Mechanics, Heat, Oscillations and Waves
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Lecture – 17
Conservation of Angular Momentum

So, let us start we will left of the last time, we were looking at the Conservation of Angular Momentum.

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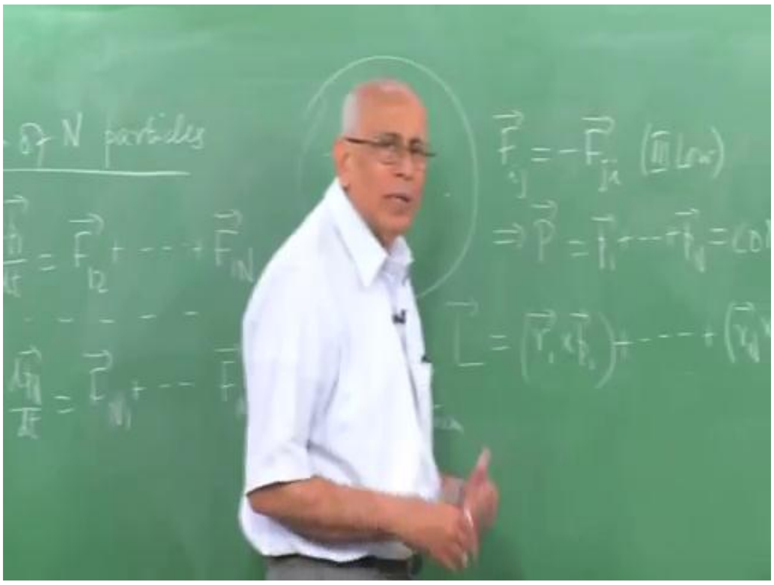

And now I have in mind, a system which consists of N bodies. So, system of N, let say particles or masses and this system moves under the effect of forces, which each body causes on the other, each mass causes on the others. So, we have a system of Newton’s equations, which is d p 1 over d t is F 12 plus dot, dot, dot plus F 1 N and so on, all the way down to d p N over d t is F N 1 plus F N 2 and so on up to F N N minus 1.

So, here is the system of these N particles, which are in interaction with each other due to some unspecified forces, which are given by F 1 to write up to F N comma N minus 1. Now, we already saw that F i j and thus equal to minus F j i, this was a third Law. As a consequence, we saw that the sum of momenta of p 1 to p N, this is a constant of the motion. This implies the p equal to p 1 up to p N equal to constant of the motion.

In other words, this combination of momenta has exactly the same value at all instance of time as the particles move and change their individual positions as well as momenta, but this sum remains a constant of the motion. What else can we say? Well, we already saw last time that if I took r cross this quantity, then if I define the angular momentum, well the total angular momentum to be equal to r 1 cross p 1 plus etcetera plus r N cross p sub N. This is the total angular momentum about the origin by definition.

And I ask how does that change as a function of time? Well, I would expect given no other information that this two, if it is a close system, not in interaction, no forces from anywhere else from outside acting on it, I would expect that this also will be a constant of the motion, because there is no net talk on the system. But, how do we prove this? Well, we need to compute it is time derivative.

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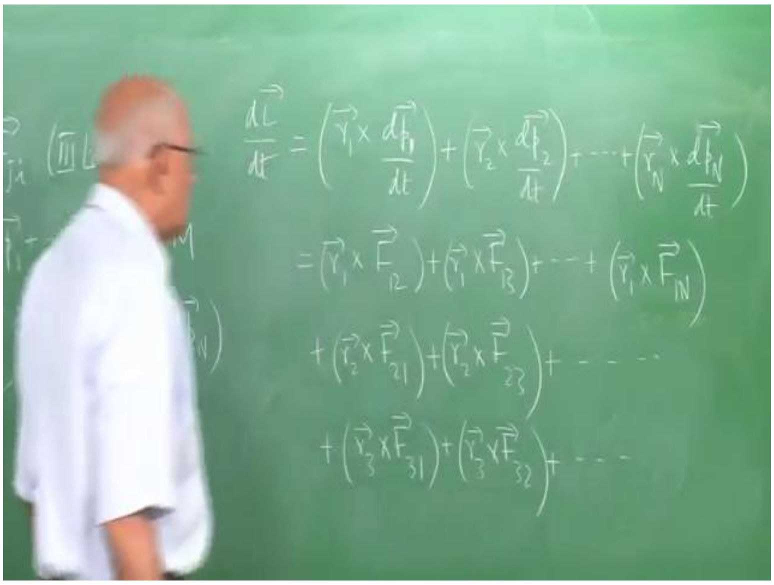

So, d L over d t equal to and now, d r 1 over d t cross p 1, which is 0, because d r 1 by d t is the velocity of particle 1 and p 1 is the mass times the velocity of particle 1. So, that part is 0, the second part is r 1 cross d p 1 over d t. So, this is equal to r 1 cross d p 1 over d t. Similarly, in the second term, the d r 2 over d t cross p 2 cancels and you are left r 2 cross d p 2 over d t plus dot, dot, dot plus r N cross d p N over d t.

In other words, the net change of angular momentum is the vector sum of the torques on particle 1, particle 2, particle N of there, but d p 1 over d t is given by this quantity here. So, let us put that in and we have this is equal to r 1 cross F 12 plus r 1 cross F 13 plus etcetera plus r 1 cross F 1 N. That takes care of just this first term plus and we have to write this for all the terms plus r 2 cross F 21 plus r 2 cross F 23 plus etcetera and so on.

We can write down all these terms, we do not need to write all of them down, because we can see the pattern immediately. Let me write just one more term and that is the next line, which is plus r 3 cross F 31 plus r 3 cross F 32 plus etcetera, and so on. But, we know from Newton's third Law that F 21 is minus F 12. So, if I put that in, this becomes r 1 minus r 2 cross F 12.

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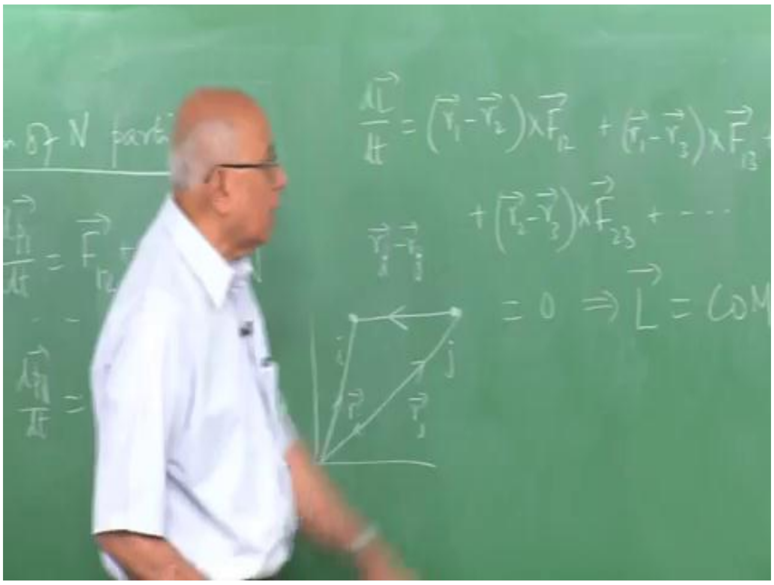

So, I can recombine this and write this whole thing as d L over d t equal to r 1 minus r 2 cross F 12. So, that takes care of this term as well as this term. Using Newton's third Law, which says F 21 is minus F 12 vectorial, I put the minus sign on this r 2 and take out of a F 12 and here it is. Similarly, this along with this, this term together with this term becomes plus r 1 minus r 3, cross F 13.

Again, using the fact that F 31 is minus F 13, similarly, each time you have one term here, you have another term with the opposite signs, so you also have terms like plus etcetera, and then you have plus r 2 minus r 3 cross F 23 and so on. But, you see Newton's third Law is applicable on the conditions is that, the force between the two particles acts along the line joining them.

So, if you have particle i here and particle j here, then this line joining the two of them, the force acts in one of the directions, F i j may act in this fashion, F j i may act in this fashion for instance or the other way about, it does not matter. But it is along this vector, along this line. On the other hand, if in some coordinate system, this quantity is r i and that vector is r j. Then, you can see that this vector here is essentially r j minus r i or written this way, it is equal to r i minus r j.

So, if you take this vector and add to r j plus r i minus r j, you get r i by the triangular law of addition of vectors. So, now we have a situation, where in every one of these terms, this force is precisely along the line of this vector, and therefore the cross product is identically 0. So, again the input has been Newton's third Law and that is all and that has happen. So, each of these terms is 0, and therefore d L over d t is 0.

So, this says now that is if a net torque on a system is 0, this implies that L is the constant of a motion. It does not say that individual else r 1 cross p 1 or r 2 cross p 2 or the sum of r 1 cross p 1 and r 2 cross p 2. It does not say that these angular momenta are the constants of a motion, just as in the case of the momenta. It says the total angular momentum is a constant of the motion as a consequence of the fact that Newton's third law cancels these terms out in pairs here.

So, as long as the force between two bodies is along the line joining them, you have guaranteed by Newton's third Law that the total angular momentum of such a system of particles or bodies is going to be a constant of the motion. Again, we ask where does this come from? We already mention that the law of conservation of linear motion arose because of the homogeneity in space, it does not matter.

The actual location of the origin of the coordinates does not matter. Where does angular momentum conservation come from? And I write this down, we would not prove it here, but I write this down.

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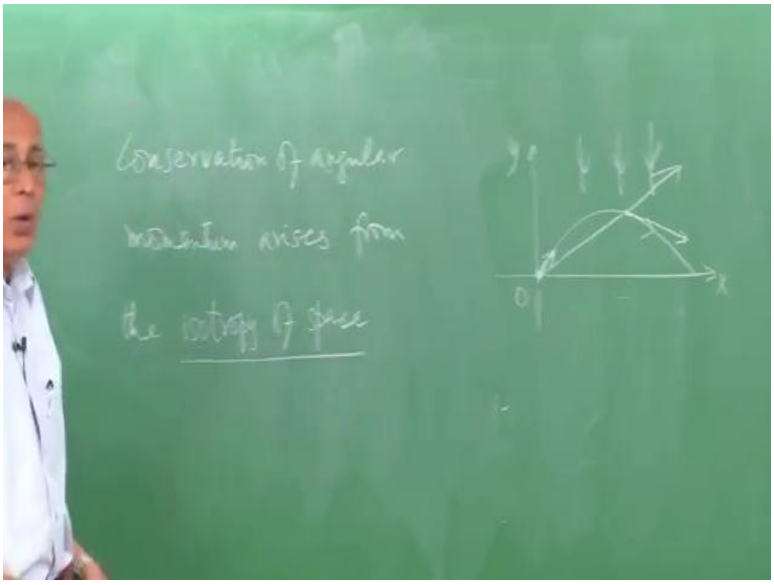

It says, you have the potential for the conservation angular momentum arises from the fact that these properties of empty space are the same in all directions. Earlier, we said space is homogeneous, in other words the location did not matter, the absolute location of the origin did not matter. Now we are going one step further and saying, space is three dimensional and it is isotropic, has the same properties in empty space in the absence of all matter and so on. It has no feels nothing of sort, the space itself is the same in all directions, it is properties are the same in all directions.

So, this arises from the isotropy of space that is the deep reason for the conservation of angular momentum, when these circumstances are right. Once again, this can be broken, this principle can be broken, the conservation can be broken or violated. Once you violate the isotropy of space by having some feel present exactly has in the case of linear momentum, we look at again at the example of the projectile.

So, here is the projectile, I project it in some initial velocity and it takes a parabolic path and falls on here and that is the origin of coordinates. Now, the question is, what about the conservation of momentum of this projectile, the answer is it is certainly not conserved. Because, you can see that at the initial instant of time, this was the momentum and that was a position, it happens that r was equal to 0 in that case.

So, L is equal to r cross p happens to be 0 initially, but when the particle is somewhere here for instance that is the direction of p, this is the direction of r and this direction and r cross p goes into the board and it is not 0. So, clearly r cross p is not conserved in this case, because the isotropy of space is broken, it is broken by the fact that this direction, the vertical direction has a field in it, the horizontal direction does not have a field in it.

So, this property of space being exactly the same in all directions is no longer true, the moment you have a force of this kind, you have broken the isotropy of space, and therefore angular momentum is not conserved. It is only when you have this isotropy of an empty space and your physical conditions do not break this isotropy, but you have the conservation of angular momentum.

In this problem itself, when I says space has exactly the same properties in all directions and if you have a force in it, which also has the same property; that is the same directions and only depends on say the distance from some attracting center. Then, the angular momentum is indeed conserved.

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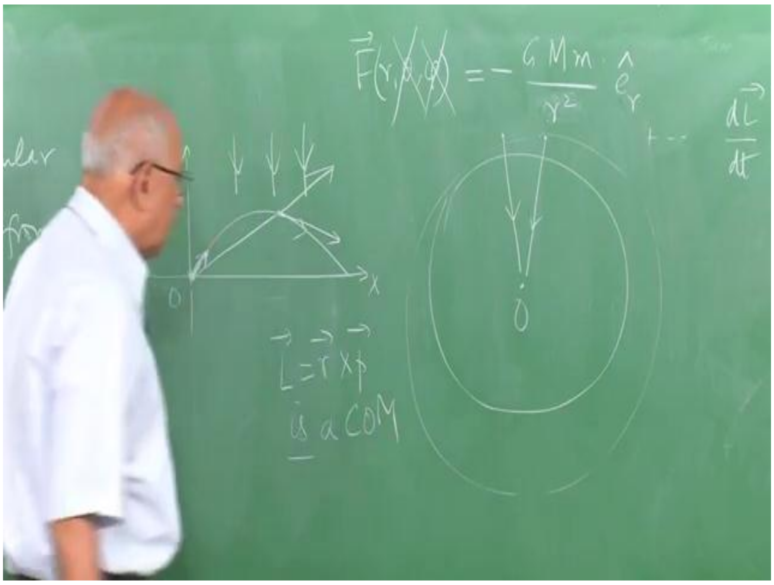

And we that in this case, this is an approximation to the fact that you have the earth, which is supposed to be a sphere and the laws of the force of gravitation at this point acts directly inwards, the force of gravitation at this point acts directly inwards. And in this case, if you take this to be the origin of coordinates, these field lines are all going to focus inverts towards the center of the earth.

And the magnitude of this force does not depend on where you are on the surface, only on the distance you are from the center of the earth and that is a radial distance, which is purely the radial coordinate. In this case, the force of gravity is equal to G times the mass of the earth times the mass of your little body divided by r squared, where r is the distance to the center of the earth, multiplied by the unit radius vector r in the radial directions.

So, let us call it r squared, the distance from wherever you are to the center of the earth with the minus sign to show that, this is radius vector and this force is opposite the radially outward vector, unit vector. There is no angular dependence, there is no dependence on the latitude and longitude in this case to this approximation of a perfectly spherical earth. And therefore, this force here is radially directed inwards and depends only on r and not on the polar angle theta or on the azimuthal angle phi. It does not depend on those quantities.

In spherical polar coordinates, it does not depend on theta, it does not depend on phi, depends only on r. This sort of force directed towards the center of the earth or center of attraction is called a central force. Now, central force does not have any angle dependence at all, and therefore respects the fact that in all directions as far as the angels are concerned, there is no dependence; the force is exactly the same.

Therefore, in such a situation, when this is the force on a body, you would expect the orbital angular momentum of that body about the center of attraction, about the origin of coordinates to be a constant of the motion. And indeed we know, that when a particle goes for example, in an elliptic orbit around the earth, we know that this quantity L is r cross p is a constant of the motion.

It is only when you make this approximation of parallel field lines and there is a flat earth and there is an origin. Here and you look at the angular momentum about this origin, that you say that it is x and y directions are not equivalent to each other, and therefore there is no angular momentum conservation. But, in reality, if you look at the actual central force of the gravity due to the earth, then indeed about the origin, about the center of the earth, there is angular momentum conservation.

It is true that as a particle moves in an elliptic orbit, it is orbital angular momentum about the origin, about the center of attraction which is at one of the force of the ellipse is a constant of the motion, definitely that is true. So, one should understand the origin of this, these conservation principles. They arise ultimately from the properties of space, linear momentum, because of the homogeneity of space and angular momentum, because of the isotropy of space under the right conditions.

Now, you could immediately ask the natural question, where does the conservation of energy come from, in those cases where it is applicable, where does this energy come from, conservation come from. The answer again is buried in a various principles and I will state it in words, but without any effort to prove it here.

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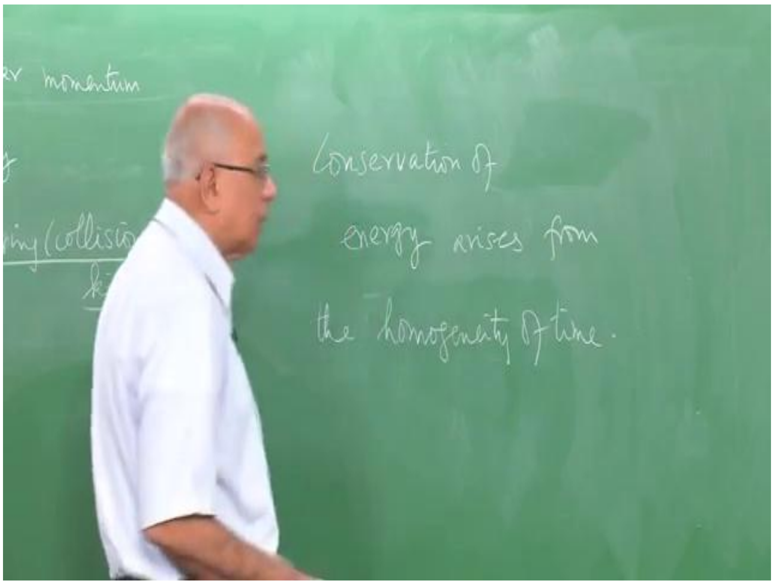

It is a good idea to know the reason why, even if you cannot prove it at a certain level and the conservation of energy, I just erase this, energy arise is from well we exhausted isotropy of space, equivalents of all directions in space and we exhausted the homogeneity of space. There is one more variable in dynamics and that is time. So, this arises from the homogeneity of time, when you have rules of evaluation or rule of change or equations of motion, which do not depend on the absolute origin of time.

But, exactly the same, even if you shifted is 0 of your clock arbitrarily, then you have the potential for the conservation of energy in suitable conditions and the suitable conditions, exactly as in the case of other conservation principles. So, again I want emphasize that these are deep principles and properties threes arise from deep property of space and time themselves. And then, those properties a refine further, then there are suitable modifications to these principles are we apply and a more general conditions and so on.

But, ultimately these conservation principles arise from the fact that the equations of motion do not change, if you make certain transformations on the coordinates, on the variables, space and time variables, provided you have in variance already built in. Provided you have the fact that in the case of conservation of energy for example, if you do not have explicitly time dependence rules of evaluation for this variables, then does in matter even is start the 0 of time, the origin of time.

And in that sense, the homogeneity of time can lead to the conservation of energy. We will see consequences of these conservation principles are, because we need a good application for this. And a very good place to look for an application is a problem that is actually very complicated, but which can be make simple as for as practical conservation are involved. It can make simple by applying conservation principles and I have in mind the standard problem of collusion of two bodies to start with let us look at that is a simplest case.

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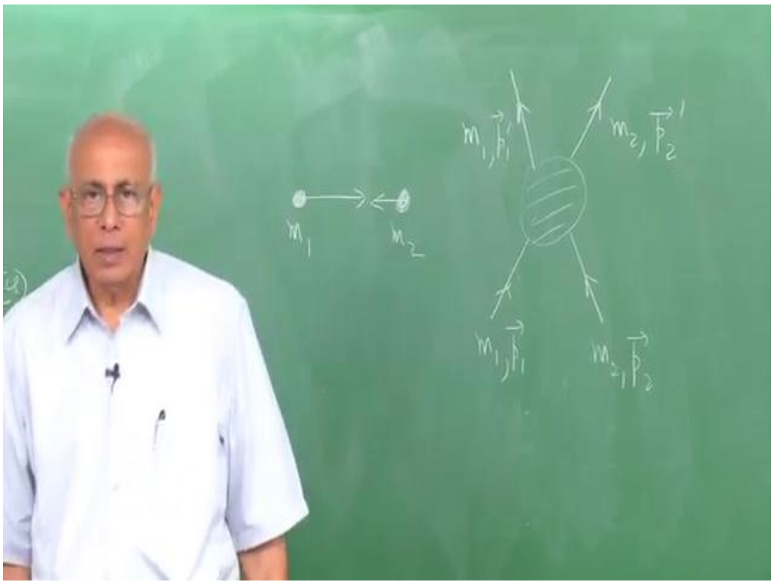

So, you have some hard mass here, some mass m 1 and it is moving along and it collides with the mass m 2. Now, to the collusion these two particles go off. So, we have an mind a situation, where the two particle, which is most convenient to think the mass particles would undergo and elastic collusion in the sense that, there is no loss of energy are anything like that. The total initial kinetic energy is equal to the total final kinetic energy. But, actually happens in the collusion process is very complicated, we do not worry about it, it happens very quickly.

But, what we do is to say that, alright here some region in which the particle collide, one particle goes with some momentum m 1 with some momentum p 1 and m 2 with some momentum p 2. The collide bounds of the each other and out go two more particles out here. So, this is m 2 with the now momentum p 2 prime and this here m 1 with some momentum p 1 prime; that is the standard collusion problem.

And we ask the following question, given p 1 and p 2 and of course, masses m 1 and m 2, can we say what p 1 prime and p 2 prime are going to be, without knowing very much about, what is going on inside this region of interaction. In a very short time a little collisions and then for some reason, the collide and this cater out, due to some complicated interaction and we move on the way and after words there is no force on them.

So, by the first law p 1 is constant till the interaction occurs as is p 2, and then after interaction p 1 prime and p 2 prime remain so for ever, in other words, the particles move in straight lines with the final momenta. So, the question is can we discover what these quantities are given these two quantities base on conservation principles. This is the problem collisions kinematics or two bodies catering, catering is another word for collisions can we discover this at all.

Well, as always we should look at a simple case. So, let us first look at the case, where this particle collides separate in the same line. So, this is a very special case and it happens on the certain circumstances.

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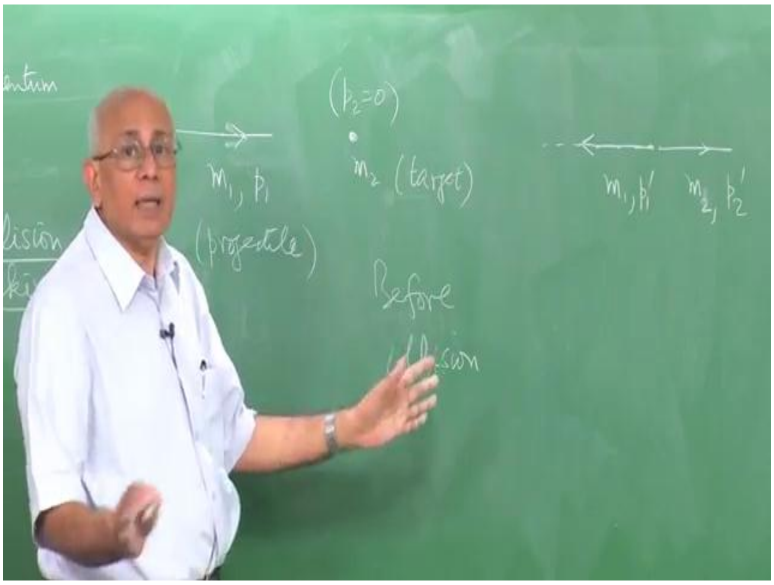

So, we look at the simplest instance, where a particle comes along in this fashion m 1 with the prescribe initial momentum p 1. Since, I am only going to look at this line, I do not even draw vector, if p 1 is positive moving this way, p 1 is negative moving the other way by convention and hits a particle, which is moving with p 2, momentum p 2 in the same direction. But, actually you can do an even simpler problem and that say this is projectile of mass m 1, this is the projectile of mass m 1 and it is stationary target from mass m 2, which is the target particle, this is before collision.

So, this particle along at a constant momentum p 1, hits this particle in infinite as interval of time, something happens to the two particles, and then the separate out once again and move along with constant velocity. So, this is before collision and after collisions, since we are assumed it to be in same direction, this was a initial direction, after collision in general you would expect this particle 1 would bounds back and transfers some of it is energy m 2, which will move out there.

So, let us call this m 1, p 1 prime and the other particle goes this way in that is m 2 p 2 prime. So, what we have done is to set p 2 equal to 0 without laws of generality, the target is stationary in the laboratory frame and the projectile comes along hits it, and then it separate all here. Now, if p 1 prime is negative, it is moving this way, but it is conceivable that if you have a very heavy particle coming along, it will push this and both will move in the forward direction, it does not matter, consider both cases one short.

So, the sin of p 1 prime determine by it is moving backwards or forwards and p to prime of course, expect on physical rounds intuitively to move forwards and not backwards, because is no initial momentum in this direction path. Now, how we going to solve this problem, we use Newton's third Law, which it says the rate of change of momentum is equal to the force on the body. Therefore, we know that the force of exacted by particle 1 and particle 2 is minus the force exacted by particle 2 on particle 1.

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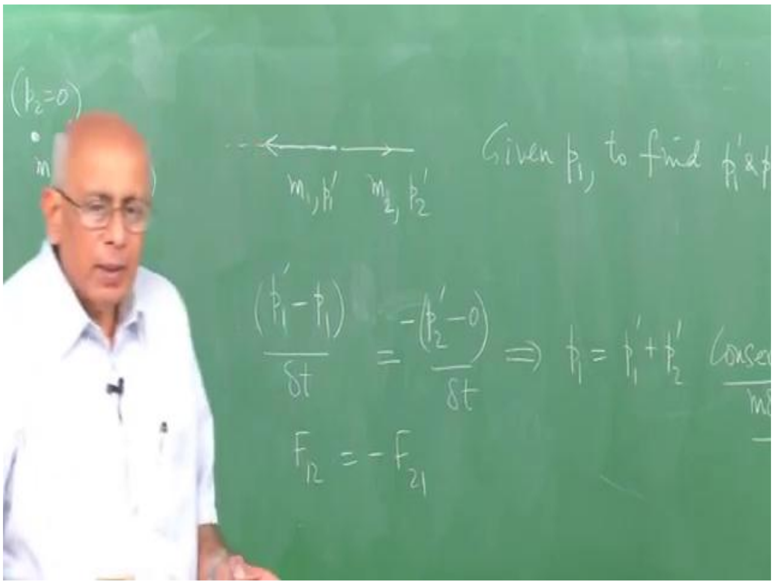

Therefore, we know that the rate change of momentum, we know that p 1 prime minus p 1; that is the change of momentum of the first particle, the change of momentum of the second particle is just p 2 prime minus 0. So, it just p 2 prime and we know that p 1 prime minus p 1 times delta t, the time of collision here this is equal to. So, rate of change and let say over delta t, it does not matter in the case over delta t is equal to the rate of change of momentum here will p 2 prime minus 0 divided by delta t.

With the minus sin, because we know that F 12 equal to minus F 21, this is F 12, the force due to 2 on 1 is the change of momentum, rate of change of it is momentum and the other guy is this. So, the delta t cancel out in this problem and this will immediately imply that p 1 is equal to p 1 prime plus p 2 prime. So, the total momentum before the collision is equal to the total momentum after the collision, the algebraic some and either side, this is conservation of linear momentum.

So, in future, I am not going to go through this argument about Newton's third Law, I am going to write the conservation of momentum down automatically. Because, you can easily convince yourself very trivially that even if p 2 not then 0, but at in some value on the same line, it do not matter, you get p 2 prime minus p 2 here and this would become p 1 plus p 2 is p 1 prime is p 2 and momentum is conserved.

So, hence fourth, we will simple write down momentum conservation in such situations without going to the argument involving Newton's third Law. But, you must remember, that it is comes this argument and it is independent of the actual details of the force, because assumption is the force acts from the infinite decimal time delta t and after that these are free particles according to moving a particle Newton's first law. That is all that is important and the third Law has we built in has been put in in writing this step and second Law has been built in.

And saying that the move freely after the collision thing straight line is given momenta is the first law once again. So, all the laws of motion have been put in understand in this position problem. Now, we are going to use this principle and we are going to use this principle that conservation of energy, because we have return that the yet in order to solve this problem and what do we need to solve, given p 1 to find p 1 prime. In other words, how is the momentum, initial momentum divided up between the target and the projectile after that collision, and then once you know the momenta, you kinetic energies. So, we could ask homogeneity is transfer to from the projector to say the target; this is what we going to do next.