the 8 meters per second, this was 10 to the minus 34 and this was 10 to the minus 11, then you end up with this is of the order of 10 to the minus 43 seconds. So, it is a incredibly small interval, because we remember for I said the last time the smallest time interval that we

can actually in directly probe through our experiments is what is called the strong interaction times scale and it is 10 to the minus 23 seconds, this is 20 orders of magnitudes smaller than that.

So, let us keep this in mind for moment that there exist an natural times scale in nature which is of the order of 10 to the minus 43 seconds in conceivably smaller than any physical process that we are familiar with and those of you few read about the big bank for instance, you might of come across the statement that we know what happen to the we have a scenario for what happen the universe after 10 to the minus 43 seconds after the so called big bank.

But, before that it is unknown the reason is buried here is buried in this, the fact is you know come back to this the very meaning of time is lost. The present understanding of time that meaning is lost below a time intervals smaller than this and we will come back to this in a moment. Now, having found the quantity of a dimensions time, how do we find the quantity of dimensions length. Well, there is a velocity here. So, if I multiply this time by this velocity, then it is automatic that you are going to get a quantity of dimensions length.

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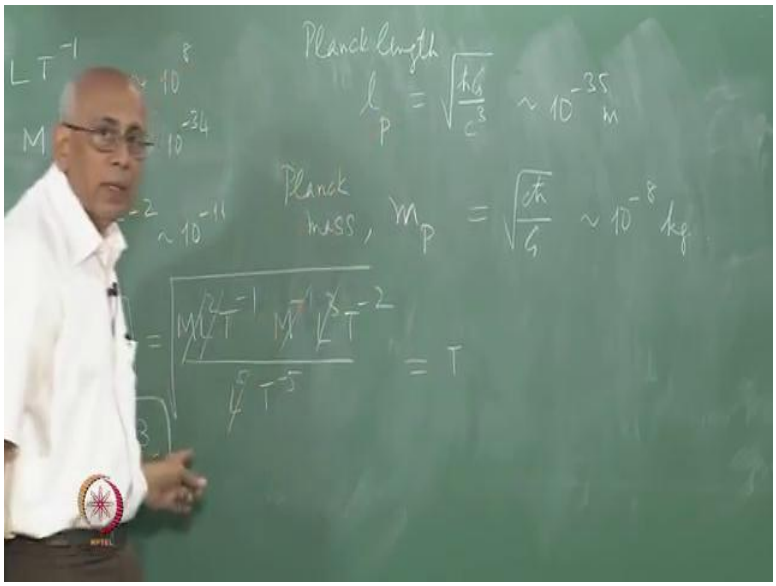

So, it is immediately obvious that you are going to get square root of h cross G divided by c to the power 5 square root and then I have to multiply by a c. So, that is the same as multiplied by c squared inside this squared root and therefore, we get c cubed here and

this is the length, this is called a plank length and this is of the order of 10 to the minus 35 meters. Once again, it is inconceivably smaller than any lengths scale that we are used to, the smallest length scale that you might be used to is for instance the radius of the nucleus, the size of a proton.

So, to speak this is of the order of 10 to the minus 15 meters a femto meter, this is 20 orders of magnitudes smaller than the femto meter. So, it is inconceivably small and once again tells you that there exist a quantity of dimensions length and natural lengths scale in the universe which is unbelievably small much smaller than any physical thing that we know about and it happens at 10 to the minus 35 meters.

So, in a sense just like in the case of time the plank time, this is the lengths scale below which our conventional notion of length is a continuous variable breaks down completely. So, these are fundamental properties of the universe as we understand that presently and it is not surprising. Therefore, these constants immerge naturally from these fundamental constants here. Well, we could ask what about a mass, can I create a mass from these.

We already have some elementary masses like the mass of the electron, the mass of the proton and so on, there is the mass that we can construct and that has the following dimensions. Once again, we need to play with these in order to create a mass, this time it is clear that h cross and G must appear in the numerator and denominators. So, that we get an n squared on top, so this quantity c h cross over G has the dimensions of a mass and it is called the plank mass M p.

And again this surprisingly is of the order of 10 to the power minus 8 kilo grams. Now, that is much, much larger than any of these quantities other than in different units. Now, 10 to the minus 30 is the mass of the electron, 10 to the minus 8 is the plank mass that is 22 orders of magnitude larger. What is this mean? This mean elementary particle with the mass of a plank mass, because that is enormous, this is almost a macroscopic mass.

Well, there are many analogies you can give for what this mass is like and many objects would have this mass for instance, certainly bacteria perhaps ((Refer Time: 13:26)) something like that would have mass comparable to this, the plank mass here. So, it does not have a significant unlike these two cases or some kind of limiting mass or the mass of some physical object or anything like that.

But, it has the following meaning which we not get into write here, if you have a point mass a single point particle with no dimensions at all. In some sense the plank mass passes the upper limit to what can happen for a point mass, more than that it cannot support a mass and gravitation collapse would occur. So, in that sense this is slightly different from the plank line and plank time, where the meaning of length and time themselves they destroyed below these case. The plank masses somewhat different as a different denote together.

But, the point of this exercise was to show that you can use combinations of the three fundamental constants to create quantities which have natural length mass and time units respectively. So, that is what this ((Refer Time: 14:30)) analysis said. So, keep this in mind, but now let us go and to the next part of it, which is to see how we can used dimensional analysis in order to drive use to information about physical problems. So, very, very important subject, so let me spent a little time on it is.

We will repeatedly try to comeback to dimension analysis some of the mention in the beginning, it is a very useful tool for instance to check the correctness of formulas various kinds. So, when we do things other formulas and so on we will always check the dimensional consistency of these equation by making sure that the physical dimension of the whatever is on the left hand side of an equation appears exactly on the right hand side is well.

In this problem I am going to choose for illustrating the power of dimension analysis, we will also see where it is going to feel but it is limitations are is a very simply problem which is familiar to you many contacts in that is the motion of a simple pendulum.

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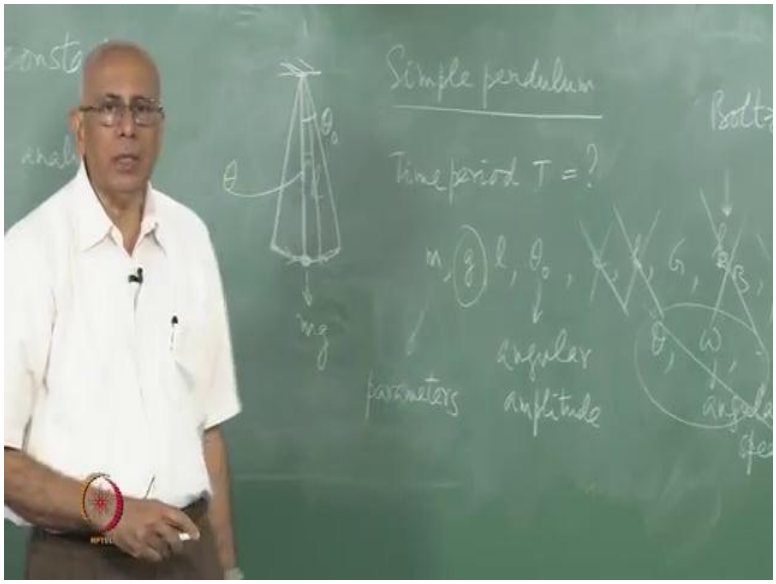

Now, what we have here is a light mass less rigid rod of length l and a heavy bob of mass m suspended from it and it is free to wide osculate about this point of suspension. So, that is what a simple pendulum is and we want to find out what is the time period of oscillation. So, this pendulum is assume to this started swanking and the net undergoes oscillate for a motion of this kind, it oscillates back and fore about pull over this point under the action of gravity in this form.

For simplicity we are going to assume that all the masses concentrated here and there is no massive. The moment you put a mass here is would be like a rod with the finite mass, then it is call another simple pendulum in becomes a compound pendulum; it is a little more interacted. But, let us analyze first the simple pendulum motion here these suppose to set it oscillating with some amplitude and let us call this angular amplitude theta and I will put not here to show that it is a constant under the given conditions, namely is started off at some angle theta and let go and by the conservation of energy this angle can never increasing beyond theta naught with stage theta naught.

And if a neglect friction and include only gravity then this oscillation will continue forever and this is the amplitude of oscillation. We will come to the roll of this theta naught here and the question asked is what is the time period and let us call T and what is this equal to. Now of course, the way to do this problem is to set up Newton's equation of motion, for the motion of this form and then solve this equation on the motion show

that it periodic motion and compute the time period and that is doing the regular dynamic of this bar, but we trying to do this by dimensional analysis.

So, we are going to see how close to the exact answer can we get by doing this. Now, the force on this pop is due to gravity and it is downward force m g ((Refer Time: 18:13)) and at any instant in it is motion there is always the constant force of gravity acting downwards, there is also tension in this. So, rigid rods, so we are not going to interested in that because the rod does not extend and this m g is the force which we have to right down then I want to write down the equations of motion.

But, I do not want to do that, I want to ask what can this possibly depend on. Now, comes a very important point, it is going to depend on the large number of factors in general and let us start writing all these down and classifying what the look like. Certainly, you are first guess would be depends on the mass m, it should depend on g, because that is the force that cause in the motion on the first place, it should perhaps depend on l length of this bob, it should perhaps depends on the amplitude theta naught.

So, this is the some given angular amplitude certainly I except physically that if I take this bob an oscillated with one amplitude and then I oscillated with the different amplitude, this no reason why the time period should be the same and therefore, they should be some dependent on theta naught, my first guest would be to say the larger they amplitude, the longer it takes the time period.

So, therefore, I would depend I would say the there is some dependents on theta naught which is to be discovered as yet. What else could this depend on, well no reason why it cannot depend on fundamental constants of nature. So, perhaps it also depends on the speed of lie, it also depends on plank constant, it also dependents on capital G may be it depends on other things as well, there other constant of the motion.

For instance, I know that in heat and thermo dynamics there is the statement that the average kinetic energy of a molecule of gas in the zoom is 3 half k t, where k Boltzmann constant. So, there is the universal constant, Boltzmann constant k Boltzmann this for as call Boltzmann constant and it is numerical values is of the order of 10 to the minus 23 joules per degree Kelvin. So, may be depends on that may be depends on the charge of the electron and whole lot of other constants the more clue at the moment and now bring in physical arguments to eliminate things.

Well, you could also ask one more thing, you could say look after all when it is moving the angular is changing at some instance of time in the bop is here, this angle is theta and then it increases up to theta naught comes back becomes negative goes up the minus theta naught comes back 0 and so on. So, may be depends on theta is well may be depends on the rate at which this data change that too is crucial. So, may be depends on the angular velocity omega this is angular speed and so on.

So, it looks like a hopeless propagation, but now comes the physical arguments, we going to make carful physical arguments to show that it actually does not depend on most of these things it depends on the very small number of these objects, these quantities and will get practically the exact formula, but carful physical argument and that is what. So, now, let us step back and ask what is the difference between these quantities and these quantities, well these are variables there dynamical variables the change with time as the pendulum swings data changes, the angler velocity changes and so on.

But, we are asking for a time period which comprises the full motion from here to there, back through here, back to this point here. And therefore, we are asking for a quantity, the time period which is not dependent directly on the instantaneous velocity for the instantaneous speed or angular speed or the amplitude. So, these are dynamical variables, the time period does not depend on this variables it will depend on them in directly in some sense, but once you finished computing the entire motion over a full time period, it cannot depend on what the instantaneous velocity is at any particular instant of time doing it is part.

So, it is going to depend on these quantities which are not variables, there parameters. So, that is the crucial difference these objects or parameters for a given pendulum of a given length, given mass and so on and on the surface of this earth, where this is given these quantities are not variables any move, even the amplitude is not a variable for a given set of motion, if you tell me how much you pulled it to left and let go the amplitude is fixed and so on.

So, the amplitude the time period could possibly dependent on any of these, but now let us eliminate these quantities. I mention that this c place a role as soon as you have either very high speeds or something like light involved electromagnetic radiation involve,

electricity in magnetic involved, this is the classical mechanical problem, this object is neutral there are no electric or magnetic forces. So, we do not expected to depended on this, this is the big object relative to an electron or an atom something like that.

So, no quantum mechanics of quantum correction come in here. So, does not depend on this, the object is electrical neutral. So, this no dependence on that it is a fix temperature we are not talking about the transfer of heat energy from one object on other everything like that. So, does not depend on that, it does depend on G, but the point is that depends on G is worried here in little G.

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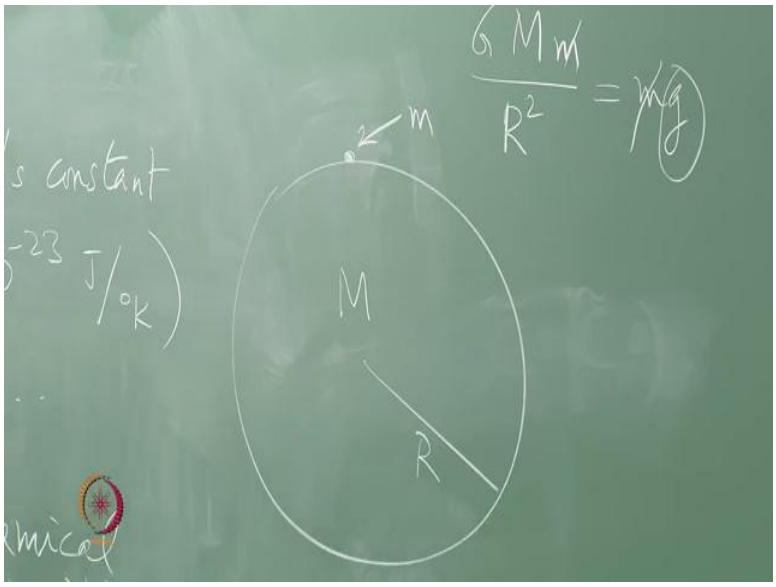

Now, what is this little G equal to remember that on the surface this arc of radius capital R, if you could a little mass here m and mass of this earth is capital M, then the force of gravity on this object is by Newton's law g M m divided by R squared is the force of gravity in surfaces of this earth at a distance capital R from the center of the earth and we saying that this is equal to m times g by definition of little g in this m cancels out.

So, you see little g is capital G m over capital R squared, where capital R is radius of the earth and capital length is the mass of the earth. So, that is already in this constant here, so we do not need to put this in separately that leaves as just these four quantities on which the time period could possibly depend. So, you see the physical arguments that have gone in here to eliminate all the other variables. So, in our going to use that in other less to be instants is as well.

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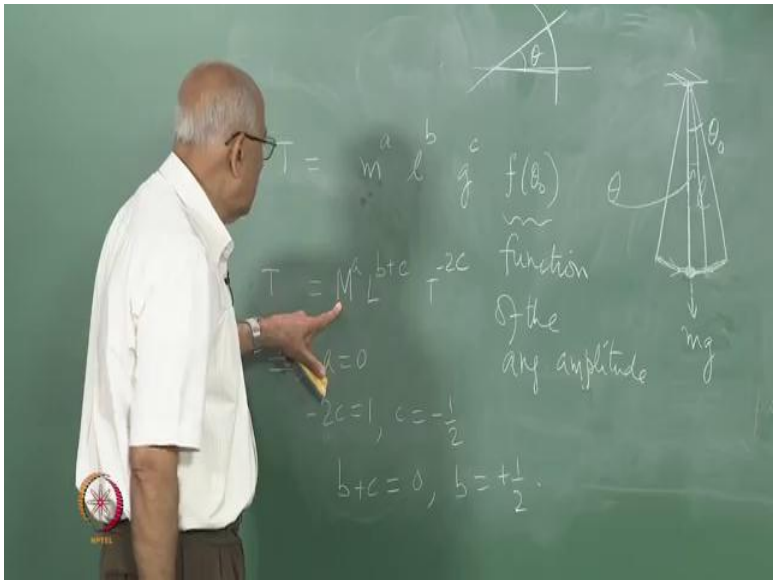

So, where does that get us it tells us that this time period must be propositional to... So, in terms of dimensions, the dimensions of this time itself this must be equal to the quantity m which as dimensions of mass raised possibility to some power a could be 1, 2 minus 1 we do not care, we do not know it could depend on l and that raise possibly to some b, then little g raise possibly to some power c which we do not know as yet and then theta naught, but you see what is theta naught it is the angular amplitude.

But, as you know the physical dimensions of angle is dimension less, because an angled is define as the r divided by the radius and an angle is dimensional less, how to you find the angle between two lines, what you do is to draw an arc of an circle and argue that the arc divided by the radius is equal to the angle between the two. Since, they arc and the radius both of dimensions length, the angle has no dimension at all.

So, this is dimensionless and by the way you measure always in physics angles in radian measure not than degrees at all. But, it is now a dimensional less quantity and it could depend here could appear any where here in this formula. So, the time period could be some arbitrary function of this amplitude theta naught some function of the angular amplitude and we do not know what this function his, no have we any way of finding out what this function is that is the interesting thing.

But, it sitting there and now what we can do is to say, so this will tell us that t must be equal to that M to the power a from this and L to the b from that and g was L theta the

minus 2. So, L to the c T 2 the minus 2 c and this should be equal to T which will of course, immediately imply the a must be equal to 0 minus 2 c must be equal to 1 or c equal to minus half and that tells you what c here is this is b plus c. So, L was b plus c which mean and b plus c must be equal to 0, this is b equal to plus half.

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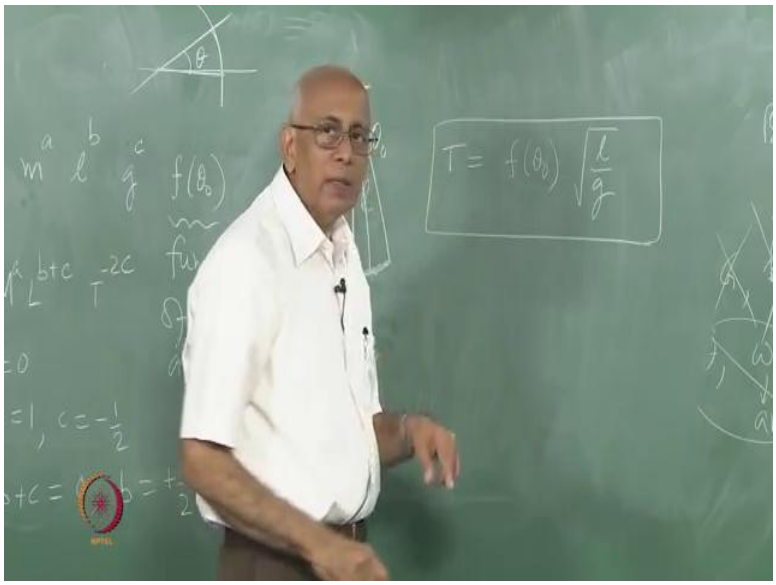

So, if you solve these three equations in write it down we have a formula here for the time period of a simple pendulum which says that the time period must be equal to some unknown function of the angular amplitude multiplied by M to the power 0. So, the mass does not appear at all L appears of course, and G appears, so square root of l divided by g, because c was minus a half while b was plus a half. So, it just a square root of l over g.

So, that is our formula for a time period of a simple pendulum in this case, it depends on the square root of this line and depends on 1 over the square root of the acceleration due to gravity here. But, it is multiplied by this unknown function of the angular amplitude and we have no way of finding out what this function is from purred dimensional analysis, because it is dimensionless.

On the other hand, we can say a little bit about it which I will and you will see how we can procedure from this. But, the fact is that no matter what this function is in this is the crucial point, the consistency of physical dimensionalities on both side of this equation says, the given a simple pendulum of length l and mass m and gravity little g given the acceleration to the gravity, the time period has to be proportional to the square root of l

over g multiply by some number which is possibly of function of the angular amplitude. Now, the next time we will see what we can say about this number.