Mechanics, Heat, Oscillations and Waves
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Lecture – 13
Cylindrical and Spherical Polar Coordinates

So, let us resume our look at Cylindrical Polar Coordinates.

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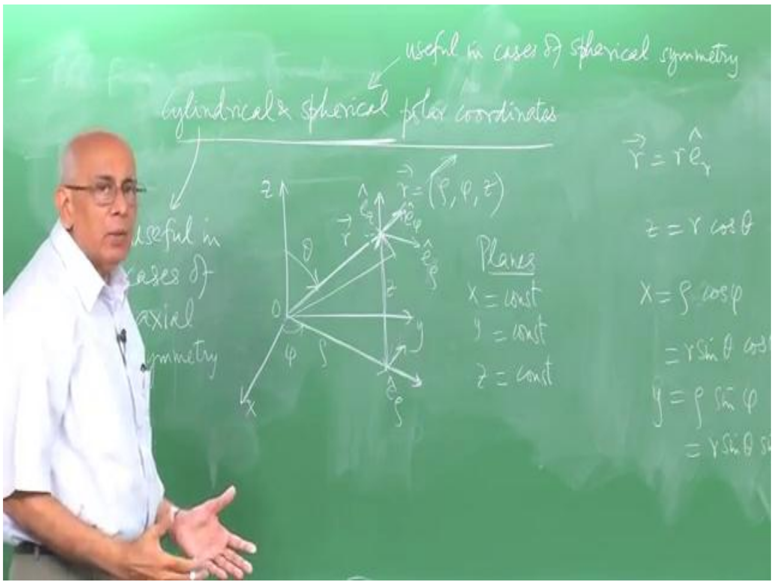

And if you recall, I mentioned that we start with the Cartesian coordinates x, y and z, the origin here and if you have an arbitrary point there with the radius vector r, make a projection of it on to the x, y plane. This distance is rho, this angle is the azimuthal angle phi, this vertical distance is z and the coordinates where r was given by rho, phi and z in this fashion.

What we need to look at are the unit vectors in this coordinate system at this, at any point like this. One of them would of course, p rho e rho itself. So, that is the unit vector e subscript rho, bring it up there, it is this direction is a rho and remember that phi increases, so this thing goes in that is e phi is. So, there is an e phi going into the board, if this point is on the plain of the blackboard. And the third is of course, the original e z in this point.

So, the unit vector tired now depends on where you are, because e rho and e phi are phi dependent and e z of course, is a constant vector. So, now, the relations that we had for Cartesian coordinates, the cross product of two members if we tired was equal to the third in cyclic type.

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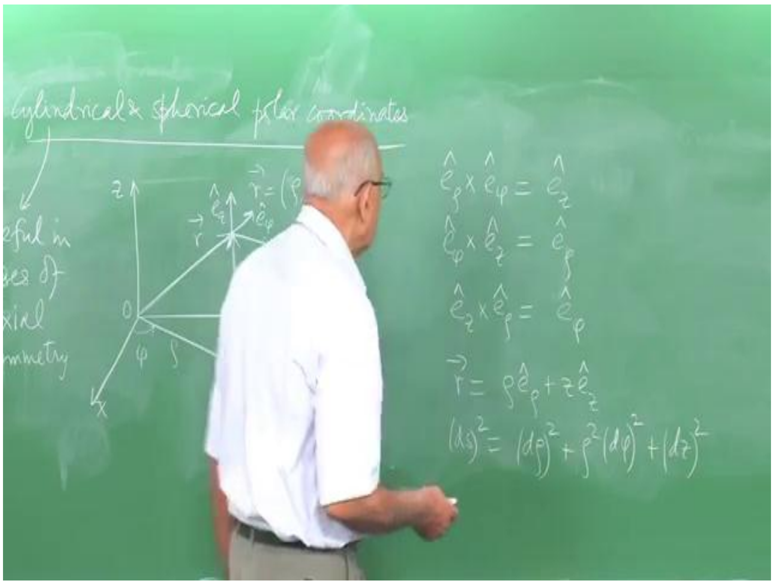

Permutation reads like e rho cross e phi at any point, take e rho this vector and drive a right handed screw driver turn from e rho to e phi, which goes in then to the plane and the screw driver moves upwards. So, this is equal to e z and similarly, e phi cross e z is equal to. So, from phi it moved to rho, the screw driver moves out along e rho and finally, e z cross e rho must be equal to e phi.

Let us check this, screw driver from phi to rho rotate and it moves into the plane at the black board, which is e phi in this fashion. And what is the position vector of any point, r is equal to the sum of these two vectors, this vector plus this vector is equal to this vector here. The r vector and therefore, this is equal to rho times e rho plus z times z, this function and you must remember that this is the function of phi with this in term, when you write it in terms of Cartesian coordinates, there was a function of phi involved there.

So, this is what cylindrical polar coordinates look like and they are useful in all problems, where you have axial symmetry, where you have some kind of cylindrical symmetry about a specific axis. And it is very useful to call that the polar axis, this is the axis, the vertical axis and then, use plane polar coordinates in a plane perpendicular to the z axis to the polar axis. So, this useful in problems this thing here useful in cases of cylindrical or axial symmetry.

If there is a preferred axis, then it is convenient to choose that as the z axis or preferred direction and use plane polar coordinates in a plane perpendicular to it. So, that is the use of cylindrical polar coordinates, we will see when come across problems, where you use cylindrical polar coordinates very profitably. And these are the relations between the unit vectors here. One final point that the line elements square is also be written down, d s squared is d rho square plus rho square d phi squared plus d z whole square.

So, cylindrical polar coordinate is the straight forward extension of plane polar coordinates, just adding the third Cartesian coordinate to it and nothing more than that. The ranges of these variables I have written down already, rho is 0 to infinity, phi 0 to 2 phi and z is minus infinity to infinity. A little more intricate in that would be the case of spherical polar coordinates here and this is what these polar coordinates look like.

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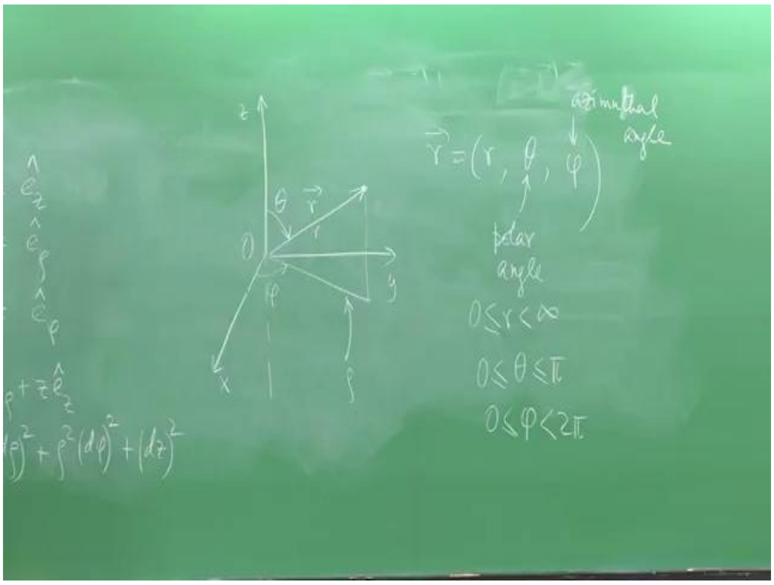

So, here is x, here is y and here is z; that is the origin. If you took an arbitrary point here and that was the radius vector to it vector r. Then, spherical polar coordinates correspond to the following, you draw a projection on to the x, y plane as before that was rho, if you recall. So, this quantity was equal to rho here, this distance is r from 0 to the point p. This angle here from the z axis from the vertical axis positive vertical axis moving down towards this point here, this radius vector here, this is called theta. And this angle here of this projection is phi as before.

So, if you like, you can imagine that a distance little r from the origin draws sphere of radius little r, then this spherical polar coordinates correspond to specifying the radius of this sphere which is r. This angle theta, which is called the polar angle as well as phi which is the azimuthal angle as before, these are the spherical polar coordinates of an arbitrary point.

It remains to say, what these angles mean in geometrical terms, it is very simple. If you took the x axis on the surface of this sphere, the point where this cuts this sphere, if you considered this to be 0 longitude, then it is clear that this phi is giving you angles of longitude. Constant phi would correspond to given meridian. On the other hand, theta the angle from the polar axis, theta would be equal to phi over 2 in the x, y plane, it would be equal to 0 on the positive z axis and on the negative z axis, it would be equal to phi.

So, theta is like the co-latitude measure from the North Pole, normal latitude is measured from the equator and we talk about latitude north and latitude south, taking this to be 0 latitude on the equatorial plane. Here, what we do is to start with the polar axis, the vertical axis, the positive z direction to have angle theta equal to 0 and then, from there you move down all the way to phi, which is down here in the negative axis.

So, the range of these variables is obvious, 0 less than equal to r less than infinity, the point could be either at the origin or anywhere in space in which case it is called a finite length distance r. But, theta transform 0 up to pi not 2 pi, but pi, because every point between 0 and pi over 2 has a theta between in the Northern Hemisphere and every point between pi over 2 and pi, the theta is equal to the point is in the Southern Hemisphere. Now, it is interesting to ask, what constant values of various coordinate says and I will come to that in a minute, phi as usual has got this range here.

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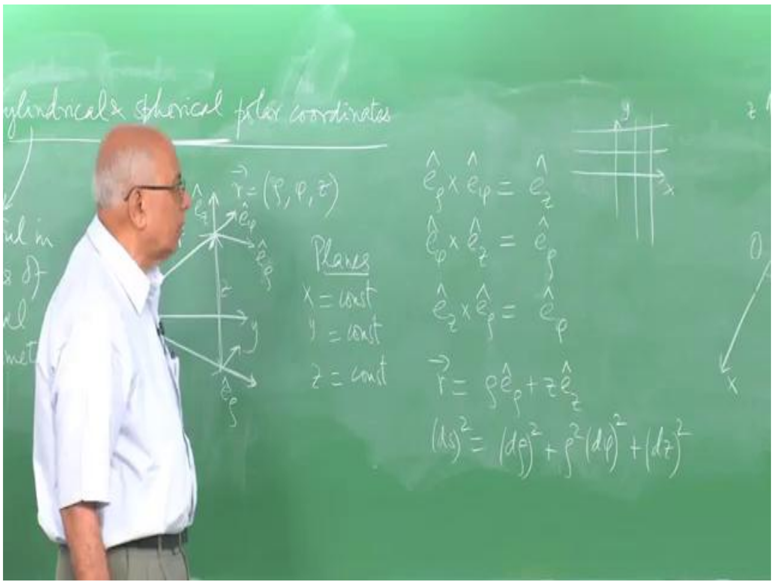

Now, it is convenient to picture these things in terms of, what it means when one of these coordinates is constant, in the case of the plane, here is x and here is y in the planer case. If I say x is constant like 2; that is a line, x is 3 is here, y is 1 is here, y is 2 is out there and so on. So, these are lines parallel to one of the two axis in the common rid. In three dimensions, when I say z equal to 2, I mean a plane which is parallel to x y plane at a height 2 about the x y plane.

So, these are x equal to constant, y z equal to constant or planes normal to whatever axis you keep constant. So, if I say x is constant, it is a plane parallel to the y z plane at whatever distance along the x axis that this constant is. So, in Cartesian coordinates, this is completely trivial.

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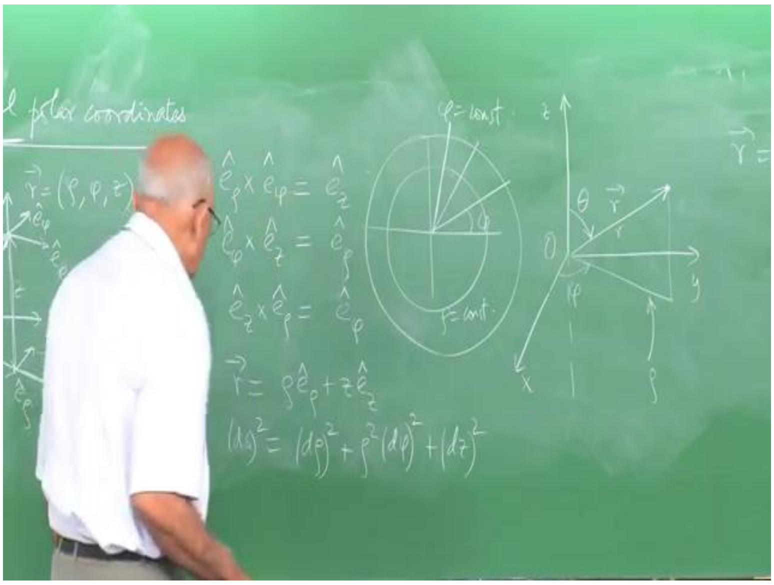

On the other hand, when I go to polar coordinates, in plane polar coordinates, if I say phi is constant, it means I am along this line, a radial lines starting from the origin going up to infinity. So, phi is a different value here, phi is a different value here and so on. So, these are radial lines or rather half lines starting from the origin going up to infinity. So, phi equal to constant, some given value is a line the locals of such points is a line.

On the other hand, if I say rho is constant, then it is clear that the distance from the origin is constant. And what is happening is this is rho equal to some constant, these are the concentric circles and they cut the lines phi equal to constant at right angles, all of them. So, these equi coordinate surfaces or in this case to lines are normal, just as in the earlier case a plane y x equal to constant is normal to a plane y equal to constant or a plane z equal to constant.

Similarly, phi equal to constant if the half line that is normal to the curve rho equal to constant, perpendicular to it that is why these are called a orthogonal coordinates. In the case of cylindrical polar coordinates, z equal to constant then again planes and rho and phi equal to constant is exactly as in plane polar coordinates. Except in three dimensions, when I say that rho is equal to constant, z could be anything and therefore, I have the surface of the cylinder, whose radial distance whose radius is equal to whatever value of rho is specified. Similarly, phi equal to constant now would also be half planes rather than plane lines, because these things would come out in this z direction and be half planes.

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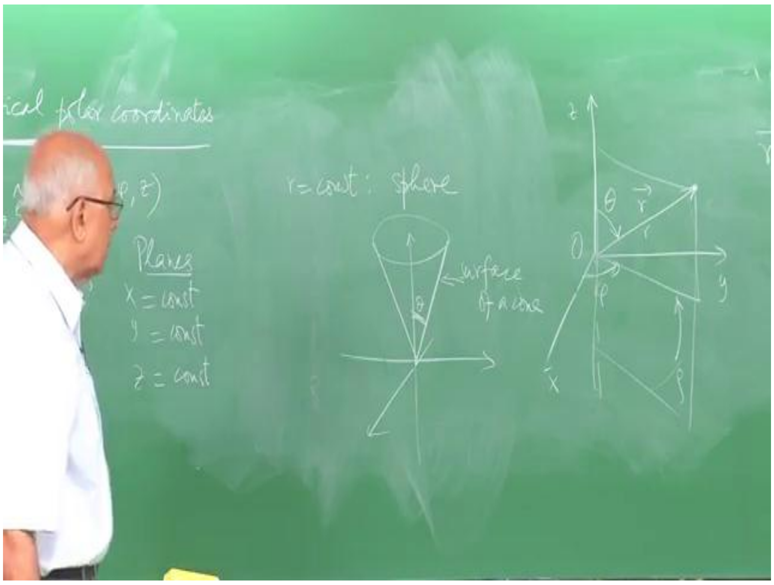

Now, what about coordinates, spherical polar coordinates? Well, it is easy to see that, r equal to constant is immediate, r equal to constant; this is a sphere of radius R, about the center at the origin. So, any time say r is constant in three dimensions, you are on the safer surface of a sphere of radius equal to that constant. What about phi equal to constant? Well, it is exactly as in the case of cylindrical polar coordinates. So, if phi is this specific value, all points in this half plane at that value of phi are involved.

So, it is like a page and this is the binding of the book. So, it is the pages of the book different angles correspond to, different planes corresponds to pages correspond to different values of phi as the book is opened out. What about theta equal to constant? Well, that is little interesting, because now, if this value of the theta is constant, then no matter what phi is and no matter what r is, theta alone is constant. And since, phi goes all the way around, the azimuthal angle goes from 0 to 2 phi.

What you have is the surface of a cone, so this is the surface of a cone, whose half angle is theta. The half angle at the base is theta and that surface which goes all the way to infinity is what theta equal to constant is. When theta equal to 0, this cone gets regenerated into the positive z axis, when theta is phi, the cone degenerates into the negative z axis. And what happens when theta is equal to phi over 2? This cone angle opens out, till it becomes the x y plane that is it.

So, in this special case theta equal to phi over 2, the surface theta equal to constant corresponds to the x y plane itself. So, it is a very simple geometrical interpretation. Now, what about the unit vectors and what about the line element? And those are interesting quantity, let look at, let us do that.

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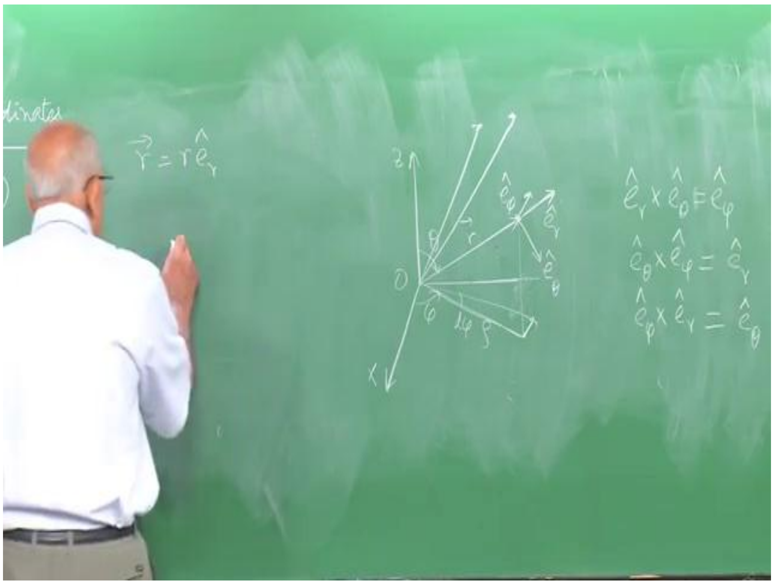

So, here is x, here is y, here is z again and we have an arbitrary point r, this is r vector and therefore, the unit vector in the radial direction is just e sub r. Just extends e sub r. It is in the direction of increasing distance from the origin without changing theta and phi. Now, this angle was theta, therefore in the direction of increasing theta, you have this vector here, which is e theta.

At this point, it is quite clear that e theta is also dependent on the coordinates and what about e phi, well if I draw my original projection as before, so I draw my projection here in this fashion and this angle is phi. And therefore, and increasing phi will take me to the plane of the board, which is equivalent to ((Refer Time: 15:51)) inside e phi; that is the unit vector, this is suppose to parallel to that.

So, now, you can see that, if I take e r and take a cross product with e theta, I am moving in it into e phi. So, I have the first of these relations e r cross e theta equal to e phi, then e theta cross e phi, so I go from e theta go into the plane of the board twist this around and it move along the radial direction, so this is e sub r. And finally, e phi cross e r. So, I start with phi and move up a screw driver turn this vector into that vector here and the screw driver moves in the direction of e theta, this x.

So, once again we have a right hand tired of vectors and we have these relations cross product relations between them of course, the dot product of every vector with itself is in unity, because an unit vectors. What about the line element itself now and what about r vector itself, well we know by definition, we know that r equal to r times e r, nothing else, that is it. But, this is position dependent, this thing here the unit vector is position dependent. So, it is derivative, they are going to be fairly complicated.

On the other hand, we could ask, what is the magnitude of a line element? So, I have two points one here and one here, this is got some distance from the origin; that is got a different distance, different r, different theta and different phi and so on. What does one do now we find out, what the line element is? Well, keep the others constant and vary one coordinate at a time and then, use the Pythagoras theorem. So, if you keep theta and r constant and shift only phi, then what you doing is to shifting by this vector is shifted to that in this direction inwards.

So, if this angle is d phi and this distance is rho, then it is rho times d phi that you shifted in the line element phi. But, remember as before we need to write down the relation between the Cartesian coordinates and spherical polar coordinates.

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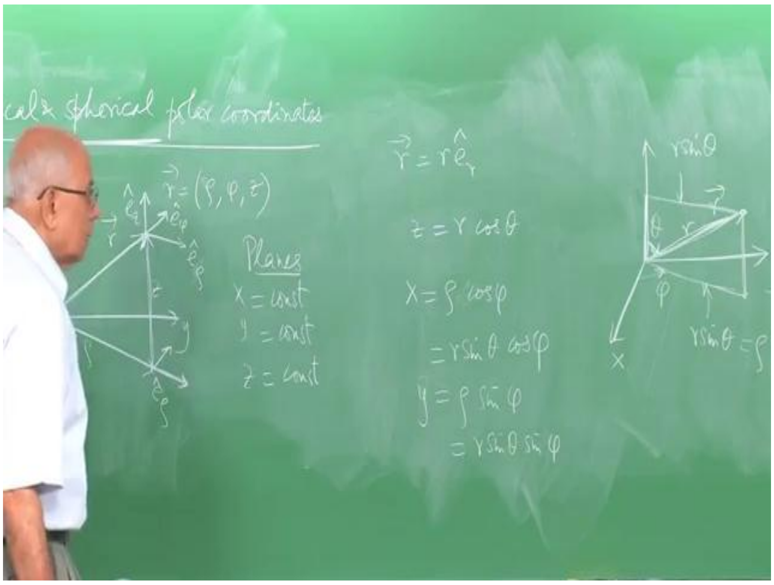

Let me do that in a minute, z is equal to r times cos theta; that is very clear from the figure itself, let us do the figure again. There is my point r, vector r, this is the distance r here; that is the projection out here, this is phi and this is theta. Now, from the way, this figure is constructed, this is the vertical coordinate is the vertical coordinate of this point about distance, about the x, y plane. And what is this point, well if this is r and this angle is theta, then the base is cos theta and that is what z is.

So, that is the first relation out here, on the other hand, this distance the vertical is r times sin theta. So, this is r sin theta and this is r sin theta, but that is what we called rho, the distance in the x, y plane to this point of projection. So, x was equal to rho cos phi, we already know that from plane polar coordinates, but rho cos phi is a same as r sin theta cos phi and similarly, y was equal to rho sin phi. So, that is equal to r sin theta sin phi.

So, these are the relations which relates the spherical polar coordinates r theta phi to the Cartesian coordinates of the point x, y, z. So, it is immediately clear that x squared plus y squared plus z squared, first of all the sin square theta phi plus cos square phi gives you 1 and then, the sin square theta plus cos square theta gives you 1. So, x squared plus y squared plus x squared is r squared, which is what you want.

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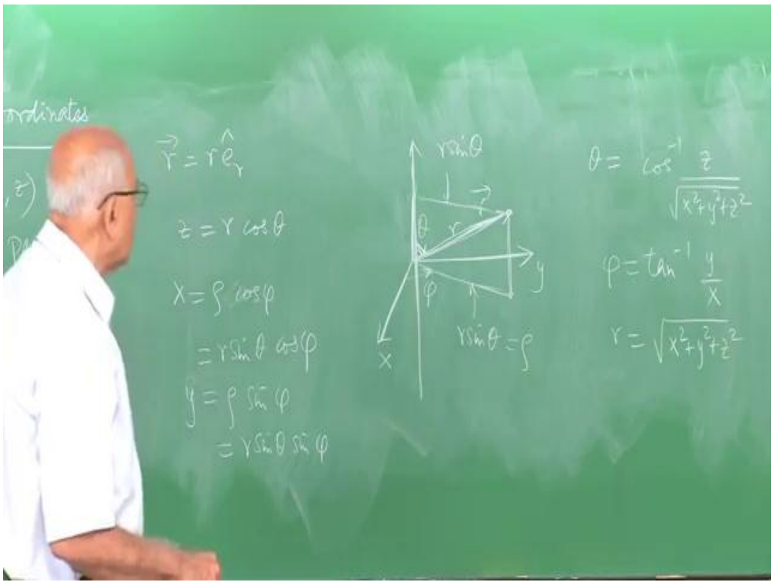

And what are the inverse relations for this, the inverse relations of equally straight forward, it is a immediately clear that the theta is equal to cos inverse, you like, if you like, does not matter, how you write this, it is equal to z divided by r. So, this is equal to cos inverse and that it in this fashion, you could also write these in terms of tan inverse. So, you could also write this as tan inverse, well from this relation, you can see that tangent of theta is this vertical divided by this horizontal.

So, it is rho divided by z and this here, which is equal to r sin theta divided over and by this quantity, by this base here. So, you can write this in a number of base, it does not matter you do. So, I will leave you to do that, phi on the other hand as before is tan inverse y over x and of course, r equal to square root of y square. And range is, we have already written down, it is important for you to remember that the polar angle theta runs from 0 to phi including the value 0 and including the value phi.

On the other hand, phi runs from 0 to 2 phi, 0 less than equal to phi less than 2 phi to make it single value here. As before in the case of plane polar coordinates, we discovered that the origin itself has no determinate, no definite azimuthal angle. There is no need to angle associated with the origin itself that rho 0. In exactly the same way, here the whole z axis, this line has in indeterminate value of phi. There is no question of saying, this is specific phi associated that this line. There are these mappings singularities, when you go from Cartesian coordinates to any other coordinate system, but we know, what they are in that quite harmless, but one has to take them into account.

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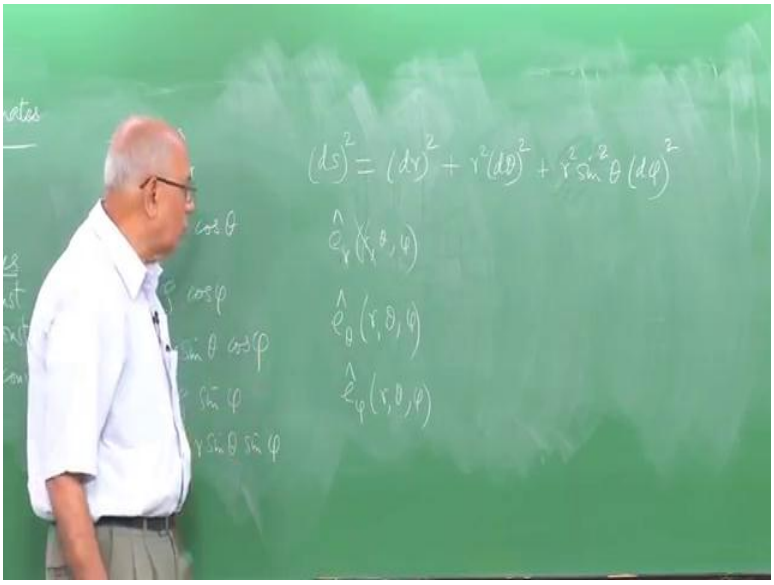

So, I was starting to say, what the line element is in this case in the way to discover the element is to use the Pythagoras theorem, exactly as we did Cartesian coordinates. So, if you keep r constant and theta constant and change phi alone, you move in you increase phi, then the distance you move by is this arc rho and then an arc d phi. So, it is immediately clear that the contribution, which you get from that portion is rho d phi the whole square, but rho is equal to r sin theta.

So, this going to a term here, which comes plus rho square r square sin square theta d phi whole square. That is the hardest term actually, the next thing is let us keep phi constant let us keep r constant and change theta alone and this is theta. So, when you increase theta you go there, this arc, this radius is r and the angle is d theta. So, the arc is r d theta and therefore, you have r square theta, whole square plus.

And now the simplest of them all, which is you keep phi constant, you keep theta constant and just change r in the same direction, you move a distance d r and that is it. We have already seen, we have a tried e r e theta and e phi and together with this expression for the line element, this tells you all that you need to do about spherical polar coordinates. It is an interesting question to ask, what is the functional dependents of e r on the position r theta phi.

Similarly, if theta on r theta phi e phi ((Refer Time: 25:22)) well it easy to guess some of these dependencies is fairly straight forwardly. First of all, if you have a point here and I have increase this that is e sub r and I go further down and increasing it still in the same direction, nothing is change. So, there is really know r dependents at all in any of these quantities.

So, write a way you see that e r cannot dependent on r, I am going to leave it you as an exercise to figure out and you will write the answer down on some stage. To figure out what is the theta and phi dependents of this unit vector and similarly, for the other two unit vector? This is an interesting exercise and it also will help you to picture, what is going on with these coordinate carefully. The important thing crucial point is the position dependents of these unit vectors. That is very important, because when you going to some another, these unit vector points we look this fashion.

So, these coordinate axis will get tilted and one as to take that into our account unlike Cartesian coordinates, where the unit vectors do not change at all. On the other hand when would you use spherical polar coordinates, just as I would use in cases of a cylindrical coordinates useful in cases of actions symmetry. Spherical polar coordinates the useful in cases of spherical symmetry, of course, where you have something which does not dependent on the actual polar angle or the azimuthal angle, but only on the radial distance from the origin.

Such as the inverse square law force, this force depends only on the distance from the origin in magnitude at least does not depend on in which direction you are in with respect to the center of attraction or impulsion. And then, it is very useful and such cases to use spherical polar coordinates. We will do so very often whenever we deal with a problem is spherical symmetric. The most important case of all in mechanics would be where this is the force of attraction or impulsion from some center of force and it is very conveniently chosen to the region and the problem is most conveniently solved by looking a spherical polar coordinates.

But, for that one needs to know how to use the unit vectors in spherical polar coordinates, how to write down the line element and so on, which is why are taken this mathematical migration to this other coordinate systems. But, we will revert to physics very shortly.