Mechanics, Heat, Oscillations and Waves

Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 15
Newton's Laws of Motion

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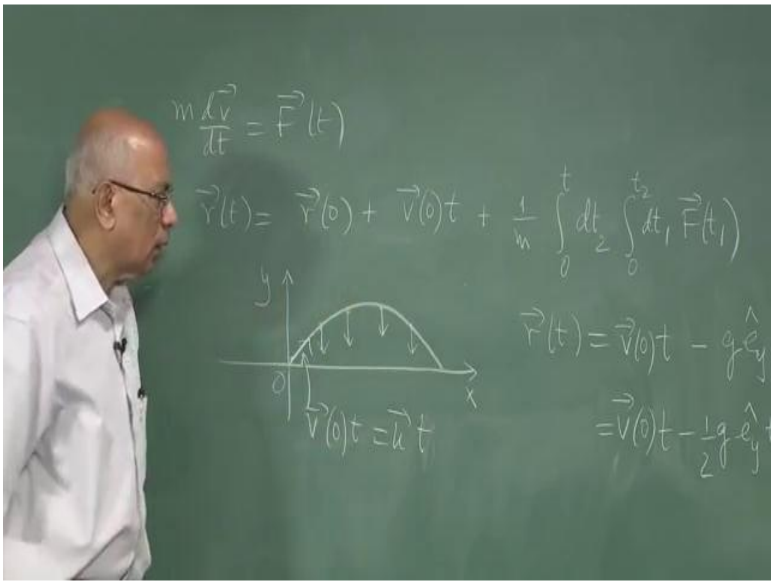

Well, we had start a stage where I said that in principle we could solve Newton's equation of motion, the second law of motions for a given body by pointing out that starting with m d v over d t equal to F of t. If we manage to specify the force on this particle at every instant of time, then it followed that the position of this particle at any instant of time was equal to the initial position plus the initial velocity multiplied by time in this fashion plus a complicated quantity which was 1 over m, an integral from 0 to t d t 2 for want by the symbol integral 0 to t 2 d t 1 f of t 1. So, you first integrate this from 0 to t 2 and then you integrated as a function of t 2 from 0 to t and in principle this was your solution.

Now, where is the difficulty in this problem? The answer is that the specification of this force at every instant of time in some sense already presumes the solution of the problem. Because, most often what happens is that F is a function of r of t, in other words the force on the particle will depend on where the particle is, the moment that happens you realize that this is useless kind of solution. Because, out here you are going to have an r dependence, but that is precisely what you are trying to find out here.

So, in that sense be able to formally integrate Newton's equation of motions is essentially useless except in very simple cases, except if this were not true, if this were a constant course for instance or a force which does not have any r dependence at all, but can be prescribed explicitly as a function of time. Then of course, in principle one can go ahead with this program and write down what the solution is all instance, but otherwise it is not going to work.

And now, let me give you an example of a case where it does work, namely projectile motion this is a classic example which you have already studied in some other context in simpler notion, but we are trying to do this in the context of vector notion. So, here we are, the problem is that if you take a particle and you project it at some angle to the horizontal and call this x and the vertical coordinate y and the system, the particle is under the action of gravity at every instant of time, then we are able to write down.

If you project from the origin for example, r of 0 would be 0 one can write down what r of t is at any instant of time. So, the idea is that the force of gravity on this particle is such that the f at all instance of time is minus m g times e subscript y and that is independent of time, it is a independent of the position as well. This of course, is a big approximation here, the approximation is that every instant of time the force is vertically downwards and it is exactly the mass times the acceleration due to gravity.

This would not be true if it were a projectile which actually went some distance like a missile for example.

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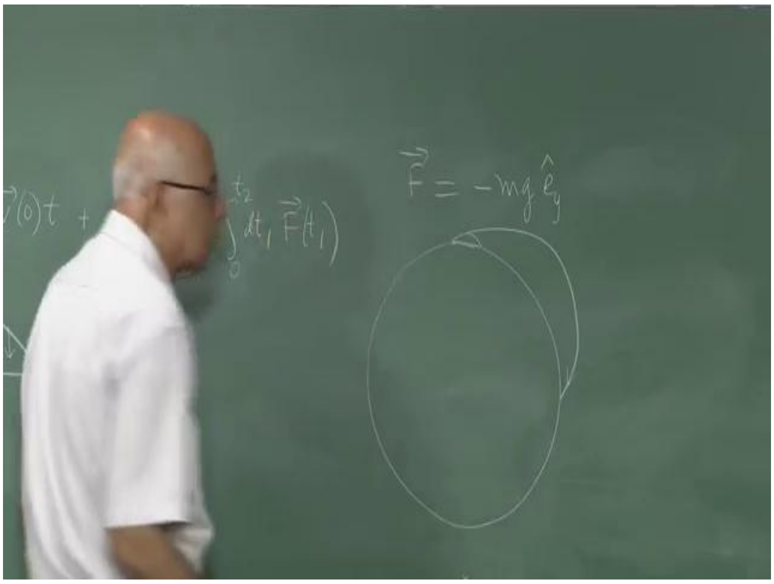

In that case you start with the surface of the earth, you project from the certain point in the side, you will launch it and you goes and land somewhere else in this fashion. As you know, this is really part of an elliptic orbit if you neglect a distance and this assumes that this is essentially and free fall. Then, this orbit here hits at this point and clearly the force of gravity on this object is not constant at all times.

Because, now you reached a height which is comparable to the radius of the earth and you gone a range which is also comparable to the radius of the earth and no longer do you have this approximation possible that the force of gravity is a constant independent of the location of the body and of time not true. But, otherwise if this distance here is extremely small and the height here is the extremely small compared to the radius of the earth, this picture is valid approximately and you have what is called a parabolic orbit.

Technically this parabolic orbit is valid only if the force is a constant independent of the location and of time. But, in practice the force of gravity is not directed the parallel lines of force, but rather they converges to the center of the earth and then you got to do more sophisticated calculation in that case. But, here in this approximation that the force of gravity is a constant, the problem is solvable immediately we could even write down the solution by plugging it in to this thing here or we could work it out laboriously one way or the other.

So, r of t in this case is equal to whatever be the initial velocity out here, the initial projection velocity v of 0 t which I believe this generally called u times t, u being a constant. So, this is u times t plus 1 over m F of t, but F of t is a constant, it is minus m g. So, that portion comes out, so this is equal to minus and the m cancels out, the g comes out all together and you have an integral and the e sub y comes out also and you have an integral 0 to t d t 2, 0 to t 2 d t 1 times 1 and that is it.

And of course, this integral this simple is equal to u t minus g e sub y, the first integral gives you t 2 and integrating t 2 from 0 to t is half t 2 squared and then half t squared if you substitute out of this there and that is it. So, that is the explicit solution and just to keep track of things let me call this v of 0, let me leave it as v of 0. So, that it is explicit but it is an initial velocity. Assuming you projected from the origin; otherwise, you add the position vector of the origin to this result here.

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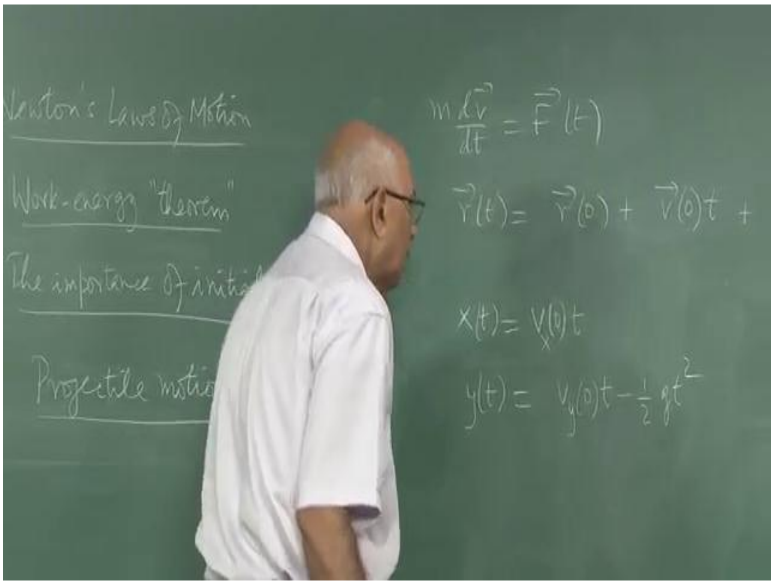

Now, of course, this familiar formula which you are normally used to for projectile motion, imagine immediately from this. If you write it in components, it immediately means in Cartesian components, if x is the horizontal component x of t is equal to v x times t at time 0 times t and the horizontal component does not change at all of the velocity, simply because this acceleration due to gravity is vertically downwards and does not affect the horizontal component of the velocity at any time.

So, it is v x 0 of t out here minus and this is in the y direction and that is it, this is all it is. So, the horizontal component increases linearly with time, if the velocity has in initial horizontal component. The vertical component is another story, the vertical component is equal to v y of 0 times t minus half g t squared and this is the expression that you plot x verses y will give you a parabola.

Because, this goes linearly with t, this goes quadratically with t and when you eliminate one versus the other, when you eliminate time between these two quantities, these two variables then you get a parabola in the x y plane and that is the path of the particle the trajectory that you get. Now, we are able to integrate the equation of motion in this case explicitly, because this force was the constant force and these integrals could therefore, be done immediately.

But, in the non trivial cases where it is the function of the coordinates in general, this could be much harder problem to do. So, this is not the way to do it, the straight forward integration generally will not work and that is important to understand. It is also important to understand the role of initial conditions and let me say about this here. If I start with this object here and I say there is a force of gravity acting on it and nothing else acting on it, I tell you what it is initial position vector is and I tell you the force on it initially and at all instance of time, it is not possible to predict the future path of this particle based on this data.

The reason is, if I let go from rest it will falls straight down towards the center of the earth. But, I give it a horizontal velocity initially when it moves in a parabolic path and falls in a trajectory like a projectile, I give it a slightly higher initial horizontal velocity, it goes further the range increases. In principle, if I give it a high enough initial velocity and there is no air resistance it would go into an elliptic orbit around the earth and in principle, if I increase the horizontal velocity further it would go in to hyperbolic orbit and escape to earth gravitational feel all together.

So, it is clear that specifying the instantaneous position and the instantaneous force on it, it is not sufficient to solve for the particles trajectory at all subsequent times that is what we are seeing, when we see that this displacement or position vector at any time is dependent on the initial position and the initial velocity.

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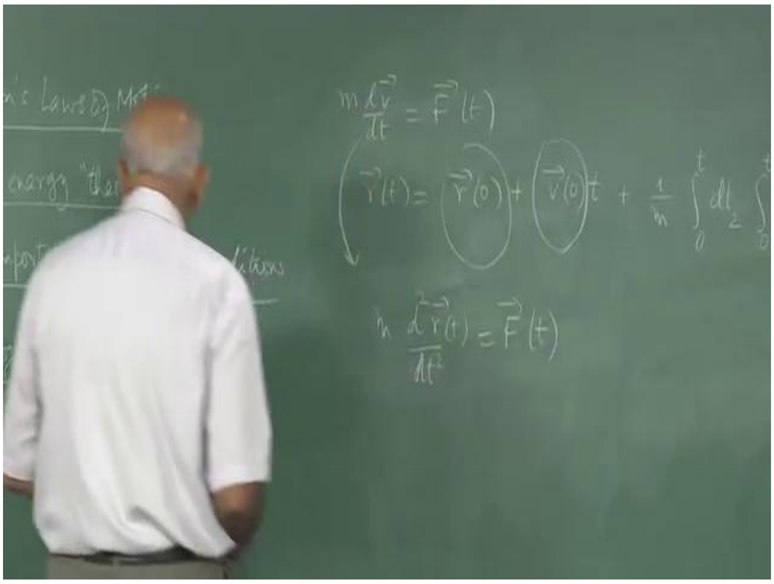

So, the initial conditions matter, not only it does the initial position matter, but the initial velocity also matters that is telling us something extremely deep. It telling us that when you solve an equation of this kind or you rewrite this in terms of the position, you write this as m times d 2 r d 2 to F of t you have two constants of the motion.

So, the speak two vector constants of the motion to deal with namely the initial position and the initial velocity before you get a unique solution for the position of the particle at any instant of time greater than 0. And those initial conditions are the initial position and the initial velocity, it is not enough to specify just this or just this, you have to specify both and when you have more than one particle you have to specify in principle, the initial positions and the initial velocities of all the particles before you can find an unique solution for the paths in time as a function of time.

This has to be a appreciated carefully and it is a consequence of the fact that Newton's laws says that the force on a particle changes it is velocity. And therefore, it is position as a consequence and does not directly become proportional to the instantaneous velocity of the particle, it is proportional to the instantaneous acceleration of the particle. This is important, very, very important, it took a long time to understand this simple this fact of nature so to speak and it is get important consequences.

It essentially says that what varies as a function of time, what dynamics is trying to tell you is how the positions and velocities of various particles change with the function of time or more generally the positions and the momentum of these particles of these objects changes as a function of time, we will come back to this idea.

So, that is has to do with the importance of initial conditions, we are going to say lot more about initial conditions as we go along. Meanwhile, one small digression we should have done this little earlier and that has to do with this, so called work energy theorem, let us go back to this equation and look at, what it has to say. So, if I take a single object mass m and apply a force f on it which could be time dependent, then I get an equation Newton's equations says that m d v over d t is F of t.

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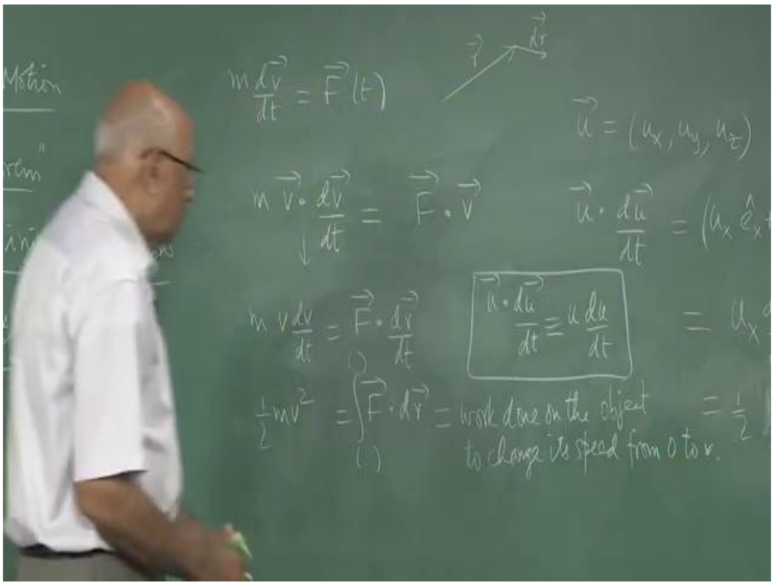

Now, let us take a dot product of both sides of this equation with v itself. Suppressing the t dependence in all these quantities, this is what this looks like after I take a dot product on both sides. Now, what is this thing trying to tell you? You have a vector dotted with the rate of change of that same vector and there is no reason to imagine that this is normal to that or parallel to that, these two could be in very different directions.

We already saw that in the case of motion in a circle, we saw that in general the motion in a circle was such that there was both the tangential acceleration as well as a radial acceleration. So, that tells you that although the velocity is purely tangential in the case of motion in the circle, the acceleration could be radial normal to it could be partially tangential, partially radial and no general principle at all, no general constraint at all, in general these vectors are not either parallel or perpendicular to each other. So, this dot product has some value or some kind here, but this is simple identity one can write down.

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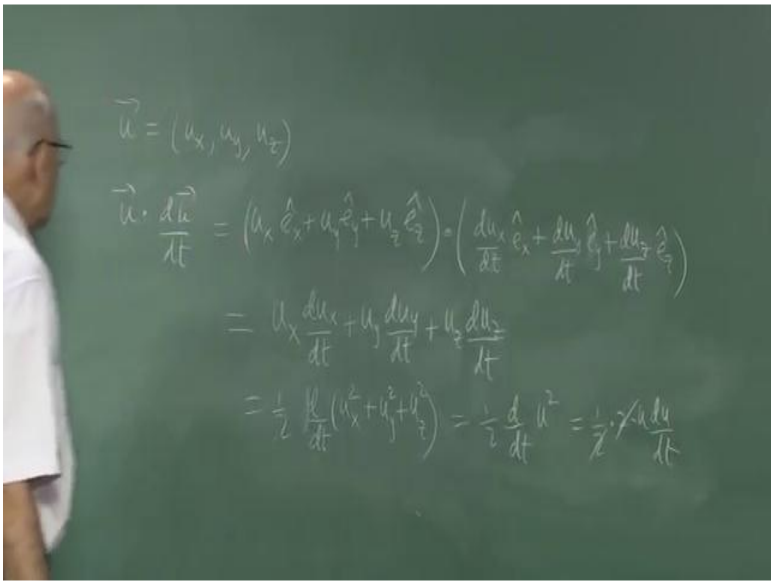

If you give me any vector u and you write it in Cartesian components u x, u y, u z, then u dot d u over d t. If u is a function of time, in other words it is components change with time, then if I write this out in Cartesian components, this is u x e x plus u y e y plus u z e z and that is multiply or dotted with d u x over d t e x plus d u y over d t e y plus d u z over d t e z, e x dot e x is 1 and e x dot e y is 0 and e x dot e z is 0, because these are perpendicular vectors.

Therefore, this thing becomes u x d u x over d t plus u y d y over d t plus u z d z over d t identically equal to that, but u x d u x over d t is half of d over d t u x squared. So, you can write this as equal to 1 half of d over d t u x squared plus u y squared plus u z squared. But, u x squared plus u y squared plus u z squared is equal to 1 half d over d t of u squared, where u squared is a square of the magnitude of this vector.

But, that of course, is equal to 1 half times 2 times u d over d t in the 2 cancels out and we are left with this identity which says the any vector u which depends on time u dot d over d t equal to u magnitude d over d t in the magnitude, this is identically true these vectors can point in arbitrary directions. But, the dot product of the two is just the magnitude of the vector times a rate of change of the magnitude of a vector that is an extremely useful identity which is used over and over again, so very simply identity.

But, you say what happens here, once I use that identity the left hand side becomes m v d v over d t no vectors and that is equal to f dot d r over d t, because the velocity is d r over d t and now I move d t to the other side and integrate. So, I get n v d v equal to f dot d r and I integrate both sides and let suppose I start from 0 velocity 0 speed and integrate up to some speed, the object attain in some speed then 0 up to v v prime d v prime is equal to the integral on the right hand side starting from wherever the initial value r is up to n's at when the velocity is v wherever, but this right hand side...

So, there are limits placed here, but the right had side f dot d r we know that if at any instant of time the position vector of this particle is r and you apply a force on it and it causes a displacement d r, then f dot d r is the work done on this particle in this infinite decimal displacement by the force. So, this is equal to work done on the particle on the object to do what to start it at rest and give it a velocity v a speed v and this side is trivial to integrate this is half m v squared.

So, it is the work done on the object to change it speed from 0 to v, so that is called kinetic energy of the body. So, we have a definition of the kinetic energy this is the so called work energy theorem cote and code, because it is just a straight forward consequence of Newton's equations here, it says the work done in moving an object from rest to some speed to give it some speed v is called the kinetic energy of this object and it happens to be half m v squared this sense.

A little later we will see on a talk about relativity, we will see that this formula is an approximation just as Newton's equation itself is an approximation, then the speeds over very small compare to the speed of light and more over when you in a inertial frame, so we will come to this brings a little later. But, right now I just want to do point out that here is an operational definition of the kinetic energy and this is how the formula half m v squared arises by the simple argument here.

Now, couple of questions one of them is the following, Newton's first law told as that an object either continuous to be at rest or moves an uniform in a straight line with uniform velocity if there is no external force on it.

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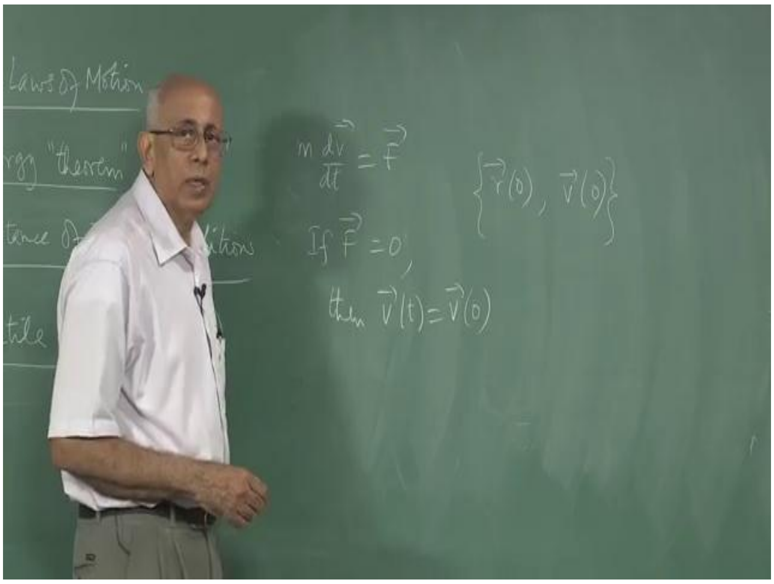

The second law on the other hands says that the mass times the acceleration equal to the force on it. Now, you can force on the body happens to be 0 then it says the acceleration of the body 0 with says the velocity is a constant. So, if f equal to 0 then v of t is equal to v of 0 whatever was the initial velocity will persist for all time if the force happens to be identically 0 at all times, but that is the first law.

So, does this mean that the fist law of Newton was the motion the first of the laws is not independent of the second law does not mean that it derive from the second law, it would look like it, it would look saying that the first law is a special case of the second law the force on the body happens to vanish is this true it cannot be, because we have three laws of motion and they have to be mutually constant with each other.

And I leave it to you to think about this and we will talk about this little later as to where the difference between the first and second laws comes. So, it is not that the first law is it special case of the second law, it is true that the law constant with each other they have to be; otherwise, they want form a set of loss of motion, but it is not true to imagine that the first law of the Newton is actually derive thus is special case of the second law not true thing about y this should be shown.

Now, that we cannot for this for let us go back once again and explore what these laws are really trying to tell us. Well, I mention a little bit about the important of initial conditions, but we need a little more than that in principle when the trying to do integrate equations of motion, there are constants of the motion integration constants. Now, the mathematical sense they may be integration constant in the physical sense their initial conditions.

So, you need to know what the initial velocity is what the initial velocity of each particle is or each object is initial position of each object is and you need to know the set of these conditions for all the objects that you deal with new dynamics problem. How does one discover these Constance of the notion, well either you specified all these initial values or you specify quantities which do not change with time and already the first law is trying to tell you something about it, it saying that if you do not have any force an objects a by the way just one small remark is a very important one though, this quantity here is not the most general from of Newton second law of motion.

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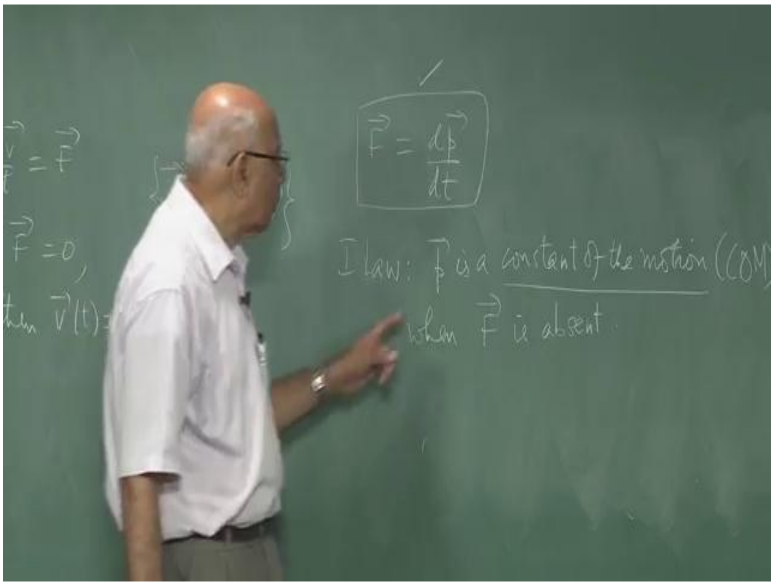

The general form is that the force is equal to the rate of change of the momentum of this body and the momentum is defined as the mass times the velocity at this level. Now, this might look like a very trivial statement is might appear that all we done is to take this m inside and since I am is constant where the differentiate p or v does not matter not so at all. Even if m change is with time this is the correct law and not this that is the first point that this is the right law.

The second point which is even more profound is that the momentum is not always defined as the mass times the velocity, we will come to examples of why this is not so and there are fairly a simple example of why this is not so. The momentum has a slightly more general definition then the mass times velocity in the Newtonian approximation that we are dealing with here implicitly, the mass times the velocity of an object is indeed is linear momentum.

But, this is not the generated formula and the fact that this is the rule the law of motion and not this is actually dramatically manifested in the case of say rocket which is carrying it is own fuel and as it goes up in it is part, the mass is changing because the fuel is bunt and put out is gas is exacted gases, in which case this is the correct law. So, if you give me the instantaneous force on the rocket I can tell you is rate of change of momentum including the fact that is masses changing as a function of time.

So, this is the right law here, we will keep this and we will use this only in those cases we are very sure that the physical situation is such that the masses a constant and you are in the reign in which this law is applicable in carful about that. Now, the first law essentially is telling you that in actions of a external force the momentum does not change, in other wards the momentum is constant.

So, it is really telling you the first law p is a constant of the motion and I am go to use that an abbreviation c o m for it. Because, it is so important as a concept, p is a constant of a motion, when if f is absent, if there is no force on a body then according to Newton’s first law is momentum is constant in time; that means, it is constant not only in magnitude, but also in direction.

If it is started out in 0 momentum it remains at 0 momentum, it remains at the 0 momentum, it remains at rest. If it is started out with the finite momentum in some direction then the momentum continuous to be the same in magnitude and direction as a function of time with for all time it remains so. And therefore, p is a constant of the motion, actually a vector constant of the motion. So, there are three components here each of which is constant in time.

So, there is a deep symmetric principle involves here, we talk about why these constant of motion arise. where do constant of the motion arise from, why are certain combinations of these dynamical variables constant time, this is very deep in dynamics itself it has to do with the symmetry in the problem, just to cut a long story short and we will talk about this elaborate on this an momentum conservation arises, because of the homogeneity of space in space is assume to be homogeneous and there is no difference in space between one point and another, then we have the possibility of the conservation of the momentum.

Similarly, the conservation of energy comes, because of the homogametic in time, if you say that the origin in time is a relevant the law does not change if you start the clock at different instance of time, then you have the possibility of the conservation of energy. And the conservation of angular momentum also has geometrical or a physical explanation it arises.

Because, space isotropic in all mentions in other ward in all directions of space is a exactly the same properties, then you have the possibility of angular momentum conservation. When these quantities are not conserved, then there is some physical effect which is spoiling the symmetry, which is spoiling breaking this a symmetry and that is what is responsible for the conservation law to break down.

I will elaborate on this starting with the example of the projectile itself it will be easy to see where this conform and what these constant of the motion mean and this is very, very basic and important constant and dynamics. And if you understand what the role of consents of motion is about the role of symmetries is in dynamics, then you gone a long way in understanding the structure of Newton’s equations and there solution is specific cases.