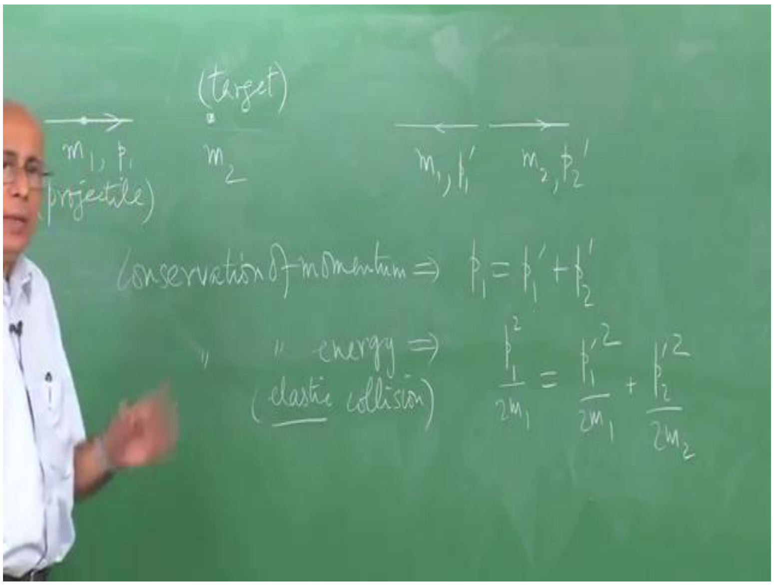
Mechanics, Heat, Oscillations and Waves
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture – 18
Two – Body Scattering

Let us look at the case of two particles or two bodies scattering of each other by some kind of contact interaction in a very short time as an example of the application of a conservation principle of conservation of momentum and conservation of energy. To see whether we can understand this scattering process to extend we can without detail dynamics. And it turns out that we can practically we can go great distance using just the conservation principles.

(Refer Slide Time: 00:51)

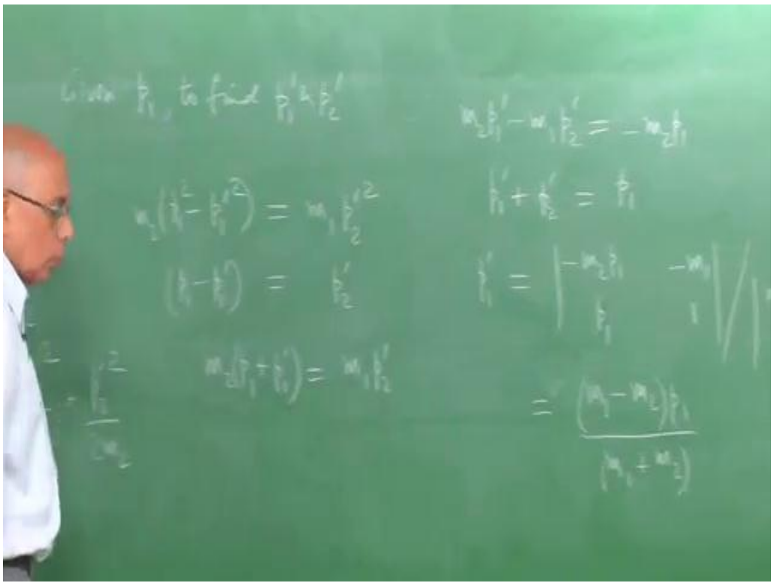


As I started saying in the first case, let us look at the simpler instance, when you have target particle, this is the target of mass m 2 and a projectile coming in of mass m 1. So, this is the projectile and that is the target and the projectile comes in with some momentum p 1 directed in this right to this right. And after the collision, you have a particle 1 moving with some momentum p 1 prime and 2 moving with some momentum p 2 prime and it is possible that p 1 prime is positive as well.

The two in the particle pushes this along and moves with the different momentum in the same line and that would show up if your p 1 prime is positive, but it is intuitively clear that if m 2 is much heavier than p 1 then m 1 this particle is likely to recall after heating it and give a small amount of momentum to the particle m 2. So, let us quantitatively write this down, conservation of momentum implies that the initial momentum should be equal to final momentums.

So, p 1 prime is, p 1 is p 1 prime plus p 2 prime the algebraic sum of the two and I do not need to put vector signs in this case, because it is one dimensions case, everything is moving along given lines, say the x axis. The conservation of energy on the other hand would imply that the total initial kinetic energy should be equal to the total final kinetic energy; there is no loss of energy anywhere in this elastic collision. So, this is because elastic collision and the initial kinetic energy of course, is p 1 squared over 2 m 1 and that should be equal to p 1 prime squared over 2 m 1 plus p 2 prime squared over 2 m 2.

(Refer Slide Time: 03:14)



So, we have two equations for two unknowns, given p 1 and of course the masses of the particles, the parameters m 1 and m 2 are supposed to give to you and you given the value of this variable p 1 and you need to find p 1 prime and p 2 prime. Actually, just using dimension analysis it is immediately clear that the only quantity of dimensions momentum that you have here is p 1. So, it is immediately clear that p 1 prime p 2 prime must be proportional to p 1, they cannot proportional to anything else; they cannot be proportional to the speed of light for example.

Because, this is non relativistic in physics, the only quantity that you given of dimensions momentum is p 1, and therefore p 1 prime and p 2 prime should be proportional to p 1 directly, we can make that statement immediately. But, let us see how this is born out from these two equations, multiplying through by m 1 m 2 or something like that. You discover that p 1 squared minus p 1 prime squared and subtract this, multiply by m 2 times this is equal to m 1 times p 1, p 2 prime squared.

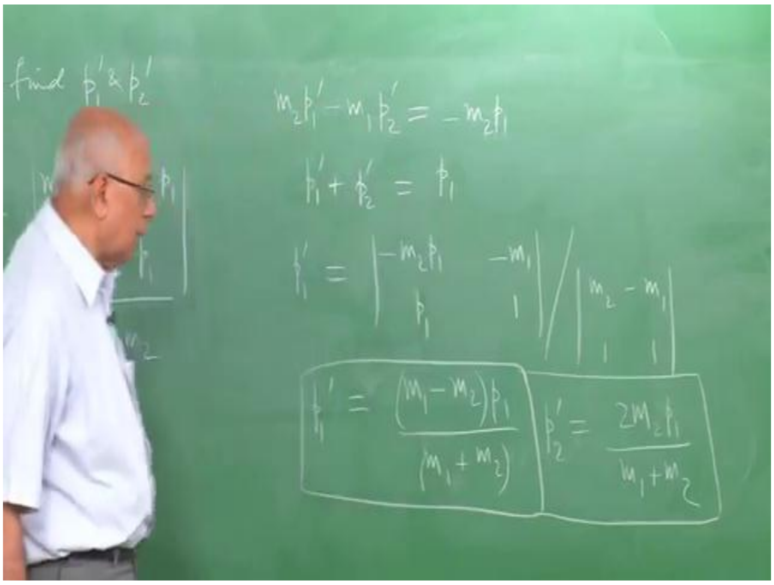
On the other hand, we also know that p 1 minus p 1 prime is equal to p 2 prime. So, dividing this by this and canceling out p 1 minus p 1 prime we have m 2 p 1 plus p 1 prime is equal to m 1 p 2 prime. So, we have two simultaneous equations in two unknowns which made them linear equations now and one of them is m 2 p 1 prime. So, let us work this out minus m 1 p 2 prime is equal to, bringing this to this side, taking this across is minus m 2 p 1 and we also have the original equation which is p 1 prime plus p 2 prime equal to p 1 on the right hand side.

So, there are two simultaneous equations for which the solutions are easy to write down p 1 prime therefore, is equal to minus m 2 p 1 1, p 1 these are the inhomogeneous terms. And then, the coefficients here are minus m 1 and 1, this divided by the determinant of coefficients here which is m 2, minus m 1, 1 and 1. So, that immediately gives us m 1 minus m 2 here times p 1 divided by the determinant is m 1 plus m 2 and that is the solution for p 1 prime, which is intuitively very satisfying.

Because, it says that if the masses are equal, if this mass is equal to this mass then we know already that if this comes along with a certain velocity, all it does after the elastic collision is to give this particle in that velocity and stay at rest. This is how Newton’s Cradle works for instance, for equal masses. So, it immediately tells you that you are in the right track, it also tells you that if m 1 is bigger than m 2, then p 1 prime is positive, which is what you expect.

If this is heavier than that, if it is more massive than that particle and this is at rest, when this comes along and hits it, it is clear that this particle is going to continue to move in the same direction, which means p 1 prime is positive. On the other hand, if m 1 is less than m 2, then p 1 prime is negative, it says if you have a lighter mass here and heavier mass here and you go and hit it and you go and recall backwards here, which is exactly what is happening there. So, this looks like, it immediately tells you this solution is on the right track. What about p 2 prime? Well, that is equally easily written down, we have same equations.

(Refer Slide Time: 07:39)

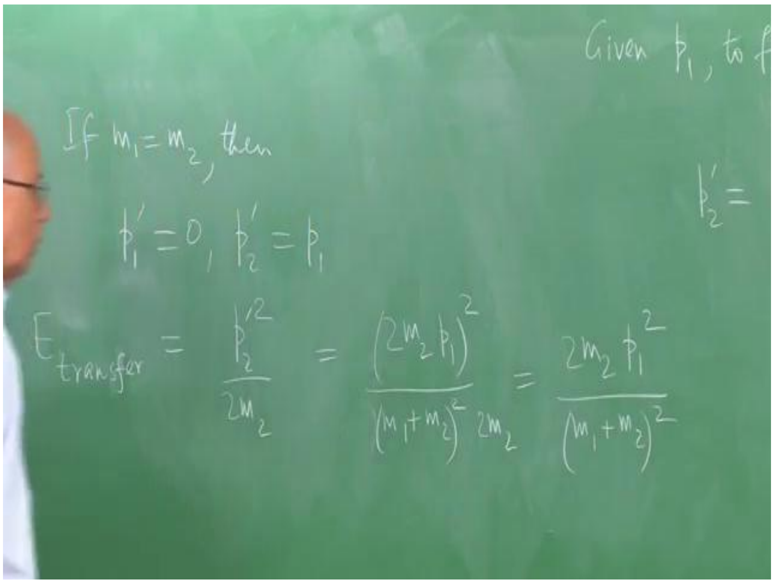


So, that is the solution number 1, p 1 prime is this thing here and what is p 2 prime, p 2 prime this is what the target momentum, the momentum that target acquires after collision. That is again equal to the coefficient here is m 2, it is 1 and here, we have minus m 2, p 1, and then p 1 divided by the same determinant, which is m 1 plus m 2 and this time, it is equal to m 2, p 1 plus m 2, p 1. So, it says p 2 prime and let me write it next to this, p 2 prime is equal to twice m 2, p 1 divided by m 1 plus m 2.

So, that solves the kinematics problem completely tells you regardless of the details of the interaction between these two particles when they are in contacts. If that happens in a very short term time, then after the collision, when these particles move off these three particles, particle 1 moves with this momentum, particle 2 move with this momentum. This is the momentum acquired by the target after the collision.

Again you see that p 1 prime and p 2 prime are proportional to p 1, exactly as we expected on dimensional grounds, because these are dimensionless ratios, the ratios are masses, and then you have a p 1 here and a p 1 there. So, this completes the solution in some sense of this simple one dimension problem. We can actually extract some useful information from it.

(Refer Slide Time: 09:35)



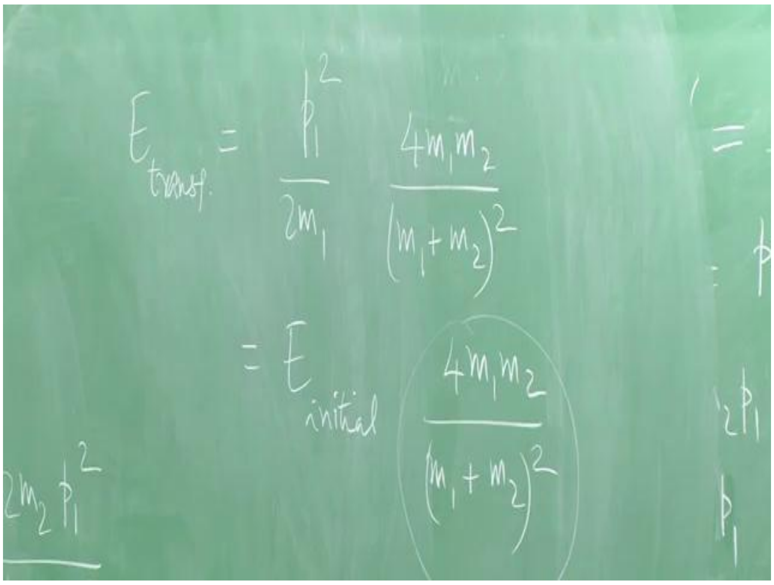
Incidentally, notice again that in this problem, so if it should so happens that if m 1 equal to m 2, then p 1 prime is equal to 0 and p 2 prime is equal to p 1. In other words, this numerator to denominator cancels out and you get just p 1. So, there is an exchange of momentum, the projectile momentum is transferred completely to the target and the target moves along in the projectile remains at rest; that is what you expect for equal masses.

And as I said earlier, if it should so happen that m 1 is less than m 2, then there is a recall backwards of the projectile, while target moves a little bit forward in a forward direction. What interesting is to ask, what is the amount of energy transferred in this process to the target? How efficient is it to transfer energy from the projectile to the target by giving the initial projectile a certain amount of energy?

Well, we know that the energy transferred, energy E transferred is equal to p 2 prime squared over 2 m 2, because the initial kinetic energy of the target was 0. And now, it is acquired the amount of energy p 2 prime squared over 2 m 2. So, that is the amount of energy transferred from the projectiles initial state to it is final state, the difference between the two is precisely this quantity.

But, what is that equal to; that works out to twice m 2 p 1 squared over m 1 plus m 2 the whole squared over 2 m 2. So, this is equal to twice m 2, p 1 squared over m 1 plus m 2 the whole squared. Now, what would like to do is to compare it with what the initial kinetic energy was, which is of course, p 1 squared over 2 m 1.

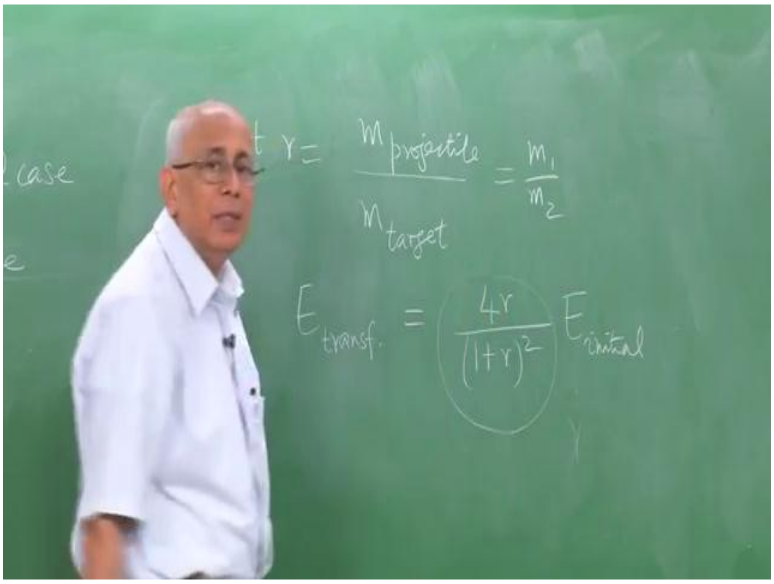
(Refer Slide Time: 11:55)



So, let me write this relation in the following way, let me write this as E transferred is equal to p 1 squared over 2 m 1; that is the initial kinetic energy. So, I divide by 2 m 1 and multiplied through by, and then up there you get times 4 m 1, m 2 divided by m 1 plus m 2, the whole squared. But, this is equal to E initial; that is the initial energy times 4 m 1, m 2 over m 1 plus m 2, the whole squared.

So, it is this ratio here that quantity which tells you how efficiently this energy transfer is taking place and look at what is happening, it is quadratic in the two masses here. So, all that it is relevant parameter here is not the individual masses at all, but rather the ratio of masses. So, let us introduce a quantity, which is the ratio of these two masses.

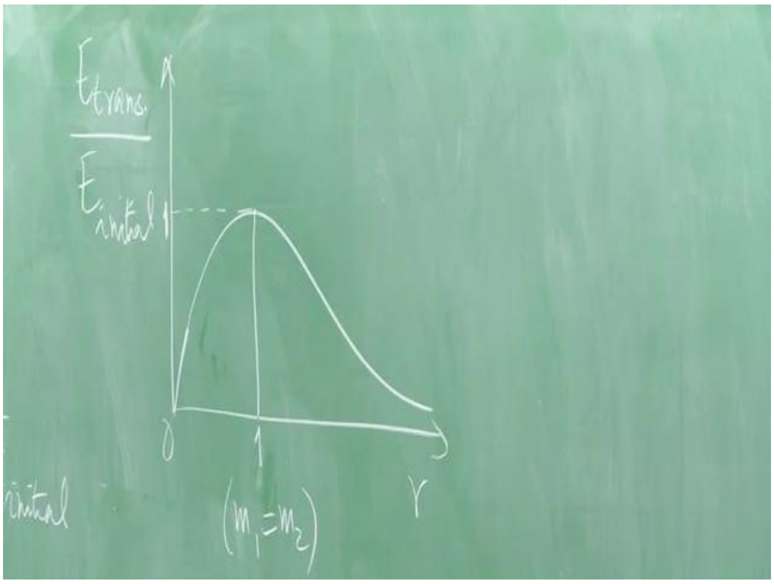
(Refer Slide Time: 13:05)



So, let r equal to the mass of the projectile to the mass of the target, which is of course, m 1 divided by m 2. When our relation here, the fundamental relation says, the energy transferred is equal to the initial energy multiplied by I pull out an m 1 here or an m 2 here and write this as m 1 divided by m 2. So, if I do that this is equal to 4 r divided by 1 plus r, the whole squared times E initial.

So, in terms of this ratio, the energy transferred is this quantity multiplied by the initial energy. So, all we have to do is to examine, what this quantity looks like for various values of r, r is a ratio of masses, and therefore it is non-negative.

(Refer Slide Time: 14:19)



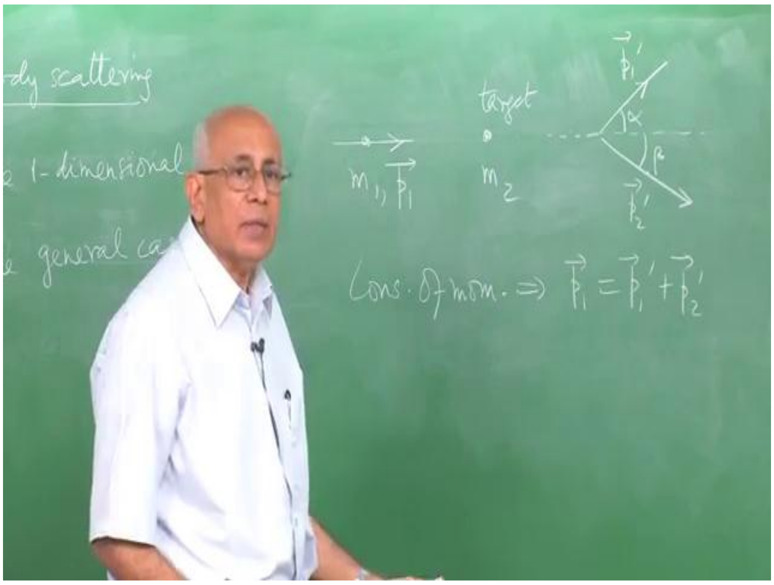
So, let us plot therefore, E transferred divided by E initial, this ratio, let us plot that as a function of r from 0. We need to plot the quantity 4 r over 1 plus r squared. Now, for very small values of r compare to unity, this is negligible and the answer is 4 r. So, it is starts out like a straight line with slope 4 in this fashion, and then for very large values of r compared to unity, the numerator goes like r, but the denominator has in r squared, and therefore the ratio goes like 4 divided by r and drops of like 1 over r.

So, in between it must have a maximum at some point and drops off in this fashion and it is obvious on inspection or differentiation that the maximum occurs, when r is equal to 1, when r is 1, the numerator is 4, r the denominator is 4 and the ratio is 1. So, you have complete transfer of energy at the point 1 and this is 1 and this is the situation, when m 1 equal to m 2. As we saw already, when the masses are equal, when the projectile has same mass as the target, it transfers all it is energy to the target and the target goes off with the full amount of initial energy.

So, therefore, you have efficient transfer of energy as long as m 1 is comparable to m 2. If it is much smaller, the energy transfer is very small, if it is much larger, it is again very small on this side. So, it immediately tells you the best way, the most efficient way, purely using kinematics, conservation of energy and momentum and nothing more than that is, if the projectile and target masses are approximately equal to each other, and then you have very efficient energy transfer.

So, we arrived at this conclusion without too much input, just the input of these conservation laws and let see, what the more general cases. So, let us look at general case of scattering of two particles, but we can again do the same thing, we can again say that the target is at rest.

(Refer Slide Time: 16:35)

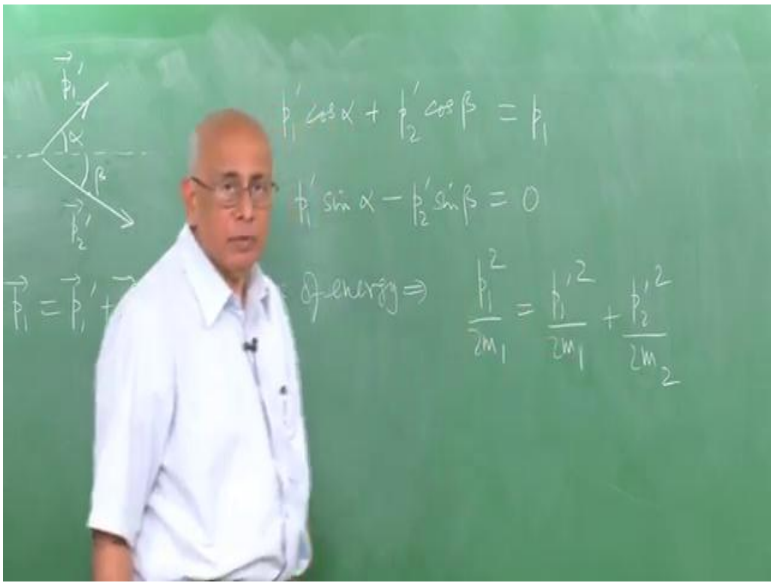


So, that is the target mass m 2 and along comes m 1, mass m 1 with initial momentum p 1 for vector now. What happens after scattering? Well, here comes in initial vector and one particle goes off in one direction and the other goes off in some other direction all together. So, if after collision with respect to the initial direction you have the projectile going off like this with momentum p 1 prime and the other particle going off like this, the target going off like this with momentum p 2 prime, p 1 prime and p 2 prime form a plane.

And conservation of momentum now tells you, implies that p 1 is equal to p 1 prime plus p 2 prime as vectors as three dimensional or any dimensional vector, vectors in dimension greater than 1. But, these two form a plane and this relation tells you, it is a linear combination of these two vectors, and therefore is in the same plane and without loss of generality I take that to be the plane of the black board. So, this is really a three dimensional collision problem if you like, but this is what it reduces to.

And then, with respect to the initial direction of p 1, let us suppose this move up some angle alpha and that moves at some angle beta measured from the x axis downwards in this case. So, that is the general geometry p 1 plus p 1 primes to p 2 prime and we have now got to impose this conservation of momentum in terms of a components along the initial direction and perpendicular to it and we also have the conservation of energy.

(Refer Slide Time: 18:51)

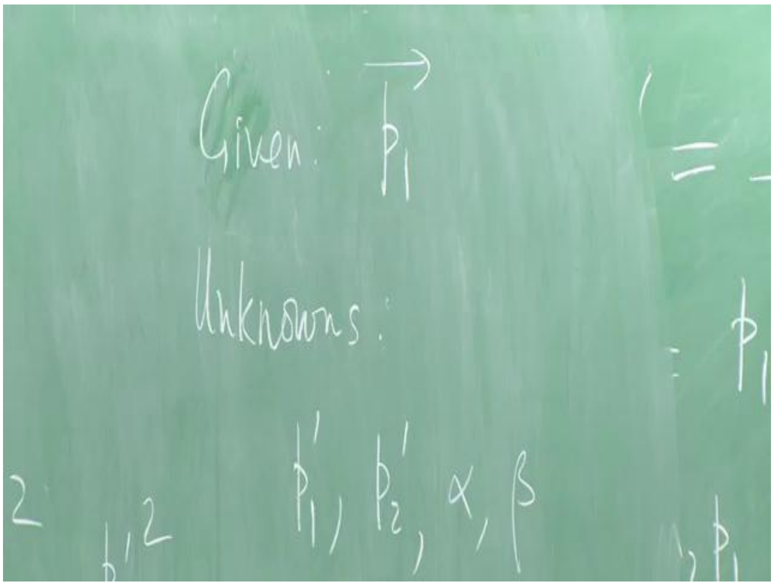


So, let us write these equations down and see, where it gets us and what the unknowns are. So, resolving this, these vectors along the direction, incident direction and using symbols without arrows for the magnitudes, it follows that p 1 prime cos alpha plus p 2 prime cos beta. In the projection of this vector in this direction is like that and it is p 2 prime cos theta and this one is p 1 prime cos alpha here and that must be equal to p 1, the initial momentum p 1, magnitude of the initial momentum.

Similarly, the vertical projections, this vector has a component upwards, this vector has a component downwards and taking that minus sign into account p 1 prime sin alpha that is the upward component minus p 2 prime sin beta is equal to 0. Because, the initial vector has no vertical component, and therefore I put 0 on the right hand side, but this one points downwards, the projection of p 2, therefore I put a minus sign here. So, these two imply conservation of momentum.

The conservation of energy on the other hand is a quadratic relation, conservation of energy implies p 1 squared over 2 m 1 must be equal to p 1 prime squared over 2 m 2 plus p 2 prime squared over 2 m 1 plus 2 m 2, since the masses are unequal. Whereas, I said I use these symbols without any arrow for the magnitudes of the vectors, so the non-negative quantities. Now, let us takes stock that is it, we do not have any further information, we have used up conservation of momentum, we used up conservation of energy and that is all the kinematic information, we have from conservation principles.

(Refer Slide Time: 20:47)



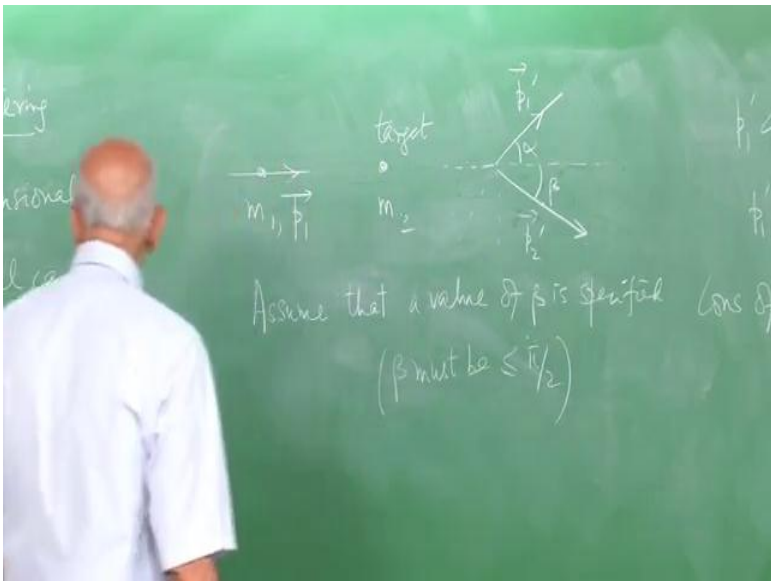
What can we now say beyond this; well we have to take stock of how many unknowns there are, there are four unknowns. So, the unknowns, the known are the given is p 1, in fact, the vector p 1 is given to you and we have chosen the vector p 1 to be along this direction here. So, given that, we need to find the unknowns at p 1 prime the magnitude of particle 1's momentum, p 2 prime the magnitude of particle 2 momentum, alpha and beta, but we have only three equations.

So, we have a situation here, which is actually at the root of, this is the beginning of classical chaos actually, but it turns out that in this problem, these three do not suffice to solve for these four unknowns uniquely. Without further dynamic information on how the interaction took place, what exactly took place in the little timed delta t, we cannot say, what these quantities are going to be uniquely.

All you can says given any one of these quantities, we can then deduce for the other quantity r, and then the task of the dynamics is to put in for the information in about the interaction. And then predict of various values of any one of these quantity, such as say beta, the angle beta of the given beta, what the other quantity is turnout to be. And then, actually when you do this problem in terms of quantum mechanics, you would assign probability is to different values of alpha and beta here in call them scattering cross section, differential cross sections.

But, here we are not concerned with that, we are concerned with geometric problem that the algebra problem of actually finding out, what these unknowns are given any one of these quantity. And then, we will look through all the possibilities for one of these any one of these quantities and see, what results we can get.

(Refer Slide Time: 22:42)

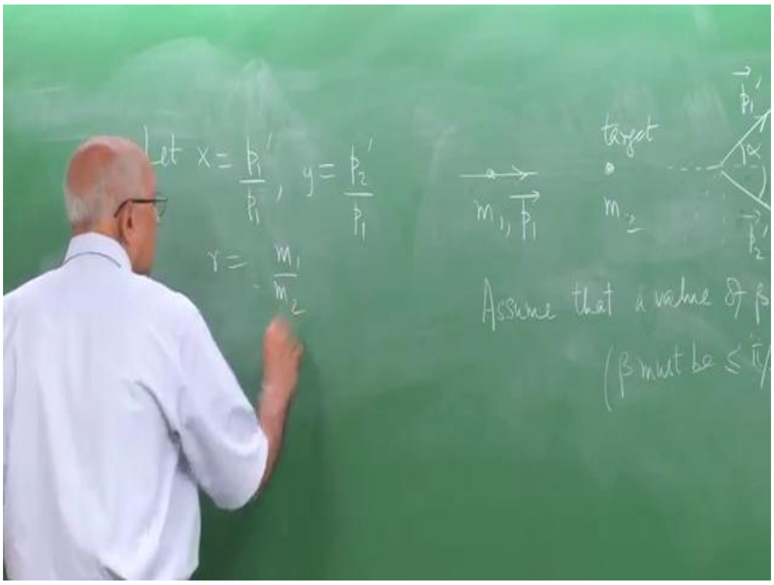


So, what I am going to do is to assume that we know, so assume that, a value of beta is specified. So, we will compute the other three quantities in terms of a given value of beta, which could run from 0 up to wherever. Now, it should be obvious to you by looking at this diagram immediately that once you have a projectile coming in this direction, it is not possible for this mass m 2 to go backwards in this direction.

At best, it should been in forwarded half, and therefore it is clear from this already that beta must be less than equal to pi over 2, we cannot have it move out here for instance, but you could have it up to this point. So, that we already know, we have some intuition for this business and you already know that, there must be a limitation on the data, such that it does not go beyond this. But, given a beta what are all the other quantities is what we going to ask, and then we will see, what limitation, we can put on this angles here.

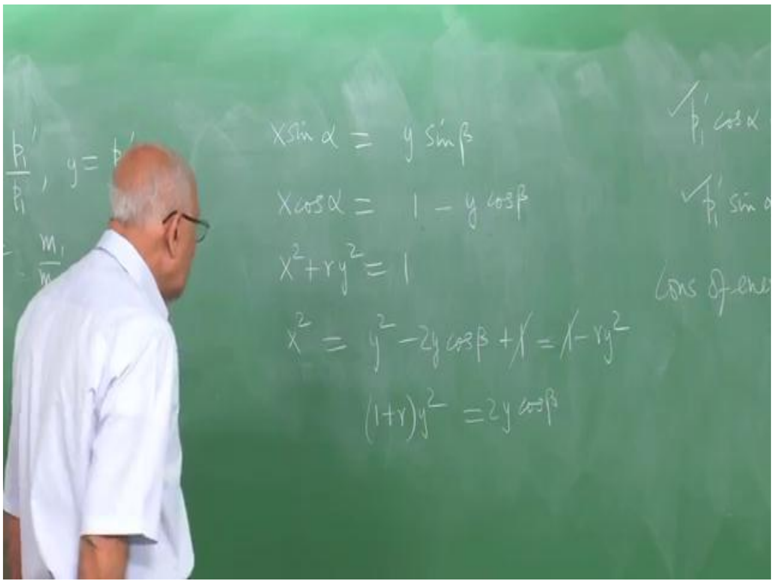
So, the task is clear given beta, so this is no longer in the list of unknowns, we need to find p 1 prime p 2 prime and alpha given these three equations with beta and p 1 as given treated as given. So, this is the general coordination problem for which we are going to try to find the solution. But, first little bit of simplification of the algebra, erase this, follows at once, because again you see the only quantity of dimension momentum is p 1 in among the given prime in variables. Therefore, it is clear immediately from this, that p 1 prime and p 2 prime must be proportional to p 1.

(Refer Slide Time: 24:45)



So, let us make life simpler that suppose that, let x be equal to p 1 prime over p 1 and let us put y is equal to p 2 prime over p 1 once again. So, it is only the ratio that we are interested in and similarly, this is the only relation, where the masses appear and the masses appear by multiplying through as you can see the ratio of m 1 over m 2 appears here. So, it is immediately clear that I could also write r as before equal to m 1 over m 2 as before, and then what do these equations look like, well this simplify quite a bit.

(Refer Slide Time: 25:34)



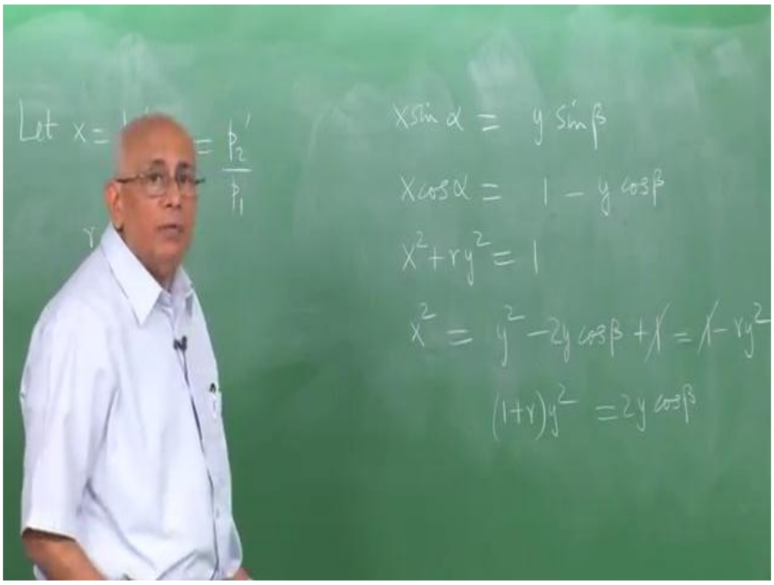
So, here we are, we have this first equation, this equation here, if I divide though by p 1, it says x sin alpha equal to y sin beta; that is the first this equation is gone. Then, this equation says x cos alpha equal to I am dividing through by p 1, so there is an one the right hand side, and then this becomes minus y cos beta; that takes care of this equation. And what does this equation give you, it says p 1 squared, I divide by p 1 squared all the way through.

So, let us multiply the this let us multiply through by m 1 and remove the 2, so this goes away, this goes away and you have plus r out here and I divide through by p 1 squared. So, it says x squared plus r times y squared equal to 1, those are the three equations and that takes care of all three equation that we put in. Now, for the solutions for the simplest way do this the many ways of doing this, but the simplest way of do this is to notices that if I square these two fellows, I immediately get x squared and on the right hand side, I get an equation, which you can use.

So, let us square these and the says x squared is equal to square of this plus this, square of this square this give you y squared straight away minus 2 y cos beta plus 1, which is the term here, when I square it, but for a x squared, I write 1 minus r y squared. So, this at once gives you 1 plus r y squared. So, this one cancels out 1 plus r y squared is 2 y cos beta. So, y is not 0, y is magnitude of the second momentum divided by the magnitude p

1, and therefore the only solution is y is 2 cos beta over 1 plus r. So, we have a first solution.

(Refer Slide Time: 27:56)



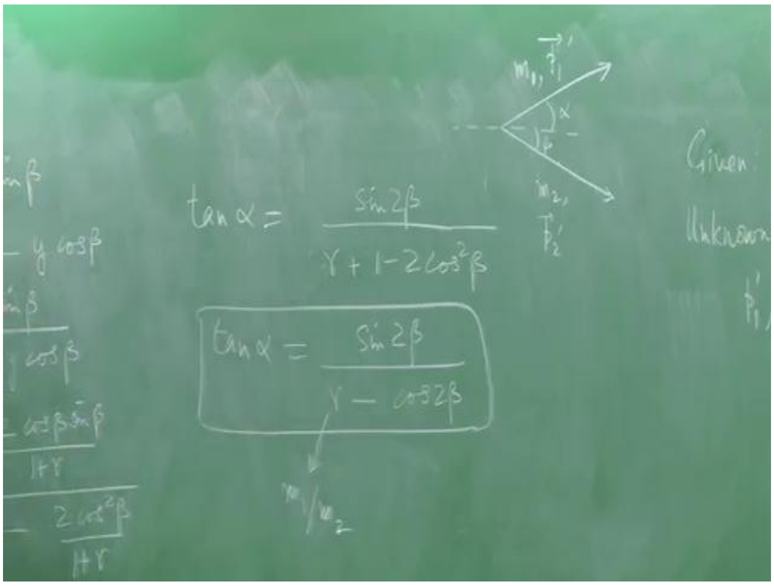
By the way that solution tells us the beta cannot be greater than pi over 2, because in cos beta in negative, but y is a ratio of the magnitude of the two vectors and cannot be negative. So, this automatically says, the beta cannot the scatter in the backward direction. So, it is nice that it is coming out automatically. So, once you have that, we can then go on to write down.

What x is, because we have a solution for y and we use this relations, it says x squared equal to 1 minus r times y squared, which is equal to 1 minus r times y squared is 4 cos squared beta divided by 1 plus r the whole squared, which is equal to 1 plus r the whole squared multiplied by 1 plus 2 r minus 4 r cos squared beta plus r squared. And therefore, x is equal to the square root of this whole thing r 1 plus r this fashion, but we can simplify that a little bit, I can do that by writing this as 1 plus r squared, and then a 2 r.

So, it is 1 plus 2 r 1 minus 2 cos squared cos squared beta that is minus cos 2 beta, so this is equal to minus 2 r cos 2 beta plus r squared to the power half divided by 1 plus r, let us see that is right. So, I takeout the 2 r and I write 1 minus 2 cos squared beta is minus 2 cos beta cos 2 beta. So, that is it; that is the solution for x and remember that, this is x, which is equal to p 1 prime divided by p 1.

On the other hand, this was solution for y, we already wrote down and this was equal to p 2 prime over p 2 in this fashion, we all set to find alpha 2. So, let us do that from these two equations, the tangent of alpha is immediately found.

(Refer Slide Time: 30:28)



So, these two imply the tangent alpha equal to y sin beta over 1 minus y cosine beta and this is equal to again we use this solution for y as 2 cos beta. So, you have 2 cos beta sin beta divide by 1 plus r, but that is going to cancel on both sides and down here, you have 1 plus r 1 minus 2 cos beta 2 1 plus r, 1 plus r minus 2 cos squared beta, but 1 minus 2 cos squared beta, 2 cos squared beta minus 1 is cos 2 beta.

So, fact it 2 cos beta sin beta over 1 plus r divided by 1 minus 2 cos squared beta over 1 plus r. So, therefore, tan alpha is equal to 2 cos beta sin beta is sin 2 beta divided by 1 plus r minus 2 cos squared beta, let us r plus 1 minus 2 cos squared beat, this is equal to sin 2 beta over r minus cos 2 beta. So, that is the solution. Therefore, this r remember is m 1 divided by m 2. So, there is simple relation which connects this scattering game to draw the geometry this was the incident direction that is the direction of the mass m 1, the projectile and that is the direction of the mass m 2, the original target.

This angle is beta; that angle is alpha and we have a solution, which tells you what each of this momentum r, p 1 prime and p 2 prime, we also have solution for angle beta. So, what we need to do is now analyze the solution and see, what all this special cases are, we will do that next.