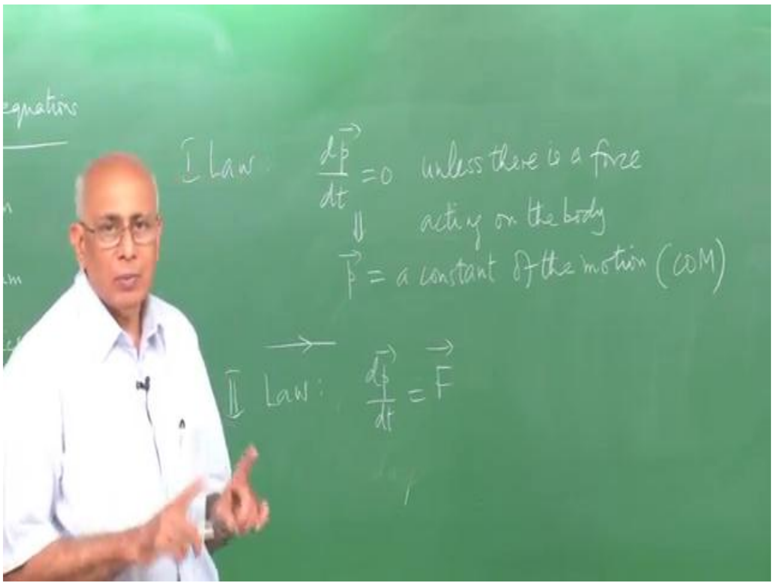
Mechanics, Heat, Oscillations and Waves
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 16
Conservation Laws and Newton's Equations

Today we are look at some important consequences of Conservation laws or conservation principles which are very fundamental in the study of motion itself, these principles have a long history. But, you understood very carefully where they come from, what the limitations are and what the consequences are at some level and I would like to explain, how these things are consistent with Newton's laws and how they indeed support and corroborate these law and help us solve problems in dynamics.

Now, you are already familiar with these terms such as conservation of energy, conservation of linear momentum, conservation of angular momentum and so on. But, it turns out these principles are saying something very fundamental about space and time themselves and we will see, how these principles come about at least qualitatively. Now, Newton's first law which says that a body that is addressed or in a state of uniform motion in a straight line will continued in the state of motion or state of rest, unless acted upon by an external force. This law already has in it a potential conservation law.

(Refer Slide Time: 01:30)



Because, it essentially says, the first law essentially says that the rate of change the momentum is equal to 0 unless there is a force acting on the body. And that is the reason why it continues in the state of uniform motion, because it essentially says that this will imply that p is equal to a constant vector of course, and we say it is a constant of the motion which are often abbreviate as COM and this is a conservation principle.

So, it says that the momentum once you specify it at some initial instant of time continues to be the same for all time unless there is an external force acting on the body. So, if this is the momentum at t equal to 0 that remains the momentum for all time as the body moves. So, the position changes, but the momentum does not change and that is the conservation law, it says the momentum of this object this conserved both in magnitude as well as direction.

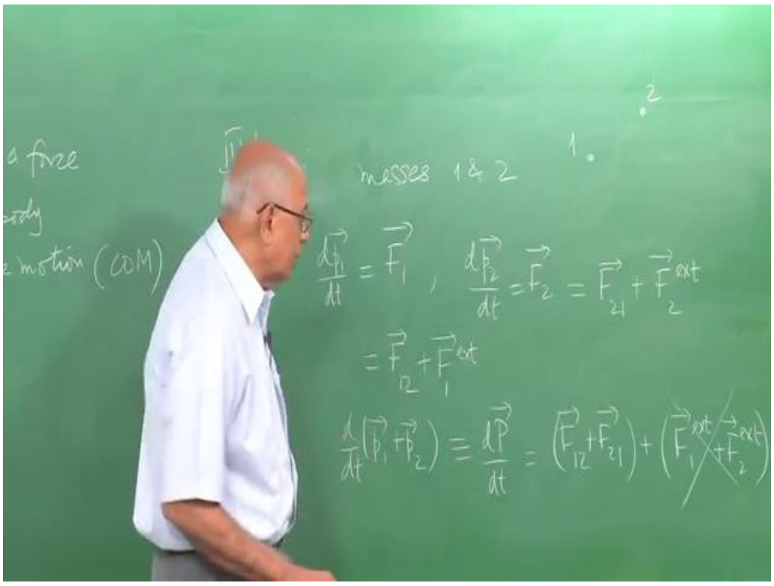
So, already you can see that the first law is essentially saying that there is a certain quantity, a dynamical quantity which does not change as time goes along, even though other quantity such as the position would change. Now, the second law is completely consistent with the first law and the second law says that in an inertial frame d p over d t the rate of change of momentum of the body is equal to the force acting on the body, the net force acting on the body for a single body.

Now, this law is consistent with this law, because it says if F is 0 then p is a constant of the motion. So, it is completely consistent, but you must imagine as it sometimes done that the first law of motion is a consequence of the second law, not at all. The first law of motion essentially helps you to define in inertial frame and the second law says that in v is inertial frames, the rate of change of momentum is equal to F and of course, when F is 0 happens to be 0, then p is the constant of the motion.

So, the second law is completely consistent with the first law, but the fact that it applies in inertial frames and the first law essentially helps you to define in inertial frame, means that these two laws are independent of each other, even though it parent looks as if the second law is a more general case of the first law whatever. So, this constant of motion in the absence of an external force, now the third law carries this one step further and this was entirely Newton’s contribution.

The first law essentially was known into Galileo and the second law was around at that time, but the third law is entirely Newton's creation and what we said was that if you have two bodies in interaction with each other.

(Refer Slide Time: 04:46)

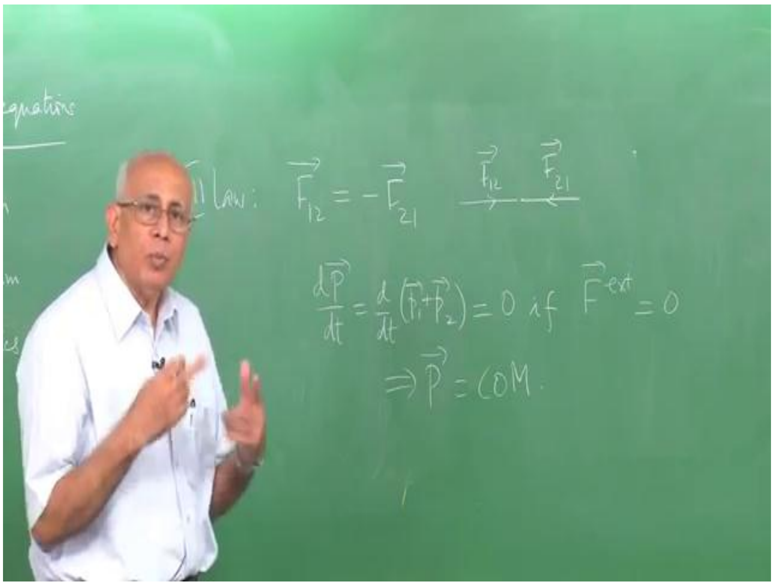


And these two bodies or objects or masses 1 and 2 in interaction with each other, then the rate of change of momentum of the first by Newton's second law is equal to the force acting on the first body and the rate of change of momentum of the second is that net force acting on the second body. Now, the forces acting on these bodies could arise from a large number of consequences of reasons.

So, here is body 1 and here is body 2 and let us suppose they exact force on each other and in addition there is some other force as well on acting on both these bodies. So, this is equal to F 1 2 by this I mean, the force on body 1 by body 2 due to body 2 plus any other force which is acting on 1 and let us call it some external force due to some age and other than body 2. And similarly, this is equal to F 2 1 which is the force exacted on body 2 by body 1 plus F 2 external, the force acting on body 2 due to any other agency.

Then, simply by adding these two equations you immediately see that d over d t p 1 plus p 2 which is the rate of change of the total momentum, let us call that capital P. The capital P is just p 1 plus p 2, this is equal to F 1 2 plus F 2 1 plus F 1 external plus F 2 external that is just by adding the two equations. Now, Newton's third law says that if there is no external force, if this term should happens to be 0 then and only then this momentum is constant. So, it equivalently saying that action and reaction are equal and opposite, so the statement is the dynamic statement is F 1 2 is equal to minus F 2 1 so what is that mean.

(Refer Slide Time: 07:41)



So, the third law says F 1 2 shall be equal to minus F 2 1, in other words the magnitudes of these vectors are the same, but they are oppositely directed. So, if F 1 2 looks like this then the vector F 2 1 is in the opposite direction F 2 1 with the same magnitude and this is the content of Newton's third law. But, immediately as a consequence it follows that d p over d t equal to d over d t p 1 plus p 2 by definition this quantity is equal to 0, if F external equal to 0.

In other words, if you have a system of two bodies, here is the system and there is no external force on this system, except the forces on 1 and 2 due to each other. Then, by Newton's third law since action and reaction are equal and opposite F 1 2 is minus F 2 1 implies that the sum of the momentum of 1 and 2 is a constant of the motion. So, this implies that p equal to a constant of the motion for two bodies.

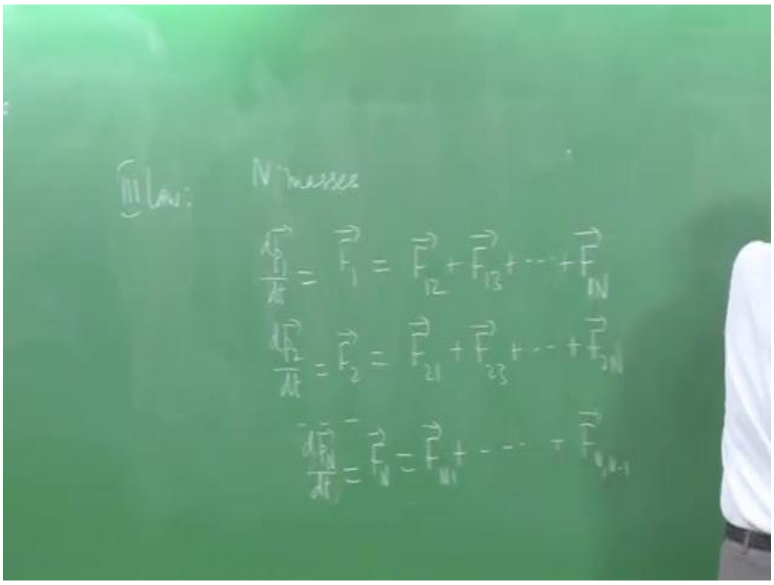
Now of course, it is still remains to ask can we say something more under what condition is this going to be true, other cases where it is not true. It will turn out as we go along, but under a very wide variety of circumstances, this is indeed true, this relation is true and this is indeed a constant of the motion provided there is no external force acting on these two bodies.

This includes the force due to other particles that may be present in the universe, when all of those are gone and you look at only these two particles, then the total momentum is a constant of the motion and that can be a good approximation in many cases as we will see. So, the consistency of a quantity arises when it is time derivative is identically 0 and you can now see that both the first law and the third law actually heal constant of the motion, they are saying something about constants of the motion.

Now, what is the meaning of these constants of the motion? What is the significance? Well, we already saw that in principle when you integrate Newton's equations of motion, you need integration constants which are determinant by the initial conditions. So, every time you discover a constant of the motion, you have gone a step further in integrating and finding explicit solutions of the equations of motion.

So, that is the reason for the importance of constants of the motion and we will have a lot more on this. As we go along when we see an application you will see immediately the role played by, the important role played by these constants of the motion. What happens, if you have more than two bodies and that is the generalization that follows again from the third law and it goes as follows.

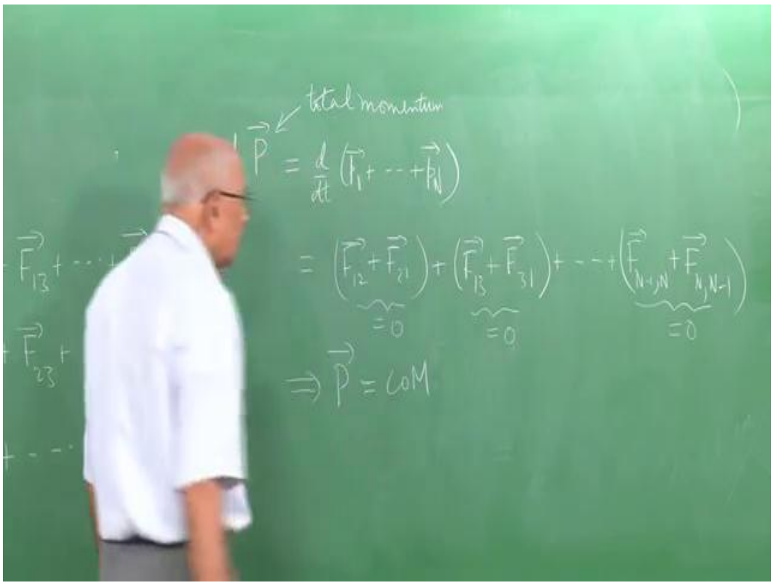
(Refer Slide Time: 10:49)



So, let suppose you have N particles or N masses, do not care what these masses are in general M 1, M2, M 3 up to M sub n and you have equations of motions for each of them. So, d p 1 over d t is equal to and now let suppose that there is no external force on these particles except what they apply on each other. So, this is equal to F 1 the total force on body 1 which comes due to body 2, due to body 3, 1 N due to the body N.

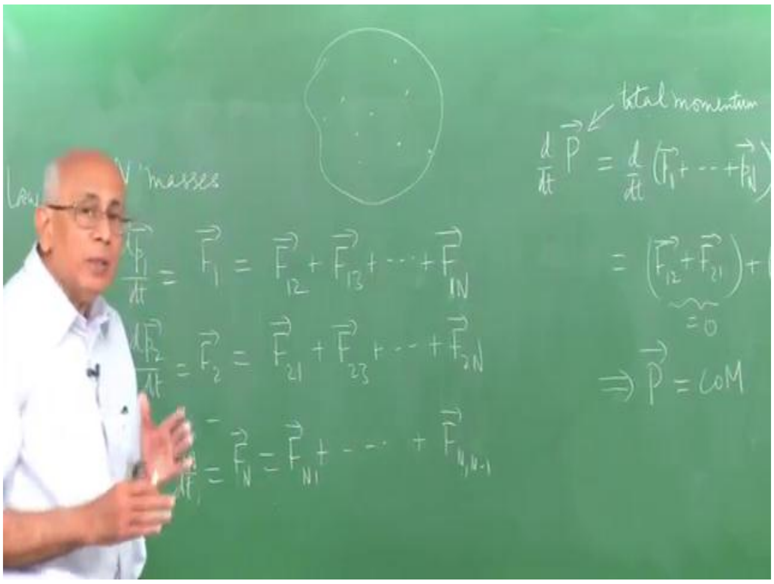
And similarly d p 2 over d t is equal to F 2 which is equal to F 2 1, the force on body 2 due to body 1 plus F 2 3 and a dot, dot, dot plus F 2 N and so on all the way up to the N particle will goes equations of motion. This is equal to F sub N, this is F N 1 plus all the way up to F N N minus 1. There is no self interaction; obviously, each body sees forces due to the other particles or the other bodies and that is the set of equations and we add them all up and use Newton's third law.

(Refer Slide Time: 12:40)



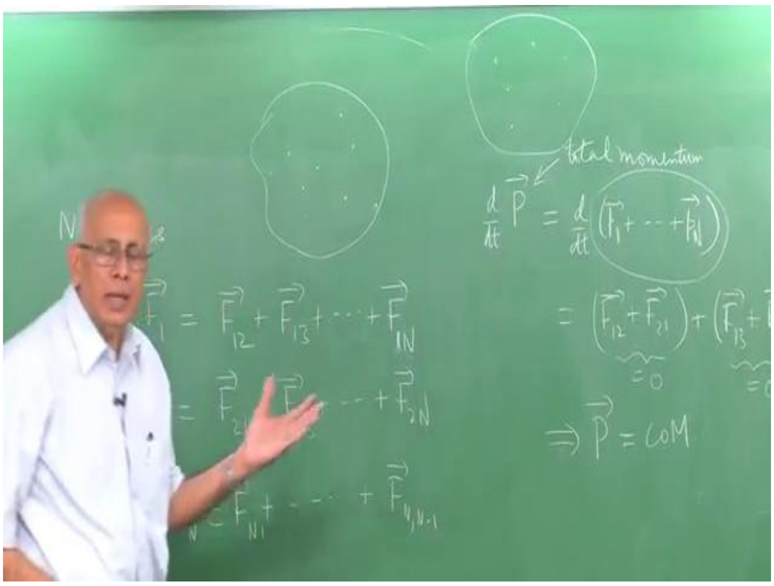
And the result therefore, is d over d t p which is equal to d over d t p 1 up to p N, this is the total linear momentum of all the particles added up pictorially. This quantity is equal to, on the right hand side F 1 2 plus F 2 1, F 1 3 plus F 3 1 and so on. So, this breaks up into pairs which goes like F 1 2 plus F 2 1 plus F 1 3 plus F 3 1 plus etcetera, etcetera write up to plus F N minus 1 N plus F N N minus 1. All the terms N square terms add up and break up in pairs, and then it will immediately tells you that by Newton's third law this is equal to 0, this is equal to 0, this is equal to 0 which implies that p equal to constant of the motion.

(Refer Slide Time: 14:07)



So, it says that if you have a collection of particles in this fashion and then interacting with each other I call that as a system, the total momentum of the system is a constant of the motion, unless there is some external force acting on the system as a whole. But, as long as their saying forces each of the bodies saying a forces due to the other bodies in the system and only that and Newton's third law is applicable. Then, the total momentum of the system is the constant of the motion, it is important for you to understand that in this situation p 1 plus p 2 is not a constant of the motion.

(Refer Slide Time: 14:43)



So, what is being said is that for each pair of objects particles F i j is equal to F j i with the minus sign pictorially. For each i and j, i not equal to j there is no self interaction involved here, while that is true it is not true that for a given pair p i plus p j not a COM in general. So, the sum of that momentum any two particle in the system is not a constant of the motion that is because each of the momentum is changing due to many forces as long as all these where absent and you had just two bodies and this was equal to minus that, then the sum of the two momentum is a constant of the motion.

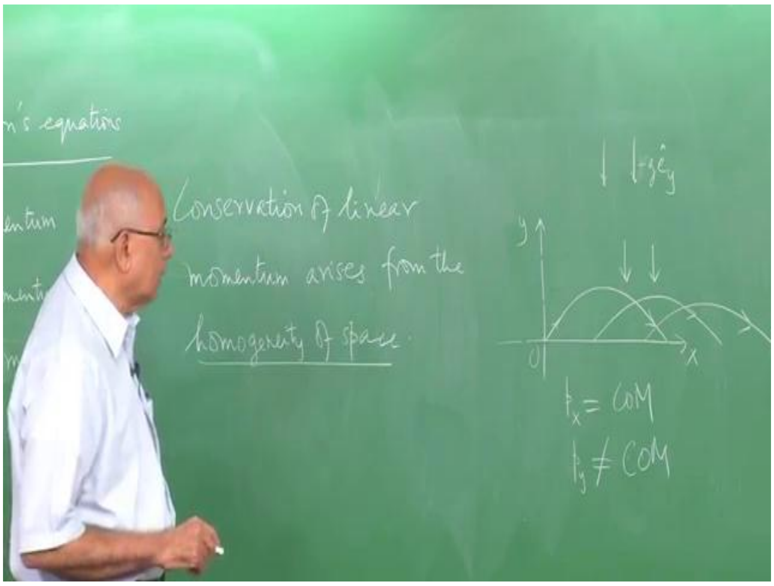
But, the moment you have other forces on the system due to particle 3, 4, 5, etcetera then the sum of any two momenta is not a constant of the motion. But, this still remains a constant of the motion, this quantity here the total momentum is still a constant of the motion. So, this is the consequence of the fact that there is no external force on the system. Now, where does this law of conservation of momentum come from?

Well, by the way this fact that it is the total momentum that is the constant and not individual pairs of momenta also say something about the limitations of the conservation principles. It is not giving you all the information about the system, but it is given in over all constraint on the system and that is the useful piece of knowledge to have. Now, where does this whole thing come from? What is the origin of the law of the conservation of momentum in these situations?

The answer lies in little deeper in mechanics and we will not going to that here, but I will state the answer in words. The answer has to do with the fact that empty space is regarded as homogenous, to the fact that you could take this system and the same thing could be shifted to some other location in space without changing anything else and the physics of what is going on in it will not change if there is no overall external force on the system as a whole, it should break this invariance this translation invariance as we call it.

The shift of any location does not change the properties of the system and this shift, the fact that things do not change when there is a shift of the origin of coordinates for instance is what is ultimately responsible for the conservation of linear momentum, as long as the conditions are right. As long as you have the statement that there is no external force or net external force on the system and system only says forces due to the different components of the system acting on each other pair wise they cancels out due to the Newton's third law and then you left with the conservation of total momentum. So, this is an important idea that let me write this down.

(Refer Slide Time: 18:00)



Conservation of momentum of linear arises from the homogeneity of empty space when we have nothing going on in this space, then we believe that space is exactly the same at different points in space nothing changes. But, then the presents of external forces the presents of feels and so on we will break this homogeneity and here is the example show how the familiar with the fact that in project I will motion when you have gravity acting down wards there is acceleration to gravity which is minus g times, the unit vector in the y direction, this is y and that is x.

If I project a projector from here from the origin in this fashion along this velocity vector, this moves in a parabolic path and falls down there. Exactly the same thing would happen, if I projected it from here with exactly the same initial velocity it would be a trajectory of this kind and so on it is easy to see that apart from the shift in the coordinates, whatever is going here is the same as what is going on here and what is going on here.

So, in the horizontal direction there is complete special homogeneity and that is why in this problem it turns out that the x component of this momentum of this projectile is a constant of the motion. The y component is not, because there is a force in the y direction, there is gravity acting downwards in the y direction given by this and because of that what happens here is different from what happens here is different from what happens here as you can see.

And this is entirely a consequence of a fact that translational in variance in the vertical direction is broken by the fact that you have a filed in the vertical direction, you have a gravitational field and this is true even if the field is exactly the same at all points within this approximation it is clear that the gravity here and the gravity here and gravity here the force is exactly the same. So, even though the force is the same, the existence of the force is enough to cause the momentum in the y direction to change.

Because, it says d p y over d t is the vertical component of the force and that is not 0. So, the moment you have a force in this direction, the translational invariance of this direction is broken and you have no conservation of that momentum. So, p y is not a constant of the motion, but p x is exactly as saying that there is a piece of chock as for as this plain the x y plane is conserved the vertical direction is a z direction, there is translation variance in the x y plane, there is no force in the x of y directions only gravity axis and the x in the vertical direction.

So, the transitional symmetry is broken in the vertical direction and retain in the horizontal directions which is why no matter how I launch it this projectile the horizontal component of it is momentum is going to be a constant of the motion. If I do it in the x z plane then the x component, if I do it in the somewhere between the x and y directions then of course, the horizontal component whatever will be a constant of the motion.

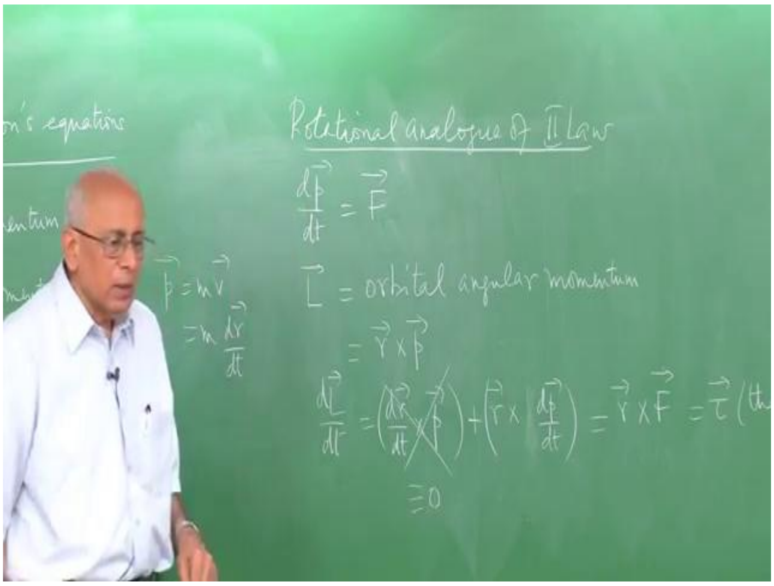
So, this is important to understand that not only have a conservation principle, we also know where it comes from, it comes from the fact that empty space by itself is taken to be homogenous have the same properties at all locations and that is the reason for the conservation of linear momentum provided the circumstance are right and this is the important point to understand.

In the same projectile problem it could ask what happens if I have the full problem of a gravitation occurring due to the force of gravity in the earth and the field lines of the gravitation feel not taken to be parallel lines pointing downwards of constant magnitude. But, we taken to be the full inverse square law of gravitation what would happen then well in that cases it is very clear, if you have distances which are comparable to the radius of the earth, then it is very clear that the motion is not in a parabola of this kind.

The motion is actually part of in elliptic orbit and then it is certainly true that you do not have this conservation in either of the two directions and we will see this, we will see this in a minute and you talk about conservation of angular momentum. So, so much for the origin of the linear momentum conservation, we will use the conservation of linear momentum very shortly, when I talked about and important application to what happens when two bodies collide and separate and go their own way.

And we will see that would good extent the conservation of linear momentum and the conservation of energy will together help us solve this problem to great extend without knowing the details of what happens to during coalition. Now, what about the angular momentum, conservation of angular momentum what is fact comes from.

(Refer Slide Time: 23:51)



Well, let us go back and look at what happens to Newton’s law of single body which has d p over d t equal to the force on one body. So, we are going to talk about the rotational analog of the second law as before I start with the single particle, single object and say the rate of change it is momentum is the force acting on it. And now I am interest in the so call arbitrary angular momentum l this is the arbitral of the body about the origin of coordinates.

So, whatever that origin b and this has a formula which is r cross b. So, it is perpendicular to r the instantaneous position of the particle, perpendicular to p they instantaneous momentum of the particle and it is in a plane directed in the plane and not in the along the direction normal to the plane form by r and p. And the magnitude of r and the magnitude of p times the magnitude of the sin of the angle between them.

Now, what about d l over d t, so let us multiply this, let us differentiate this d l over d t is equal to d r over d t cross p plus r cross d p over d t. So, the same change role of differentiation applies whether you have a dot product or a cross product you have vectors in does not matter a same roll applies. So, we have this cross plus that I should write like this, but in the cases we are looking at this quantity is the velocity by definition and in the cases we are looking at it is the momentum divided by the mass.

So, essentially p equal to m v equal to m the r over d t and what we have here is the situation that d r over d t the vector the velocity cross product with itself. But, the cross product of a vector with itself is 0, because the angle between the two vectors is 0 and therefore, this term is identically 0 and you left with r cross d p over d t, but that is equal to r cross the force on the body by the second law of motion and this is what you called the torque.

So, we have the rotational analog of Newton's equation of motion second law for a single body, it says that the rate of change of it is arbitral angular momentum is equal to the torque on the body r cross f, the moment of the force about the origin and therefore, as in the case of the second law l is the constant of the motion if that torque on the body is 0.

So, we have conservation principle with says that if the torque on the body is 0, then it is angular momentum does not change, this is the exact rotational analog of the law for linear momentum and this case. But, we can go a little further and we will do that next when we consider a set of bodies and ask what happens when you have the third law operating and you have a collections of particles in we will do this we will take this immediate future.