Mechanics, Heat, Oscillations and Waves
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Lecture - 14
Motion in a Circle – Acceleration

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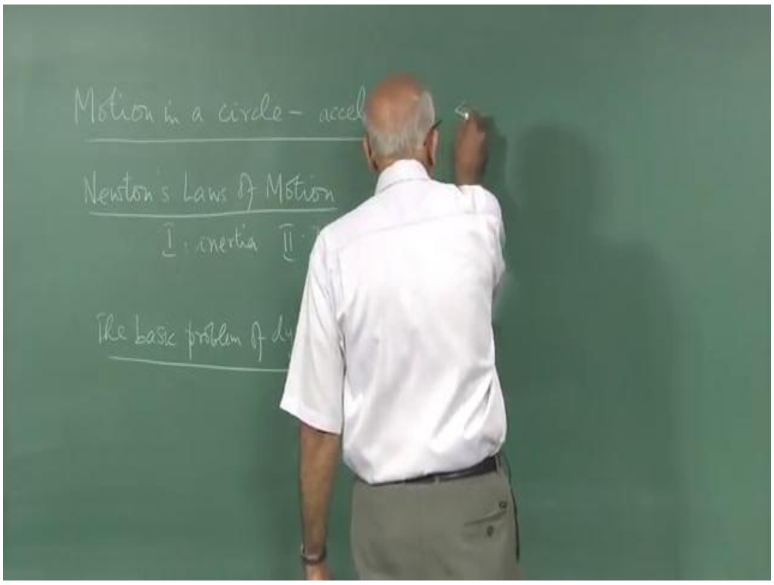

We are ready now to come back on our study of Newtonian mechanics, one small preliminary a special case of what we have already looked at a little earlier, but the appropriate. And this has to do with motion in a circle and what the acceleration require to maintain such a motion is, we will come back to this topic a little later. But, I thought I should mention it at this juncture, because it gives you good instance of how the unit vectors in coordinate systems other than Cartesian coordinates are useful in description of motion.

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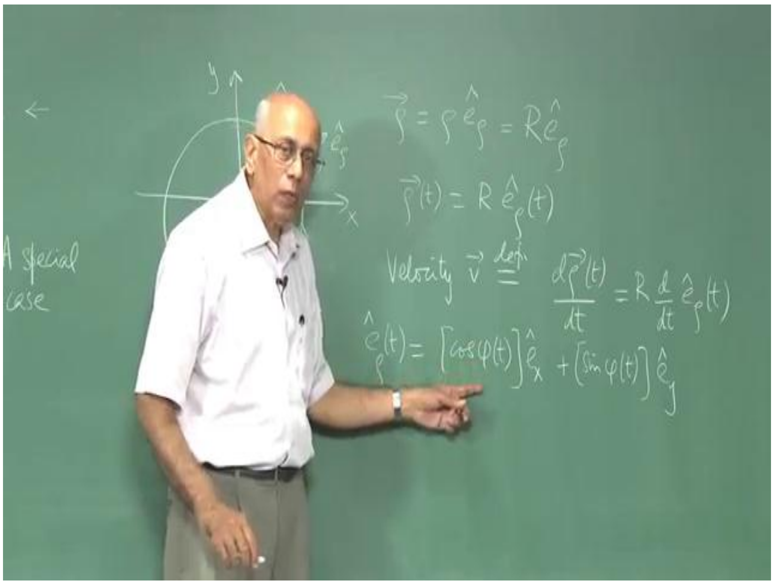

So, we have in mind, I am going to look at this topic, we have in mind a particle which is moving in the x, y plane is it the origin in a circle. We do not say anything more about the motion at the moment such as, what is the force crossing this motion or anything of that kind. We observe that it is moving in a circle, it is restricted to a path which lies on this circle and moves in a circle with some angular speed some kind.

And then the question is, how does one describe the kinematics of this particle? Well, as usual let us draws slightly better figure than this. The radial distance in the x, y plane from the origin to this point is some rho and this vector rho is a position vector of the particle at some instant of time t. The radial unit vector e sub rho is in this direction and the tangential vector e sub phi which increases direction of increasing azimuthal angle phi is normal to e sub rho and points in that direction at this point and of course, these two vectors change direction as the particle moves along the circle.

Now, the starting point is to say that the position vector of this particle at any time is equal to rho times e sub rho. And remembering that rho is a constant in this case, it immediately tells us that since the radius of this circle is prescribed to you, this is equal to r times e sub rho. Therefore, any change in this position vector rho comes entirely from the change in the unit vector in the radial direction and let us call that a function of

t. So, we have rho of t equal to R times e rho as a function of t.

Therefore, the velocity of the particle v which by definition is the rate of change of it is position vector over d t that is the definition of velocity, instantaneous velocity. This must be equal to R times d over d t of e rho of t. So, notice that we have the time derivative of a unit vector which is constant in magnitude unit magnitude, but whose direction is changing as a function of time in some fashion of the other, which depends on the force on this particle.

But, at the moment we are interested in evaluating this d over d t e rho of t and we can do that in many ways, one of them is to say that e rho at any instant of time t. So, let me put the t in, we know is equal to cosine of phi which is a function of times e x and no argument of t or anything like that here, because that is the constant vector plus sin phi which is a function of t times e y. Therefore, the derivative of e rho comes from the derivative of cos phi with respect to time and sin phi with respect to time.

So, that would be a standard way of doing this, but we can write the answer down more easily by just the geometry of it and let us draw another figure to see how this works, slightly magnified figure.

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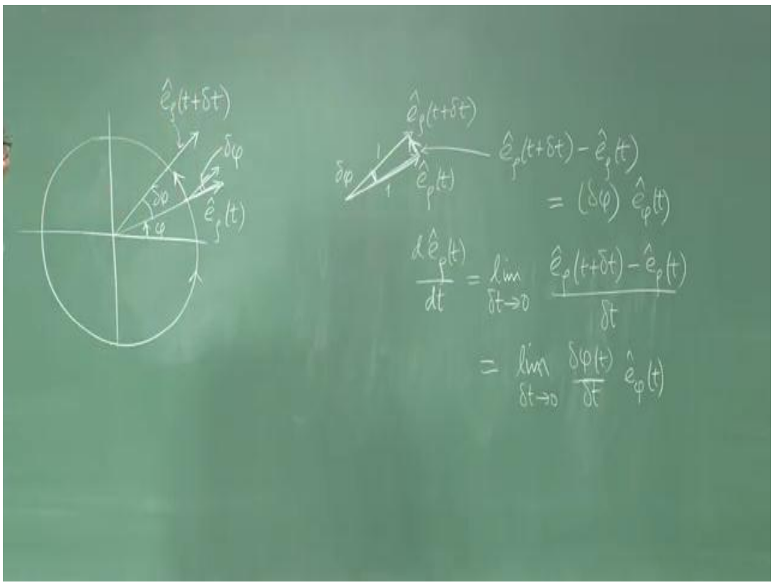

So, here is the circle, at some instant of time this is rho and the radius vector points in this direction, so this is e rho of t and at a later instant of time. Infinite has mainly later instant of time t plus delta t and I am drawing this in an exaggerated way, so that you begin to see what is going on. This thing here is e rho at a later instant of time t plus delta t, since the particles moving in this direction counter clockwise and you can see this vector is not parallel to this vector, so it is change in direction.

And what we need to do to find the change in the vector is to translate this vector parallel translated here. So, make it parallel to it is original, make it parallel to itself and bring it to this point here and since this angle is phi and that angle is delta phi where incremental increase in phi by the law of corresponding angles. This angle is also equal to delta phi and each of these vectors has unit magnitude. Now, if I draw this figure in a little exaggerated way, things become clear.

So, you have unit magnitude e rho at time t, you have an angle delta phi between the two that is this angle. Another unit magnitude 1 here and this is e rho at t plus delta t and it is quite clear by the addition of vectors that this vector here pointing this direction is the difference between e rho at t plus delta t and e rho at time t. So, this vector here is e rho at time t plus delta t minus e rho at time t, because e rho at time t plus this difference is equal to e rho at t plus delta t by the triangle rule for the addition of vectors.

So, once we have that in place this quantity is equal to the length of this arc or this vector infinite decimal vector which is equal to what, there is an angle delta phi and there is a base or radius 1. So, it is clear that the magnitude of this vector arc divided by the radius is equal to delta phi, therefore arc is equal to delta phi multiplied by 1. So, this is equal to delta phi times the vector in this direction normal to e rho. If it is infinite decimal delta t, then this vector becomes essentially normal to e rho and that is the direction of e phi, as you can see it is normal to it in this direction.

So, it is delta phi times e phi of t and therefore, if you divide about sides by delta t and take the limit d e rho at time t over d t is therefore, limit delta t goes to 0 e rho at time t plus delta t minus e rho at time t divided by delta t. The numerator is a vector, the difference of two unit vectors and that turned out to be this quantity here and therefore, this is equal to limit. Let me write this out explicitly that at t goes to 0 delta phi of t over delta t multiplied by e phi of t.

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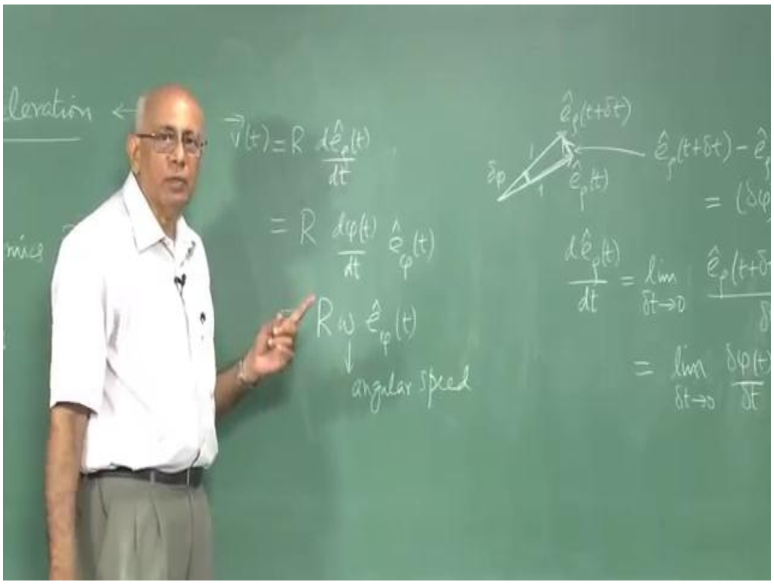

So, this tells us that the velocity v at time t which was equal to R times d e rho t over d t is equal to R times the limit of this quantity which is the angular speed. It is the rate of change of the azimuthal angle, it is phi dot of t or d phi over d t. So, d phi of t over d t times e phi of t which is as same as saying this is equal to R times omega times e phi of t, where omega stands for the angular speed. Of course, we could have derived this more simply by writing e rho in Cartesian coordinates and noticing that the derivative of e rho of t would be just e phi of t times whatever this quantity is t phi over d t.

We could have done it that way, but this geometrical illustration makes it very clear notice, that this is independent of whether omega is constant in time or not. So, we are not restricting ourselves to uniform circular motion, you have motion in a circle, but the particle could be accelerating, the angular acceleration could be non zero along this circumference of the circle. So, this is not restricted to uniform circular motion, it is a general relation v of t is always tangential that is obvious. Because, when you move in a circle, it is clear that the instantaneous velocity is always tangential to the part of the particle.

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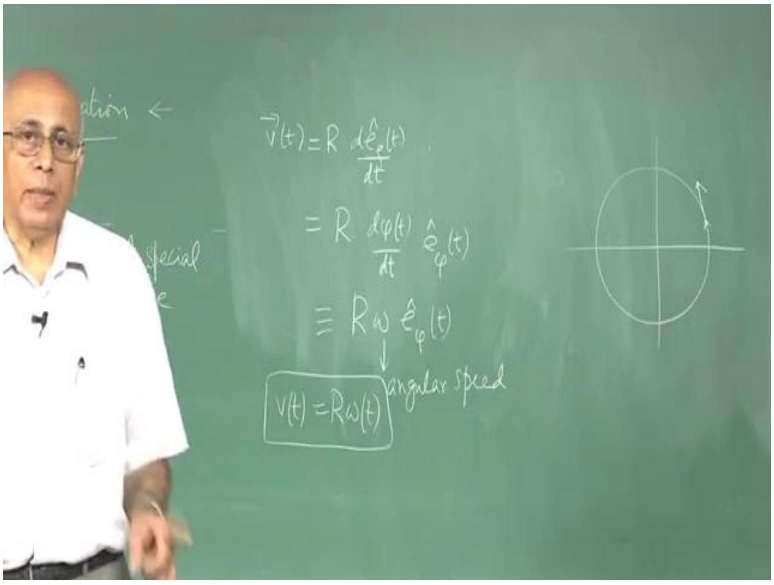

So, again there is a path of the particle and at any instant of time the initial, the instantaneous velocity is tangential along e phi at any instant of time and that is indeed what is happen and the magnitude is proportional to R omega. So, we have our important result which says that this speed which could depend on time is equal to R times omega of t in general. So, for circular motion, uniform or otherwise the speed is the radius of the circle multiplied by the angular speed.

I emphasize again that this is independent of whether omega of t, omega is a constant or not. Even, if the angular acceleration is not zero, it is still true this relation is still true, so purely geometrical relation here. So, we have our first result which says what the velocity is in terms of the tangential vector and the angular speed here. The next thing to do is to ask for the acceleration and there we need now to differentiate these quantities.

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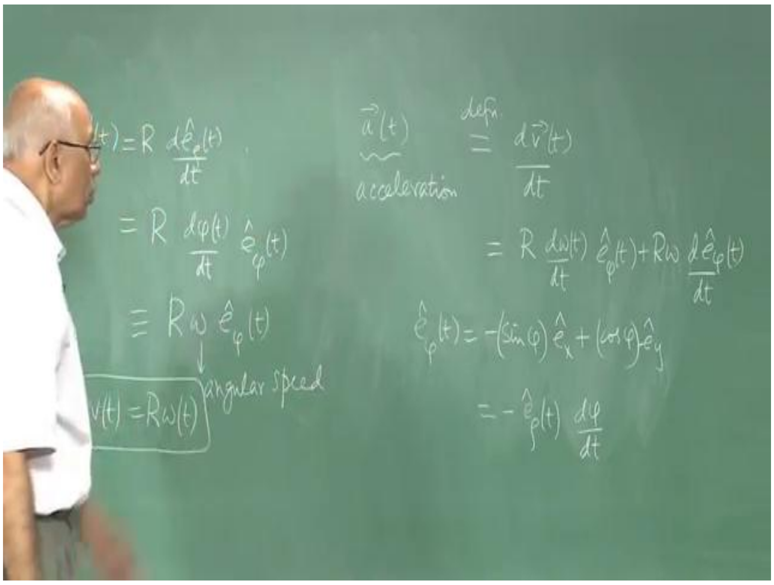

So, it says the acceleration a of t this is the instantaneous acceleration is again by definition d v t over d t and we need to differentiate this combination here, but notice that capital R is a constant. So, that does not get differentiated and by the chain rule this is therefore, d omega over d t that could possibly be a function of time times e phi of t plus R omega and v e phi over d t.

So, there are two components to it, one of them coming from the fact that the angular speed might be time dependent. And therefore, they might be an angular acceleration and the other coming from the fact that this vector e phi of t changes in direction as the particle moves along the circle. So, there is an instantaneous acceleration due to the change in direction of this unit tangent vector and we need to compute that quantity.

As before, same argument as before one could write down explicitly e phi of t is equal to minus sin phi times e x plus cos phi times e y and differentiate in this. Remembering that e x and e y are constants, the only thing that gets differentiated is this and that, this immediately tells you is equal to minus cos phi e x plus sin phi e y which is simply e rho. So, this immediately says this is equal to e rho multiply as a function of time multiplied by d phi over d t of phi dot.

So, when we differentiate sin phi with respect to time, first you get cos phi and then d phi over d t, because phi is a function of t. So, we put that in there and this immediately tells you that you have two terms here.

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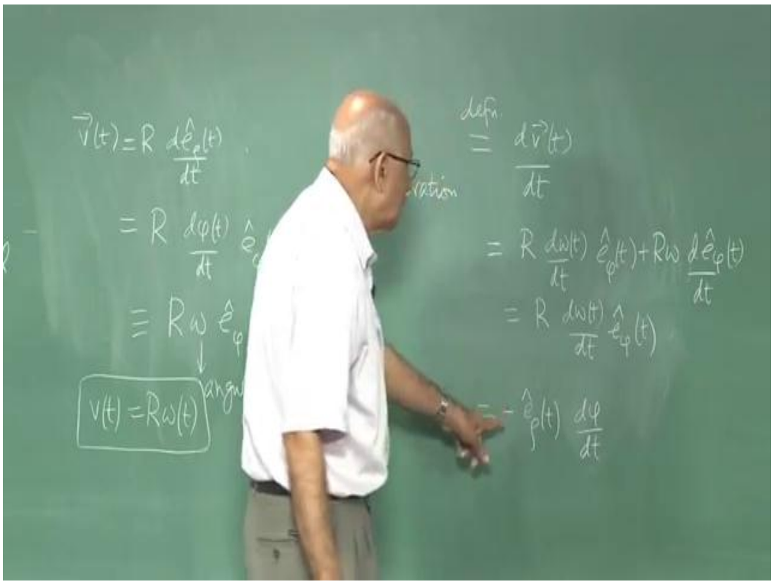

This is equal to R times d omega t over d t e phi of t and there was a minus sin here, so this is equal to minus R omega squared, because there is another omega from this, this quantity is omega times e rho of t.

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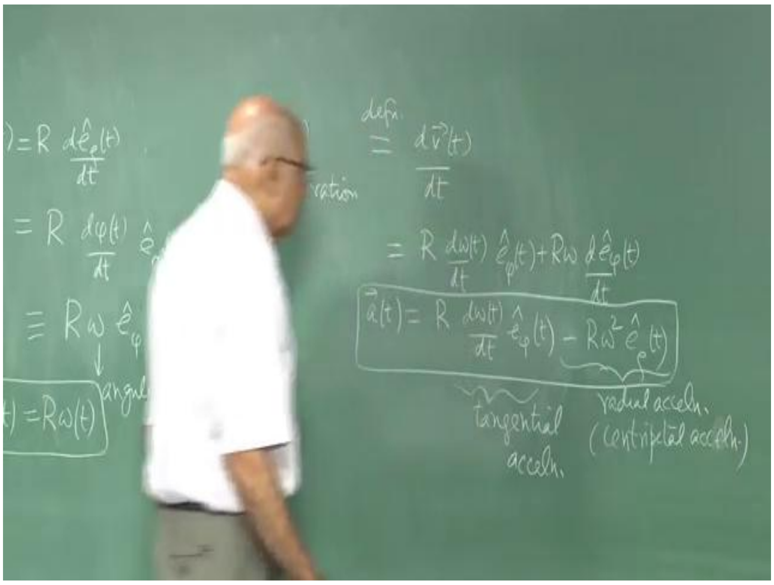

So, that is our final answer for the general acceleration of a practical moving in a circular path. The general expression has two terms, one of them coming from the possibility that as it moves along the circle, the angular speed might change as a function of time. It might accelerated, it might decelerate and so on and that gives your contribution here which is purely tangential. But, and here is a crucial point, in order to even maintain this particle on a circle, you have to add a centripetal acceleration directed towards the origin, this has a minus sign these are positive quantities.

So, this is directed towards the center of the circle, in other words if a particle moves in a circular path uniformly or otherwise, then has to be a centripetal force which pushes it towards the center of the circle constantly in essentially; otherwise, it will not move in a circle. Now, the physical origin of this acceleration must come from Newton's laws, we must know from Newton's laws which says mass is equal to the times the acceleration is the force. To maintain this acceleration you need a force and that force has to have a component which is directed inwards otherwise, you cannot have circular motion no matter what.

So, I emphasize again this portion, if you like is the tangential acceleration and this portion with the minus sign is a radial acceleration or the centripetal acceleration. This thing is also called centripetal, because it is towards central and it exist always, even if omega is a constant it still exist, now a little paradox here. So, this is immediately telling you something which is little counted intuitive, to keep an object moving in a circle, you need to have a force that directed towards the center at all times.

You might also have a force that is tangential which will then cause and change in the angular speed. But, even to maintain it at constant angular speed, you need a force directed towards a center. Now, where is that force come from in the case of planets moving around the sun if the orbit is approximately circular? It comes of course, from the force of the gravity with the sun exits on the planets.

So, there is a centripetal force which pulls the object inwards. What happens when you twill a stone around in a circle type was string, another to one and of string. Well, the tension in the string is providing the centripetal force in this case and here is a little paradox.

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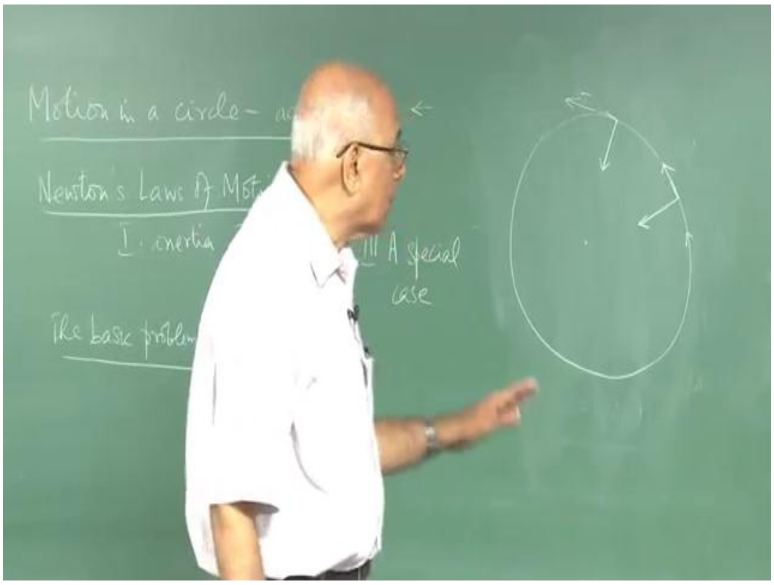

Imagine having a train moving in a circular track in this fashion. At any time here is the train, it is clear that the engine is pulling it in a tangential direction. How then the train move in a circle track? Because, it should really be moving along this tangent in a straight line, but it moves in a curved path there has to be a force on the train which is directed towards the center at all time. So, when the train is here engine pulling like that, but there has to be a force which is directed inwards in this direction.

I leave it to you to figure out, where the normal force where the force that is radially inverts on the train comes from. Even though the engine obvious to pooling it in this straight along with the length of the train, you still have a force have to have force inwards; otherwise, it would not it possible to maintain this train on a circular path. So, I leave it to you figure out where comes from.

Now, you will also notice that these formulas a special case is of the general expression we wrote down for the acceleration in plane polar coordinates, where there components which came from the fact that the radius itself could change, the radial distance itself could change. So, there was a term which depended on rho double dot the second derivative of rho the radial acceleration there was another term is depended on rho dot another combinational of rho dot in phi dot and so on.

But, this is what happens if I said rho equal to R, the constant this is special case of the more general formula from motion in a plane in plane for the coordinates the derive earlier the expression for the acceleration that are several terms and some of them disappear when rho is a constant as in this case here. We will return to this problem of motion in a circle and talk about non inertial pseudo forces a little later.

But, now I am going to go back to a main task which to a mark upon the study of dynamics, they going to have a lot to say about this Newton's laws. So, let me as you know there are three basic laws of motion propounded by Newton, the first of them is a law of inertia which is actually known to Galileo itself and it essentially it says that objects. So, suitable size not to small, no quantum mechanics, not to fast, no relativity such objects move in such a manner to satisfy Newton's equations of motion.

These three equation of motion are believe to in compose all of the dynamics that you need to know for Newton's in the regime in which these larger applicable. The first law says that an object will continue to be a trust or in a straight of uniform motion in a straight line, if there is unless acted upon by an external force, so it saying something about inertia. So, the first law is essentially a statement of, what an inertial frame is in this sense. It is a frame in which the first law is valid.

So, if you roll a Mabel on a floor and you discovered that it is essentially frictionless moves on forever in the assumption that there is no fiction it moves in a straight line forever, then we know that the floor is part of inertial frame of reference. The second law is where the meet is this is where the dynamics is, but already in the first law certain assumption of gone in. For instants, it is been assume that the space is infinity extend and has a Euclidian structure it is three dimensional Euclidean space.

It is also been assume the time flows in execrably, there is something called absolute time following in execrably in the infinite pass to the infinite future. Now, these are assumption which are not really valid, we have discover since then that these have to be taken in some qualifications in there approximation. But, in Newtonian mechanics the first law has already presume that this concepts have been incorporate that it is presume that the time is absolute quantity which flowing from the past to the future at uniform rate and that space is of infinite dimension, infinite extend and is Euclidean and structure.

The second law says something about the way a system response to an external force. So, in some sense which is the very definition force if you like what it says is that the mass times the acceleration.

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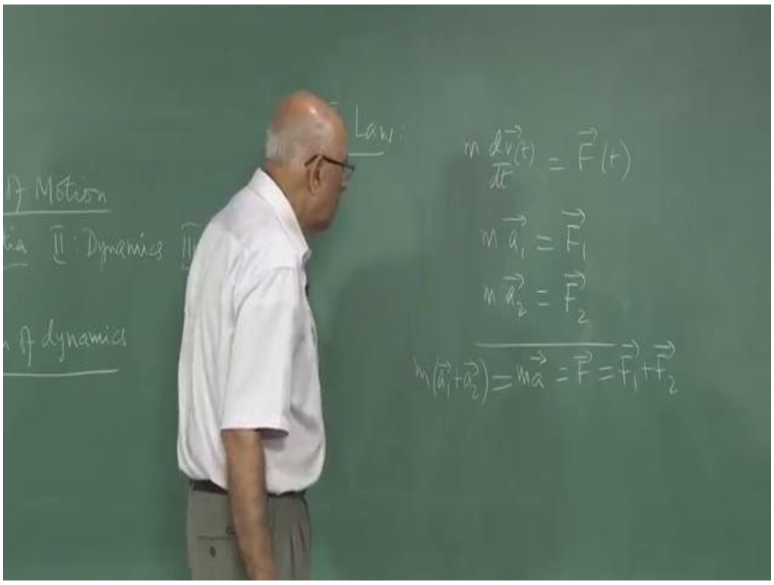

So, here is the second law it says the mass times the acceleration of a body is equal to the force applied on this body, the instantaneous force applied on this body. So, let say in compose that I by saying let say this is v of t is f of t it does say anything about where this force comes from, what are the other bodies that is act on this object in order to produce force on it does not. So, it is does not care, it says the moment is a force on the body, the body reacts by accelerating not changing it is position immediately, but changing it is velocity.

The force is proportional to the acceleration and not the displacement, the force is not proportional directly to the velocity, this is very important. So, a force of causes change in the velocity in the rate of that change is d v over d t is quantity here, the acceleration is multiplied by a certain constant property of this object call just mass. The way one should really look at this is to say, if you apply of force on a body then the responsible of the body is by an acceleration and the acceleration is proportional to the force and the constant of proportionality is one over the mass of this body.

So, in that sense this is a responds stimulus relationship here it is got many, many other interesting properties some of which we will take about. For instance, the vector equation and I do not specify what the coordinate system is in the answer as we said in the beginning is that it is does not matter what coordinate system you are in, because this is a vector equation and we know how these quantities stand form and the rotations and translation of the coordinate access.

So, if you have till at coordinates with respect to original coordinate from somewhere else and shifted somewhere else in space it does not matter, this relation is still true in form, in other words in the new frame you get m d v prime over d t is f prime of t and you still guarantee this same proportionality. So, that is the a sense of this vector law of motion if you like and this is the problem that basically one has to solve in dynamics.

So, notice also that it is a linear in the sense that if I have two forces acting on a body then the result and acceleration is guarantee to be the some of the accelerations in a vector sense. So, if for instance have a m times a 1 for a force f 1 on the body and you have m times a 2 for the force f 2 in the body, you are guarantee that if you have a force f equal to f 1 plus f 2 you have an acceleration a which is guarantee to be equal to m times a 1 plus a 2 this looks like a trivial statement, but it is a profound statement it says the response of this object it is acceleration is linear in whatever is a stimulus in whatever forces apply independent of the nature of that force.

So, if you have one force which is mechanical, another force which is electrical on this body and the first force produces an acceleration a 1, the second produces in acceleration a 2 your guarantee that one both forces acts simultaneously on the body, the result and acceleration is again the vector some of the individual acceleration cause by the two individual forces, this is not at all obvious, but it is a straight forward consequence of Newton's law of motion and it has very deep implications, it is a linear law in that sense.

It had it been something else more complicated, like proportional to the third power of the force something like that we would had very many more intricate effects going on, but we do not we have a straight forward linear super position in operation here in the second law. There are other consequences as well and talk about some of them as we go along.

But, first let we see, what is the basic problem. The basic problem of dynamics here, the fundamental problem is for a single body for instance, once you given an equation this kind and you told what this f of t is the basic problem is to find out what is the velocity at any instant of time and what is the position of this particle at any instant of time. So, that is the task to solve this differential equation and discover what is the velocity and what is the position at any instant of time. Now, in principle this is not such a format able task looks very straight forward.

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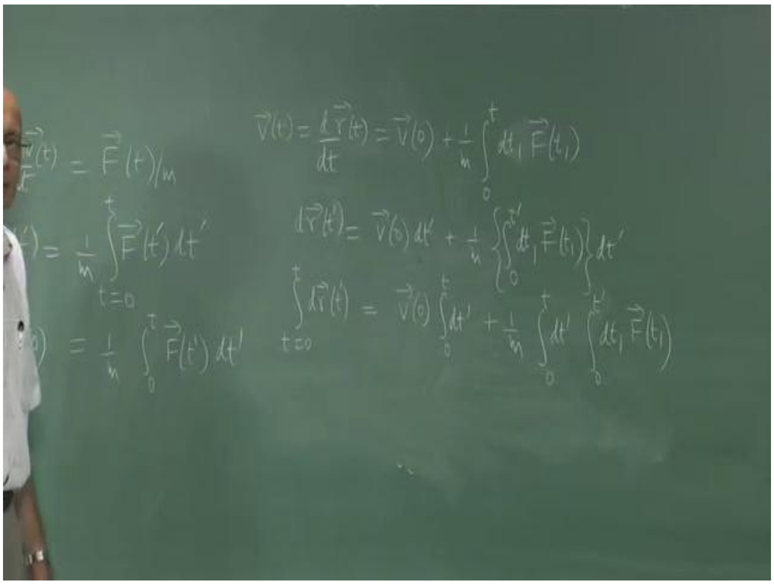

For instance, I could take this equation and m bring to the right hand side that it is a constant and then if I integrate both side will multiplying by d. So, you have d v t equal to 1 over which is a constant and then a function of t d t in this side and the integrate both sides from time 0 t equal to 0 up to sometime t and I integrate this side as well t equal to 0 up to sometime t this side as well t these are definite integrals from 0 to any given specified time t.

So, I need to use another symbol for the integration symbol integration variable. So, let us write it in this fashion it is always a good idea to keep variables distinct wish from each other and then this quantity here is a total differential. So, by definition this side left hand side is v of t minus v of 0. Notice this, you need to put in this term here, you need to tell me what is the initial velocity at some initial instant of time which you prescribe that quantity is equal to 1 over m and take well from 0 up to t f of t prime that is a vector d t prime.

I do not have write t equal to 0, because it is a integral with respect to t itself the time itself. I needed to do that here to tell you what I integrated over, what is the quantity I integrated over. So, here is the formal solution to this equation formal solution, because we do not really know what f of t on the right hand side is and you will see there are complication comes in a minute.

But, this is the formal solution and therefore, if I write this out in terms if the velocity it says v of t is equal to the rate of change of the position r of t the position vector at any instant that is the definition of the position vector, this quantity is equal to v of 0 plus 1 over m and integral become 0 to t. Since, we are going to have more than one integral let me change the variable of integration, it does not matter what I call the variable of integration it is only a label which will disappear after the integration.

So, let me call it t 1 f of t 1 and let us put the d t 1 write away here in this fashion, this is a definite integral and it depends the answer depends only on the end points. So, whatever the value of this integral is, you have the indefinite integrals, you have to substitute for t 1 the value t and subtract from it whatever you get the substituting the value 0 out here. So, I emphasize again t 1 is a variable of integration it is an index of summation if you like and it does should not appear in the final answer.

So, this is the formal solution to this differential equation here, but it still involves in differential, it still involves the rate of change of the position. And the next step should be integrate that and would depend get again I multiply by d t and integrated. So, d r t let me write it out is v of 0 d t, this is a constant vector, because it just says whatever happen to be the initial velocity at t equal to 0 that vector has we put in there in place and it is a constant vector it does not change it time.

So, this is v of 0 times d t plus 1 over m an integral from 0 to t d 1 f of t 1 this whole thing a some function of t multiplied by d t. But, it is a vector function and we should never lose track of that this quantity here is a vector function and as you know the elementary definition of the integral is like a summation, you break up this interval of integration into small infinite decimal pieces and add up all the little pieces that contribute and here you got to do a vector addition here. So, that is what is mean by saying integral over a vector.

So, that is the formal expression for d r of t and if I integrated once again then 0 to t d r of t t equal to 0 up to t d r of t prime is equal to on the right hand side you get v of 0 that is a constant vector times 0 to t d t prime. So, the first step is to call this t prime everywhere, t prime, t prime in that fashion and then d t prime and the next step is to say this is 1 over m and integral from 0 up to t d t prime integral from 0 up to t prime d t 1 f of t 1 this is not such a trivial integral this portion is trivial it is just t. So, I can get read of this plus this quantity here.

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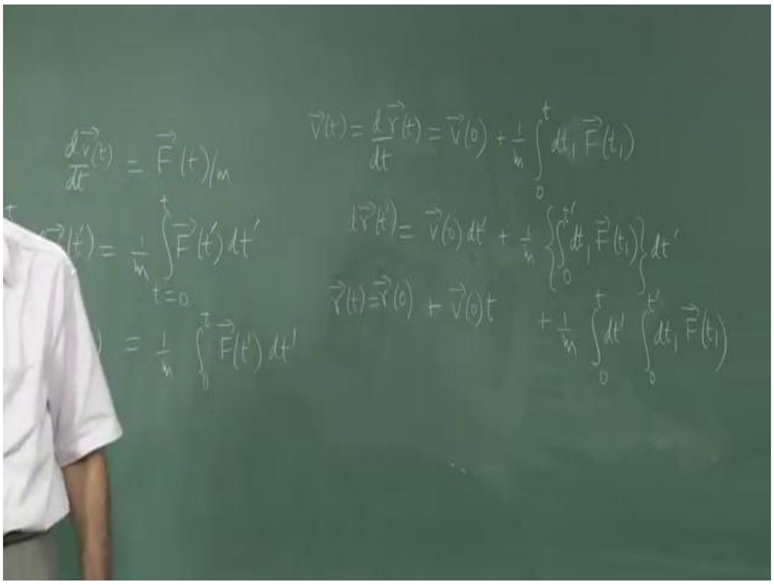

And what is the left hand side, the left hand side is r of t minus r of 0 and that is equal to this quantity in this side. And of course, we move the r of 0 to the right hand side it is equal to this plus this, and that is the formal solution to the problem of integrating Newton's equation of motion for a particle or for a body of some kind of single object. But, this is not so simple as it looks and now I will come in further and what this integral is and why it is so non trivial to evaluate.