Mechanics, Heat, Oscillations and Waves
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Lecture -19
Two - body Collision Kinematics

To repeat what I have been saying, we will be looking at the problem of Two Body Collision Kinematics, where you have a target particle at rest in the laboratory frame particle of mass m 2.

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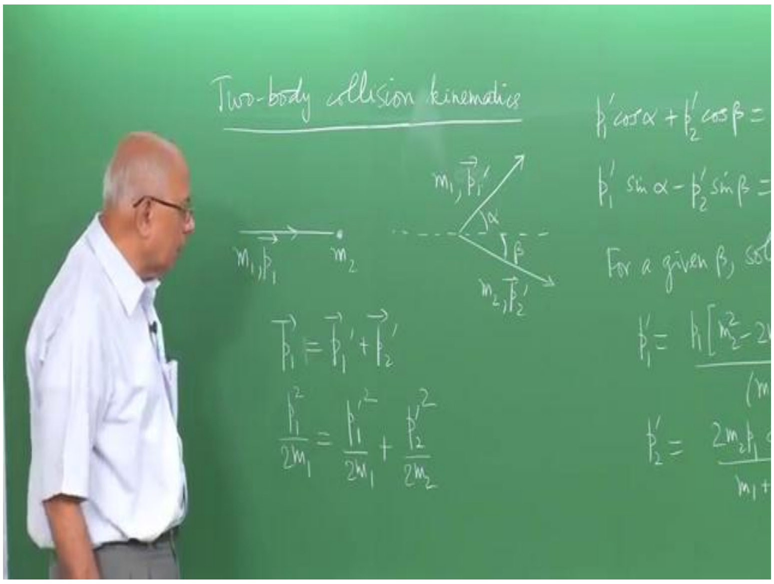

A projectile a particle of mass m 1 comes along with momentum p 1 and hits it in an elastic collision and the final outcome is that, the target particle goes off at some angle beta with respect to the original direction with the momentum p 2 prime. While, the projectile goes off at some other angle alpha with the momentum p 1 prime. And then, according to Newton's third law, therefore, action and reaction are equal and opposite.

It leads to the conclusion that the initial momentum, the total initial momentum p 1, since p 2 is 0 to start with is equal to the vector sum of the final momenta of the target and the projectile p 1 prime plus p 2 prime. This is also an elastic collision in the sense that, no energy is lost, otherwise in the total energy; kinetic energy remains exactly the same before and after the collision. And the initial kinetic energy was p 1 squared over 2 m 1, the final one is shared between particle 1 and particle 2 and it is this expression here.

So, subject to these two equations, we need to find out, for instance, what p 1 prime is and p 2 prime is in magnitude and direction. In other words, we need to find these magnitudes of these vectors as well as alpha and beta. Now, as I pointed out, when we resolved these vectors along the direction p 1 and perpendicular to it, you have two equations respectively given by these two equations here.

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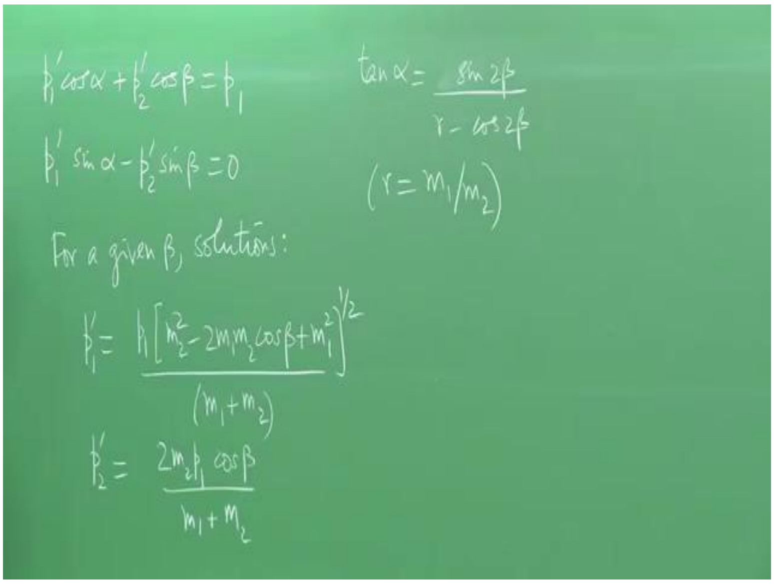

Together with energy conservation, they form three equations for four unknowns, p 1 prime, p 2 prime, alpha and beta and there is no way, you can solve uniquely for all four given just three equations. So, what one does is to say for a given value of one of the variables and conveniently taken to be beta for technical reasons, which are mentioned later on.

The solutions of these equations, simultaneous equations are straight forward, we saw this how to do this in a simple way. What you have to do is, to simplify the algebra define p 1 prime over p 1 is some x and p 2 prime over p 1 is some y, and then if you define the mass ratio r to be the mass of the projectile divided by the mass of the target m 1 over m 2. Then, p 1 prime has this solution in terms of beta and p 1 and p 2 prime has this solution in terms of beta and p 1. Moreover, the angle alpha is related to the angle beta by this expression here.

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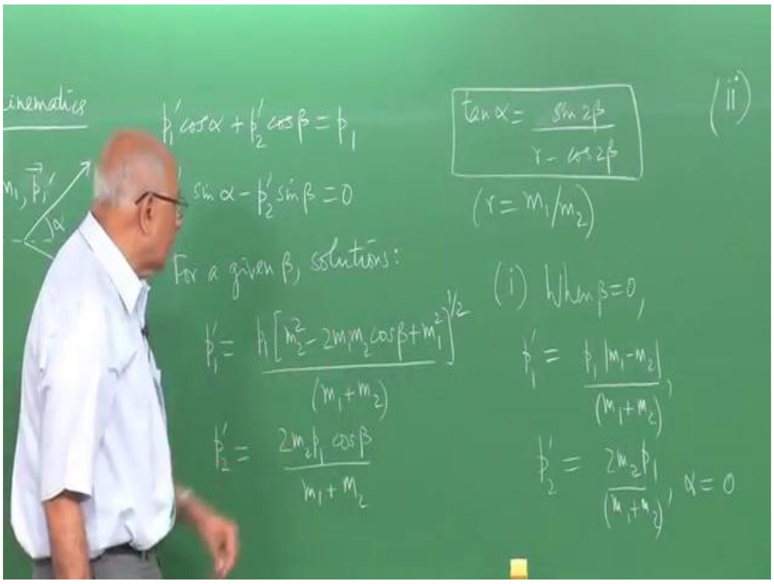

This is an important expression sin 2 beta, tan alpha is sin 2 beta over r minus cos 2 beta, where r is the mass ratio here. So, that is the conclusion we get, those are the results we get for these dynamical variables, after this scattering process. Now, the task is to analyze the solutions. The first thing we notice is the following 1. So, let me write it as 1, in the one dimensional case, in other words, when you have things moving along just one line corresponding to beta equal to 0. So, the target particle just gets pushed along the same direction as the incident direction.

When beta equal to 0, everything should reduce to the simple one dimension case that we already sort. So, it is clear, immediately from this that p 1 prime for example is equal to, it will took what the magnitude here is it is m 2 minus m 1 squared, square root, so it becomes modulus, this is the magnitude. So, this is p 1 modulus m 1 minus m 2 divided by m 1 plus m 2.

You recall that the exact solution is in fact, in the one dimensional case, m 1 minus m 2 here without the modulus sign, because when m 1 becomes smaller than m 2, you could also have it moving backwards. It could bounds of the target and move backwards as you physically expect, but these are magnitudes of vectors. So, in this case strictly when you take the limit beta is equal to 0, you get that solution. Therefore, the magnitude of the p 1 prime, p 2 prime magnitude is 2 m 2, p 1 over m 1 plus m 2.

And of course, you see from this relation that on beta vanishes so does alpha, so it immediately, so you have this, this and alpha equal to 0. So, it checks, the one dimensional case emerges as a special case of more general solution that you have got here as it should. So, everything is alright as far as this concern. The next thing to do is to check out, what happens to these limits in the equal mass case, suppose the masses m 1 and m 2 are equal that means the ratio r is equal to 1.

Then, you can immediately see that things simplify a bit, you take out an m 1 squared here and write things in terms of ratio or we set m 1 equal to m 2 equal to m and you pull it out, it cancels in it is denominator.

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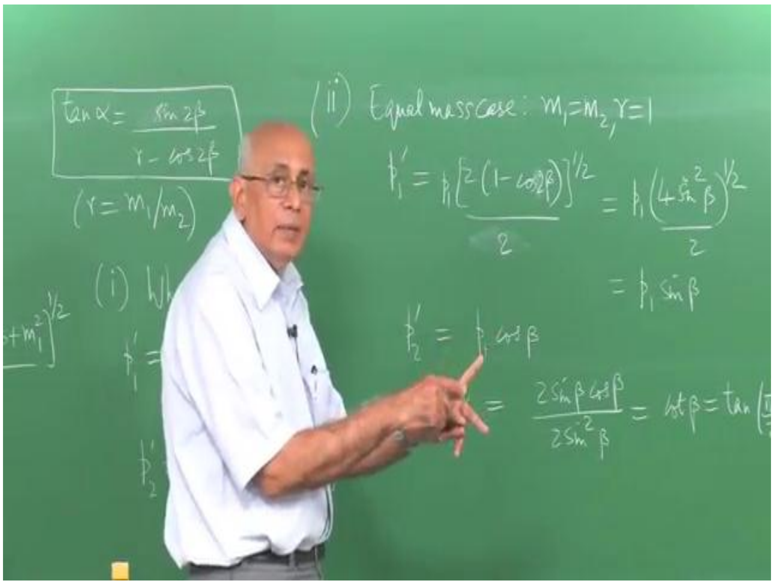

And you end up with the equal mass case, so you have m 1 equal to m 2 and the ratio r is equal to 1 in that case. Then, p 1 prime becomes equal to, well you get a 2 here and the 2 from here, so it is a twice 1 minus cos beta. Look at there at this fashion and you want a half to the power half divided by m 1 plus m 2. Well, m is go away, so it is just a 2, because the mass has been taken out common and cancel against the numerator and you have a 2 here. So, this thing here is 4 sin squared, this solution here is a 2, ((Refer Time: 07:10)) this is 2 beta of there.

And the solution now becomes 1 minus cos 2 beta and this is equal to 4 sin squared to the power half over 2 equal to with the p 1, p 1 this always sitting outside, p 1 sin beta. And p 2 prime is equal to this expression here and the m cancels out on both sides, 2 cancels out on the p 1 cos beta. So, we have a picture which is remarkably simple in the equal mass case.

In fact, you see what alpha becomes, it is immediately clear, alpha then becomes, this is sin 2 beta and this is a 1. So, this is 2 sin beta, cos beta divided by 1 minus cos 2 beta; that is 2 sin squared beta. So, this is equal to cos beta, which is equal to tan pi over 2 minus beta.

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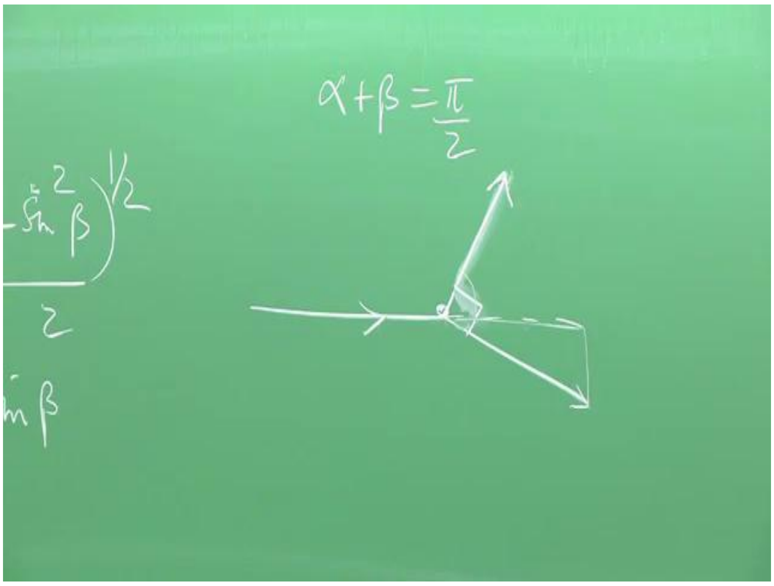

So, it says alpha pi over 2 minus beta, which is the same as saying that alpha plus beta equal to pi over 2. In other words this scattering process looks like this, in comes this fellow hits again this particle, this goes off at some angle and this goes off at right angles to it. So, that is the right angle, this is 90 degrees, since alpha plus beta is pi over 2. So, what happens in this equal mass case is that, the target and projectile emerge at right angles to each other.

Although we have not said and we have more dynamical information to say, whether it is like this or like this or like this and so on. So, it is clear that the restriction alpha plus beta equal to pi over 2 is a very simple interpretation. It says this particle has a projection of p 1, which is p 1 cos beta and the other particle has a projection, which is p 1 cos of this angle which is pi over 2 minus beta, so that is sin beta.

So, that is the reason why you get p 1 prime is p 1 sin beta and p 2 prime is p 1 cos beta and the relation alpha plus beta equal to pi over 2 in the equal mass case. So, that is the particularly simple instance, just to check that the general result is right, it helps to write down, what this does in that limit. Notice another fact here and that is, that you cannot have any value of beta beyond pi over 2.

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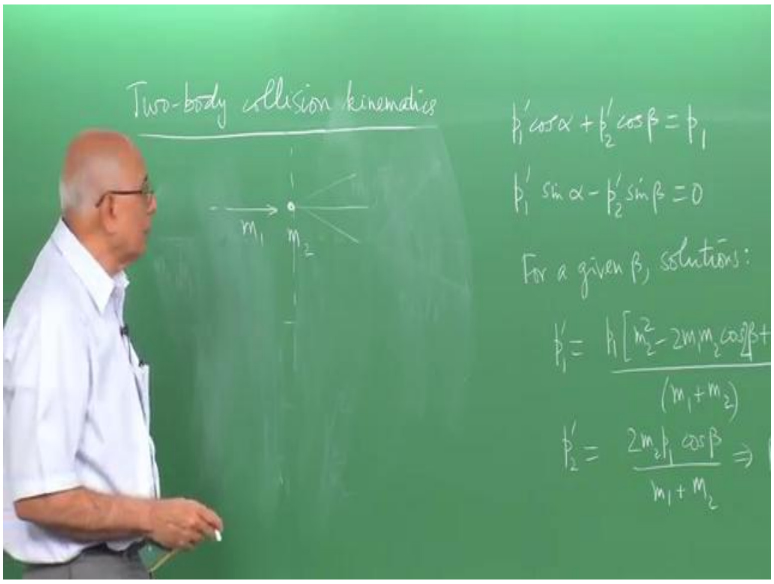

So, what is happening is that you have a mass m 2 here and you hit with the particle m 1 with the direction in this direction. Therefore, there is no likely wrote of this going backwards. It certainly cannot get to the left of this line at all, it has to move into the forward half plane in this case. And therefore, beta cannot be greater than pi over 2 and that is immediately obvious from here, because this equation implies the beta is less than equal to pi over 2. Because, if beta is bigger than pi over 2 this back scattering here, then this becomes negative, but this is the magnitude of the vector p 2. So, it is not possible.

So, there is a limit on how big beta can be tends from 0 to pi over 2 as in that case as well. However, one can go further one sees immediately that if this particle is heavier than the target particle, if the projectile is heavier than the target particle, it is unlikely that the projector will come back. So, in fact, it is much more likely that it will move in the forward direction, might not even reach pi over 2 and that is corroborated by the fact that there is a limit on alpha in the case, when the ratio r is bigger than 1.

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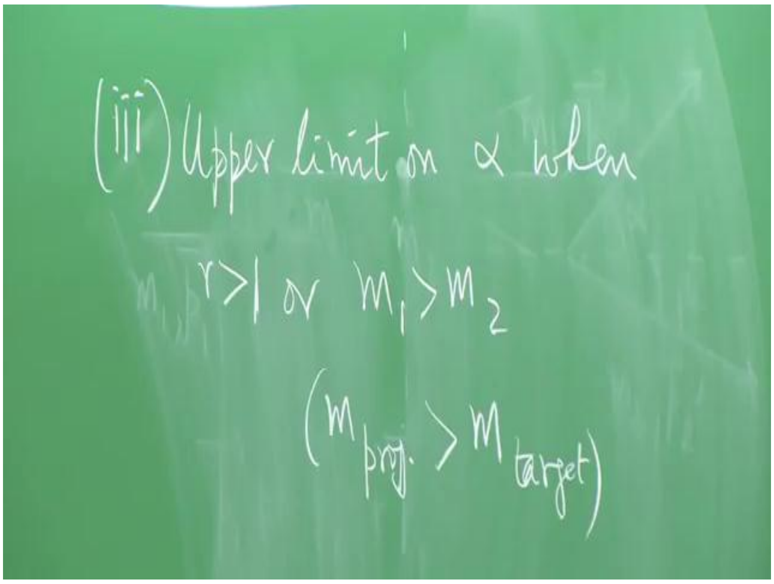

So, let us look at the case. So, this is the next special case you are look at, the next point three upper limits on alpha, when r is greater than 1 or m 1 is greater than m 2. This means m projectile is greater than m target. Normally, in scattering experiments one is used to have an very target and a light projectile hitting it, but you could have the other situation, when you have a very light target and heavy particle comes in and hits it.

What happens then? There is an upper limit on this quantity alpha, which is found by taking this object here and differentiating with respect to beta and seeing this as the maximum. So, let us do that.

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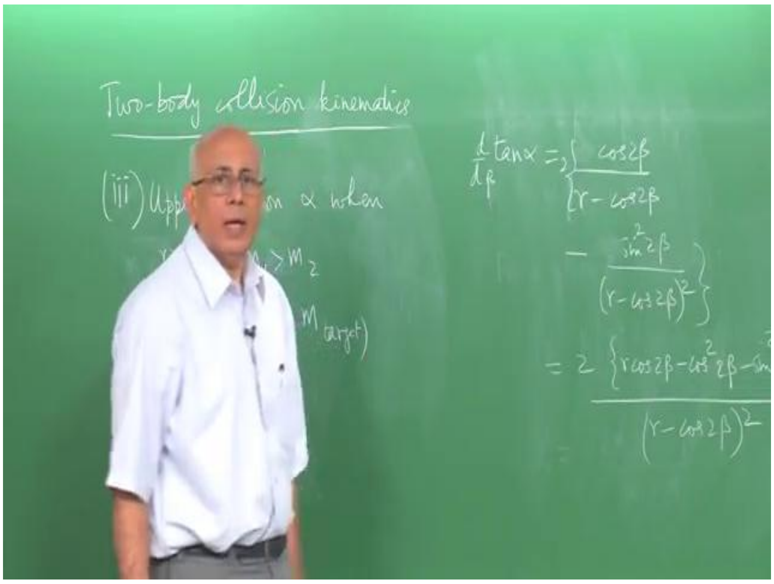

So, from this equation it follows that d over d beta tan alpha equal to the derivative of this quantity, which is 2 cos 2 beta over 1 r minus cos 2 beta minus sin 2 beta over r minus cos 2 beta. The whole squared multiplied by the derivative of minus cos 2 beta, which is 2 sin 2 beta and so let me put a square there, in this fashion.

Take out the 2, so let us take out the 2 and this is equal to twice r cos beta cos 2 beta minus cos square 2 beta, that takes square of this times that, and then minus sin square 2 beta divided by r minus cos beta, the whole squared. So, that is the derivative of tan alpha with respect to the angle beta and in the next 3 months; that is caught to be equal to 0.

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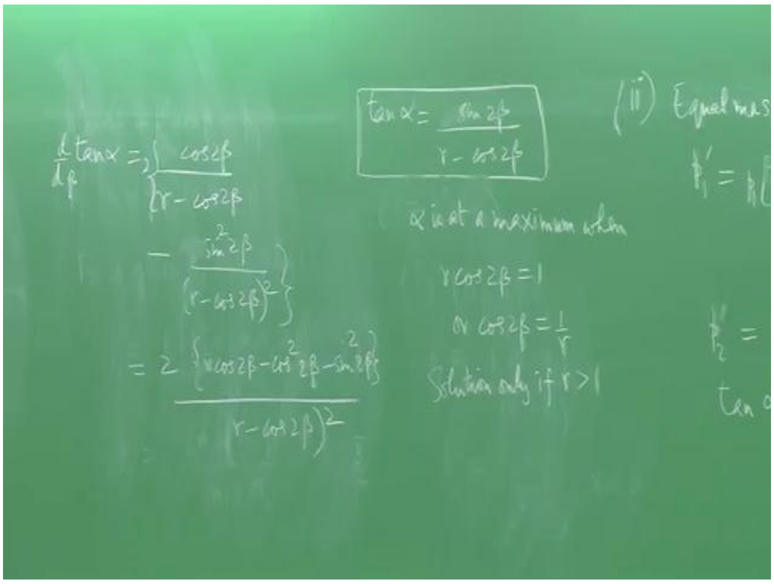

So, what is that tell you? It says alpha is at a maximum, when alpha is maximum, when this derivative vanishes and the numerators r cos 2 by the minus 1, when r cos 2 beta equal to 1 or cos 2 beta equal to 1 over r. But, cos of anything has to be less than 1 or less than equal to 1 in magnitude. So, there is a solution to this, solution a physical solution only if r is greater than 1; that is precisely the case we are looking at.

So, when the target is lighter than the projectile, the conclusion is alpha cannot go beyond a maximum value; that is the largest it can be. And what is that maximum value? It is given by cos 2 beta 1 over r, so let us write this maximum value down.

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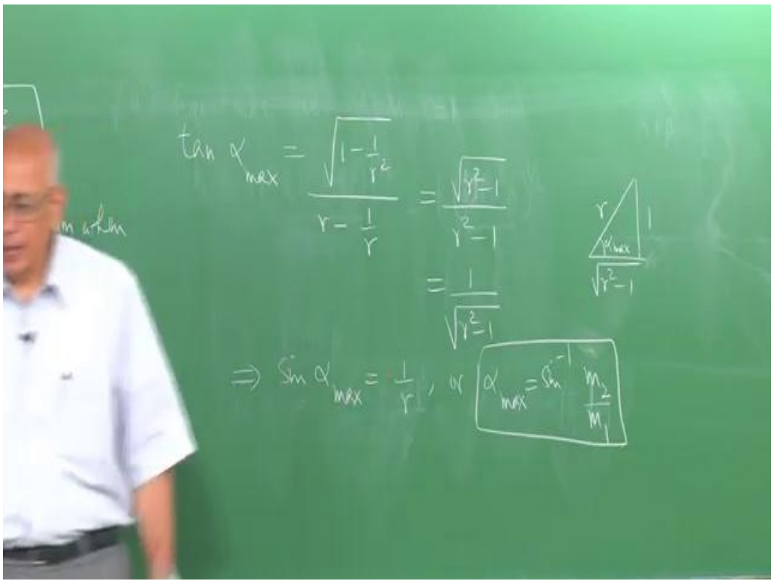

It says the tangent of alpha and let me write this as alpha max here must be equal to sin 2 beta, but cos 2 beta is 1 over r. So, the sin is square root of 1 minus 1 over r squared 1 minus cos squared divided by r minus 1 over r, which is equal to square root of 1 minus r square minus 1 divided by r square minus 1 equal to 1 over square root of r squared minus 1. So, it says and remember that r is bigger than 1. So, this is a real number, it says, this is the tangent of this angle, the largest value.

Actually is more expressively done by remembering of the cosine and the sin become. So, we had in this case of a alpha max here and this is 1, then this is 1 over r square minus 1; that is the tangent become this, and therefore by the Pythagoras theorem in this is equal to r. So, this square of that is equal to r square and this will implies the sin alpha max r from x equal to 1 divided by r or alpha max equal to sin inverse 1 over 10 m 2 m inverse.

So, we have this constraint, we have this relationship this tells you the upper bound on the angle through which the projectile can be scattered after the collision. This entirely on conservation principles and the kinematics of the collision independent of the dynamics, it says alpha cannot be bigger than tan inverse of this number and m 2 over m 1, which is the number less than because m 1 is bigger than m 2; that is when you get the solution.

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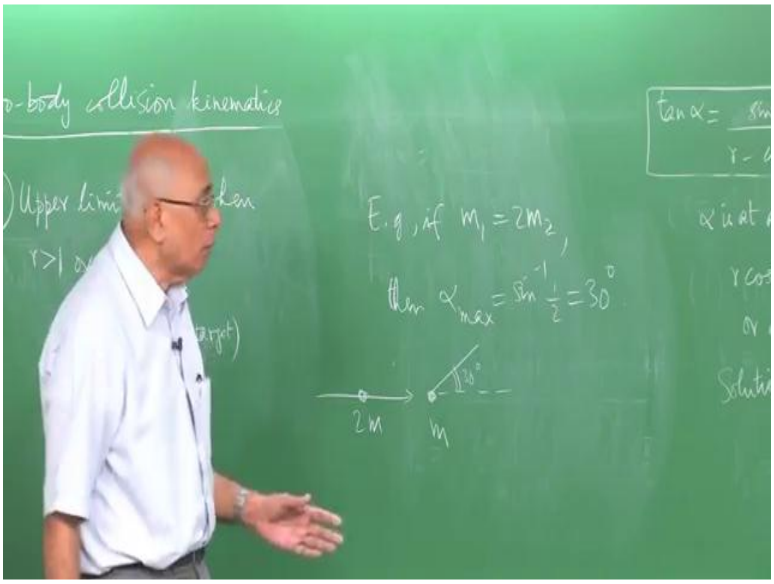

For instance, it says as an example, if m 1 is equal to twice and m 2, then alpha max equal to sin inverse of 1 over the ratio and the ratio as you can see is a half and m 2 over m 1 in this case is equal to half. So, this sin inverse half equal to 30 degrees. So, it says, that if you have a particle of mass m here and you hitting with the particle of mass 2 m. Then, under these circumstances, if the collision is elastic, no matter what that can dynamics of the collision is, this particle cannot be catered any where greater than 30 degrees, the largest it can be scattered by this 30 degrees or less.

So, that is the kinematic restriction on the possible angle scattering independent of the dynamics, follows entirely from the law of conservation of principles of momentum and inertia. So, you see you can get some fairly powerful results based on simple algebra and trigonometry using the conservation principle. So, that was the lesson, than that I want to convey that these conservations laws a very powerful in giving these limitations possible bounds on various physical quantities.

What about the energy transfer; that is another quantity, we looked at in one dimensional case, we need to look at it in the higher dimensional case as well.

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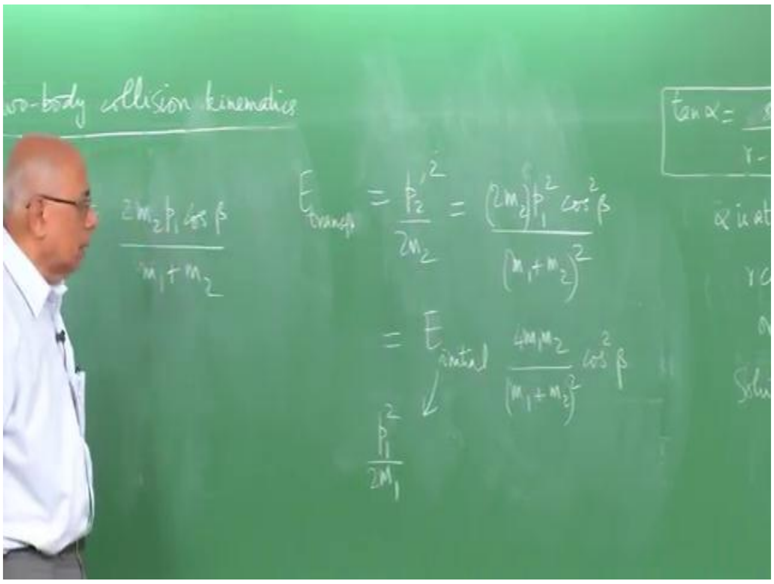

Recall that p 2 prime was equal to 2 m 2 p 1 cos beta over m 1 plus m 2. In the case, when beta was 0, it was m beta p 2, p 1 over m 1 plus m 2, we already saw that in the one dimensional case. So, the energy transferred in the E transfers, transferred is equal to p 2 prime squared over m over 2; that is the kinetic of the target, which was initially at rest. So, this must be the last in kinetic energy of that target part of that projectile particle. So, p 1 squared minus p 1 prime squared prime over 2 m 1 is equal to this quantity here.

And this is equal to as we see now 2 m 2, p 1, so m 2 squared p 1 squared cos squared m divided by m 1 plus m 2 whole square, which can also be written as equal to I multiplied divided by 2 m 1, there is 2 m 2. So, this goes away, because I divided by 2 m 2 here. So, now, if I divided by m 2, so this is E initial which is of course, p 1 squared over 2 m 1 multiplied by 4 m 1m m 2 over m 1 plus m 2 the whole square, multiplied by cos square.

So, that is the exact expression for the energy transfer to the target from the projectile, it is energy given out to the target. And now this relation this generalization of what we had in the one dimensional case, where this fact that cos beta was missing, because beta was 0 in the one dimensional case.

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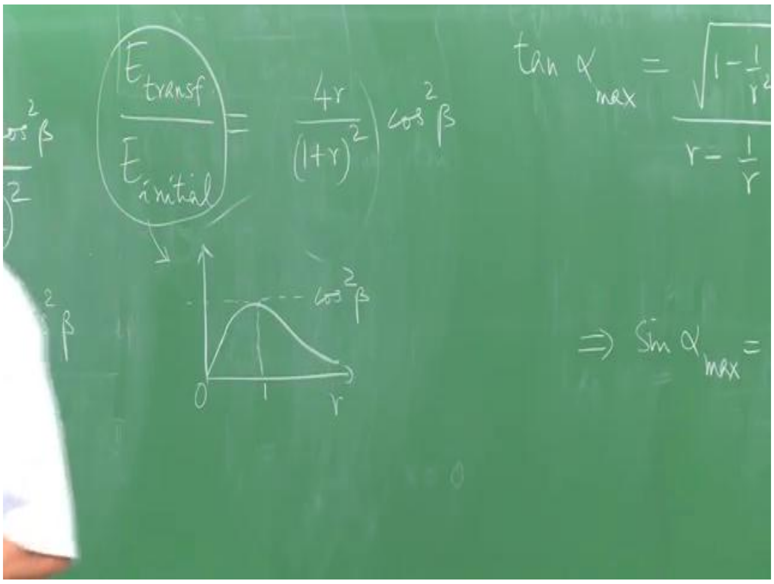

So, we immediately see that E transferred over E initial is equal to a certain ratio of masses which of course, can be written as 4 r over m 1 plus r, the whole squared. If I pull out a m 1 on either side and cancel the m 1’s, then this is what you get, multiplied by cos square beta and this quantity here just a purely function of the mass. This quantity here just purely the function of the mass ration, we saw what it did in the one dimensional case.

So, repeat that here r here is the factor at starts of linearly with the slow 4 r and then when r becomes larger compare to 1 it becomes 4 divide by r. So, drops of applied 1 over r with the maximum at the point r equal to 1 and if you plot the full thing for cos 4 beta included, then this maximum here, the value of this things. So, now, we are plotting the full quantity and plotting this quantity now here and this maximum value, this cos squared.

So, it depends what this scattering angle is and it change with the the mass ratio with the extra factor shown in here in the case in the general case as compare to the one dimensional case. So, we have a fairly complete picture of a scattering, elastic scattering under the assumptions we made, namely that the collisions are elastic, no energies lost; that it is two body scattering; that the dynamics happens in a short interval of time. We do not care with the detail is, but we use the fact that action was equal to the reaction.

In other words, we use the fact that the rate of change of momentum of the, we simply use the fact that the momentum change of the first particle is related in the simple way to the momentum change of the second particle, which is in the minus sin in this case. We select to the conservation of momentum which happens, because we are independent the whole process is independent of the absolute location in space are these objects. This homogeneity in space, and then all these conclusions follows, including formulas for this scattering angle and so on.

Now, in the actual scattering theory, there is another frame of reference as suppose the laboratory frame, which is an extremely useful one and I will just mentioned it here all the we want do that very much at this stage.

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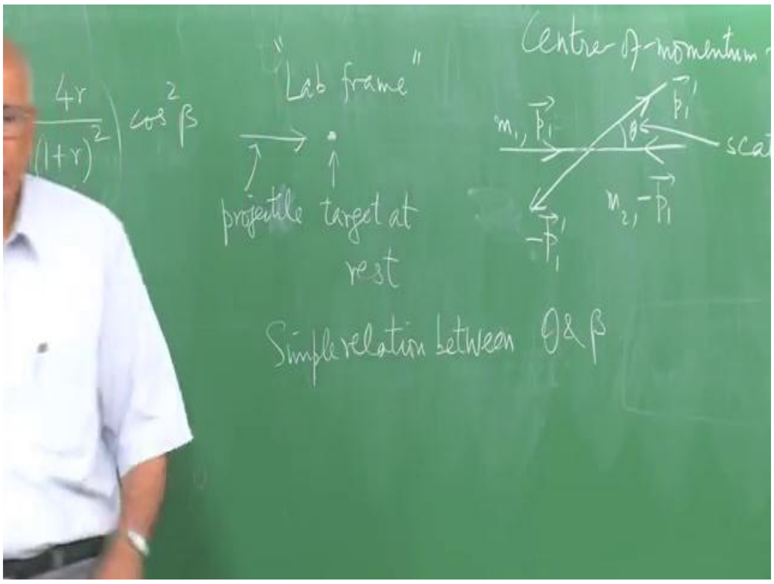

But, we might talk about it later on and that has to do with not a frame of reference in which you have one particle at rest and the other particle comes in. This is the lab frame, the laboratory frame, where that target is initially at rest; target at rest and in comes a projectile. There is another frame of reference, where what happens is, you arrange matters in such a way that by translation of this by shifting by going to a frame, which is moving with respect to these particles.

In such a way that particle 1 comes in this way and particle 2 comes in the other way and it is call this center of momentum frame, then that is frame the two particles are equivalent opposite momentum. So, this center of momentum frame, this scattering happens with the vector p 1 here and a vector minus p 1 coming in from n the other direction and this is mass m 1 and this is mass m 2. I have use the same symbol p 1, but actually it is in different frames, so you can call it something else, but does not matter, but these are the incident momentum, then equivalent opposite.

So, the net momentum is 0, then by the conservation of momentum which is still valid even in this frame particle 1, may perhaps scatter and go in this fashion and particle 2 may scatter of the collision and go off again opposite direction as to do. So, again opposite direction by the conservation of the momentum, so this is p 1 and this is p 2 prime, this is m 1, p 1 coming in, this is m 2 minus p 1 coming in and this one here is minus p 1. So, we could drop this subscript 1 and simply call it p minus p and a p prime and minus p prime.

And the problem now is to find out, what this angle is this is called the scattering angle in the center of the momentum frame. Earlier, we have two scattering angels, alpha and beta, but there were related to each other by the trigonometric formula, we wrote down. On the other hand, here is only one scattering angle, because this one scatters by pi minus theta, while this one direction is changing by theta here, relatives to initial direction.

This once momentum as change by the same angle relative to initial direction or pi minus theta with respective to p 1 here. So, the target now would be to find out, what p prime is in magnitude and what this scattering angle theta is. Now, I mentioned without proof here that, this is the simple relation between angles scattering angles theta and the angle alpha and beta that we talked about here. The reason the formula is very simple in terms of beta and not in terms of alpha is because there is a simple relation, between theta m 1 which takes a little bit of derivation, I am going to skip that.

But, because of this simple relation between these two angles one often use this beta in a laboratory frame preference to alpha, although that completely equivalent you could use either angle easier to write thing is in terms of beta. Because, of this simple relation between beta and beta and the fact is that the whole scattering process the kinematics of it by algebra becomes much, much simpler, if you work in the center of mass frame, center of momentum frame. And that is the reason why this would choose that to work in to compute quantities.

And then, after the calculation is done you transform back Laboratory frame to find out, what the physical parameters are in the variables are in the laboratory frame here. Now, is particularly important, this frame becomes particularly important, when these particles moving slow fast compare to the speed of the light moving up the fraction of the speed of the light. That you have to replace the Newtonian equation that you have here for momentum and the relation between the momentum velocity with the relativistic equations and there are little more involved in complicated.

But, once again the algebra becomes simpler, if you work in the center of momentum frame and I must mentioned here that even with special relativity, even in the presents of relativity, the conservation principles of energy and momentum continue to whole good, for the same reasons they whole good in non relativistic case. Let me the homogeneity of case and the homogeneity of time, lead respectively to conservation of linear momentum and the energy and those continue to whole goal as also the conservation of angular momentum, which follows from the isotropy momentum of space.

So, these are the some of the elementary applications of these conservation principles, but the rules in a variety of context. And when the occasion arises, we will call this back, we come back to this these principles, and see how all the use in other applications.