Mechanics, Heat, Oscillations and Waves

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Lecture – 12
The Finite Rotation Formula

We are going to continue with little more mathematical preliminaries, especially with regard to vector analysis. Because, this is going to be the sort of basic language in which Mechanics is going to be study and therefore, it is work understanding these very clearly. I am going to complete a couple of things, which I started talking about the last time and which we are not gone to the end of.

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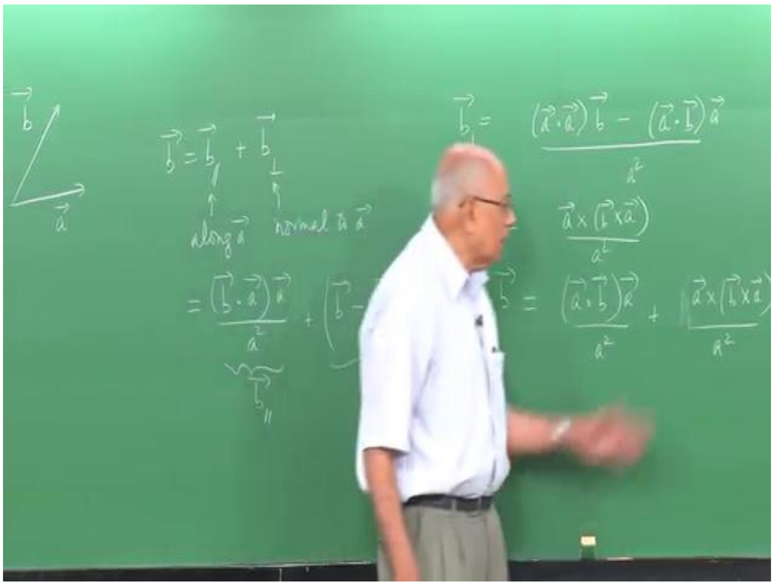

In the first of these, I have to do with resolving a vector, a given vector b along some specific given vector a. So, the question was given two vectors in three dimensional space, what is the way in which you can resolve this vector b into a component along the vector a and a component in a plane perpendicular to the vector a. And the answer was that you could write b as b parallel plus b perpendicular.

And this was along a and this component is normal to a and b parallel, we had a very simple formula for and this formula was simply to take the component of b along a, multiplied by a and ensure that, these are unit vectors by dividing by a squared. So, that is b parallel, the question is what is b perpendicular and the best we could do at that stage was to write this as b minus b dot a, a divided by a squared.

So, this part is b parallel and this part is by definition b perpendicular, this was just an identity. But, now that what we know, what the triple cross product does this can be written down in a slightly more simple form and that is the following b perpendicular is moving the a squared to the top and writing it as a dot a. We have a dot a b minus a dot b a divided by a squared taking this common factor out.

And of course, you recognize the fact that because we have a formula for the triple cross product which we derived the last time, you could also write this as a cross b cross a divided by a squared. So, we have a very elegant formula for resolving the given vector b in the form of this nice little formula a dot b, a over a squared plus a cross b cross a over a squared. So, the dot product appears in the parallel path and the cross product, the triple cross product appears in the normal path here.

So, that is the useful formula to remember, not necessary to memorize it, you can actually derive it whenever you need, it is quite easy and straight forward do so, but it is a very simple looking formula out here. I thought of I mention this just so that you know, what the resolution of the vector along some other direction means. This part is sometimes called the longitudinal component along a and this is the transverse component in a plane perpendicular to a. Having done this, the next thing I want to mention is an important formula in vector analysis, which is called the finite vector formula and the finite rotation formula and it goes as follows.

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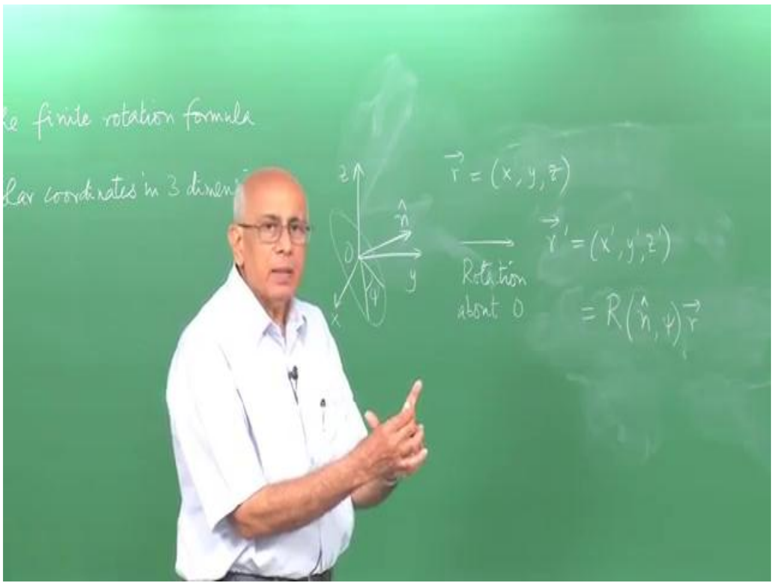

We have already mentioned, I have already mentioned several times that if you give a coordinate system O with x, y and z axis and we rotate this coordinate system to produce another orthogonal coordinate system. The new Cartesian axis x prime y prime z prime, such that any position vector r, which was originally x, y, z in Cartesian components goes under this rotation about the origin.

So, rotation about O, it goes to a point r prime or new coordinates r prime, which are given by x prime, y prime, z prime where x, y, z determine, what x prime, y prime, z prime are according to the action of some 3 by 3 matrix called the rotation matrix. So, in symbols, we said that r prime was equal to rotation matrix defined by the axis of rotation. So, whatever direction of rotation you had in space to rotate about and then, in the plane perpendicular to this n, the rotation was by some angle psi.

So, the rotation is parameterized as we say is described or specified by specifying the unit vector, which tells you the direction or axis of rotation and the amount by which you rotate, which is this angle psi. This matrix, when it acts on r written as a column vector produces the new coordinates out here. In general, this is hard to write down, it is quite a messy thing to write down.

But, it turns out that you can actually write down a simple formula called the finite rotation formula, which summaries the effect of this r, of this rotation matrix on any given point r. And therefore, on any vector which you start with and it tells you, how a vector transforms under a finite rotation about some specified axis and through an angle psi and here is the formula.

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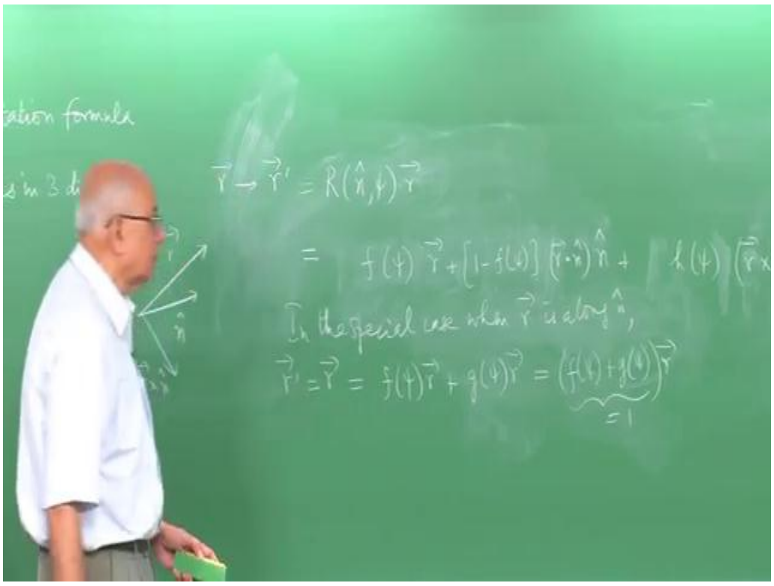

So, r goes to r prime, any r goes to r prime, any point coordinates r go to r prime, which is R specified by n inside acting on r equal to… Now, what can it possibly be equal to? Well, we know that the origin is left unchanged. So, if r is 0, namely the components of r are 0, 0 and 0, then r prime is also be 0, a rotation is a linear transformation. So, each term in the right hand side must be proportional to the first power of r, it cannot depend on r squared for instance or any more non-linear function of r. It must depend on n and it must depend on psi.

But, you see if you have an arbitrary vector r and some arbitrary direction n in space, these two in general, these two vectors would specify part of a triad, this is third vector which comes out to the plane of n and r. And that is the vector r cross n, which might perhaps come out in this fashion. Now, together the vectors n, r and r cross n would actually determine a kind of basis in three dimensional space. So, this vector is r cross n and it is normal to the plane from by the vectors r and n.

So, it is clear that this r prime being a vector in three dimensional space must have a portion which is along r, a portion which is along n and a portion, which is along r cross n and there is nothing else possible. So, there must be something here multiplying r plus something here multiplying n plus something here multiplying r cross n. Now, what goes in here is what we have to determine by the physical argument.

Now, one can derive this formula recursively completely from, have to ensure starting systematically from this starts, specifying this rotation as a proper 3 by 3 matrix. But, what I am trying to get is a simple physical argument, which practically enable you to guess the answer inside here, based on just physical reasoning. Now, this term here, I have already said should depend on r, should be linear in r, but the vector was point along n.

So, it is clear that this is the component of r along n and therefore, whatever is here must be of the form r dot n times n. This is the scalar product and it is proportional to the magnitude of r, which enables us to satisfy this requirement, that every term on the right hand side must be linear in r. In other words, you multiply little r by 6, the whole answer must get multiplied by 6, each term must get multiplied by 6.

Now, what can be inside here, cannot depend on r, cannot depend on n, because this vector n, all it dependence is already been explicitly stated. So, all it can depend on is this angle psi. So, there must be some function of this angle f of psi plus some other function of psi, g of psi plus some other function of psi, h of psi and that is what the finite formula must look like, the rotation formula must look like for any arbitrary r.

Now, what is it that we can say further? Well, we can simplify things a little bit by pointing out, that if it should so happen that r is along n itself or n is along r itself. So, looking at all those points which are along the axis of rotation, which is quite clear that these points on the axis of rotation do not get affected at all. So, if I have a point along the axis of rotation, those points do not get changed at all.

So, r prime must be equal to r in that case and what is that imply. Well, r cross n is 0, if r is parallel to n, because the sign of the angle between them vanishes, the angle vanishes. So, this term goes away, r dot n, n is simply r itself. So, in the case, in this special case, when r is along n, you must have r prime equal to r, must be equal to f of psi r plus g of psi, r dot n which is the same as magnitude r, multiplied by n which is the same as the vector r, equal to f of psi plus g of psi times r. And therefore, this combination must be equal to 1.

So, in the general case, whatever happens these two functions must be such that they add up to unity. So, we can write down now, we can eliminate one of them and simply write the finite formula as plus 1 minus f of psi times that. That is already got rid of one of the unknown functions, we have just two unknown functions now.

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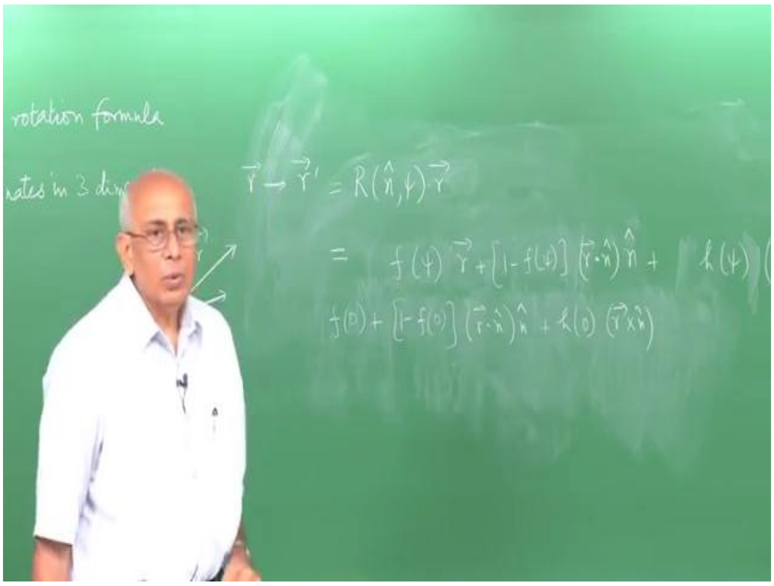

What else can we do? Well, when you have no rotation at all, it is clear r prime must be identically equal to r for every point r and therefore, if you have no rotation at all, f of 0 plus 1 minus f of 0 and in this case psi is 0. So, 1 minus f of 0, r dot n, n plus h of 0, r cross n. In this case must be equal to r itself, no matter what n is and what is that tell you, it is very clear that immediately implies certain things about f and h.

For instance, if you have no rotation at all and I am sitting here and that say it, no rotation at all and then, r prime is identically equal to r or if I rotate through 0 angle, it is the same thing as I am saying I rotate through 0 angle. You have f of 0 here, you have h of 0 and it is trivial to see that h of 0 must be 0, f of 0 must be equal to unity.

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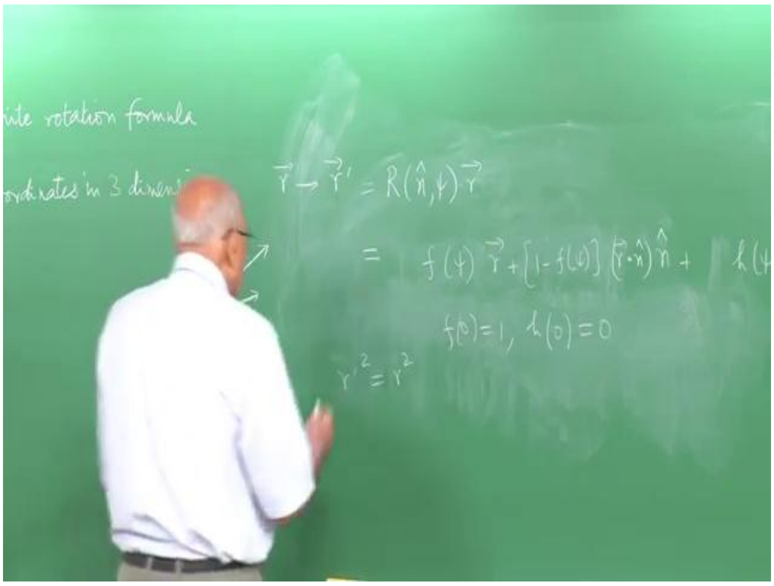

So, those are little inputs which one can put in and let us f of 0 equal to 1 h of… So, that this term vanishes h of 0 equal to 0. And then, you can use further symmetric properties such as the fact that this must be, must change sign when psi changes from plus psi to minus psi. Well, you can also ask, can I impose the condition r prime squared equal to r squared, so dot place with itself and then, we use the fact that under a rotation the distance from the origin to any point does not change at all. So, you have to put this in as well and this enables us to practically guess the answer, I am going to write the answer down and tell you exactly, what this looks like.

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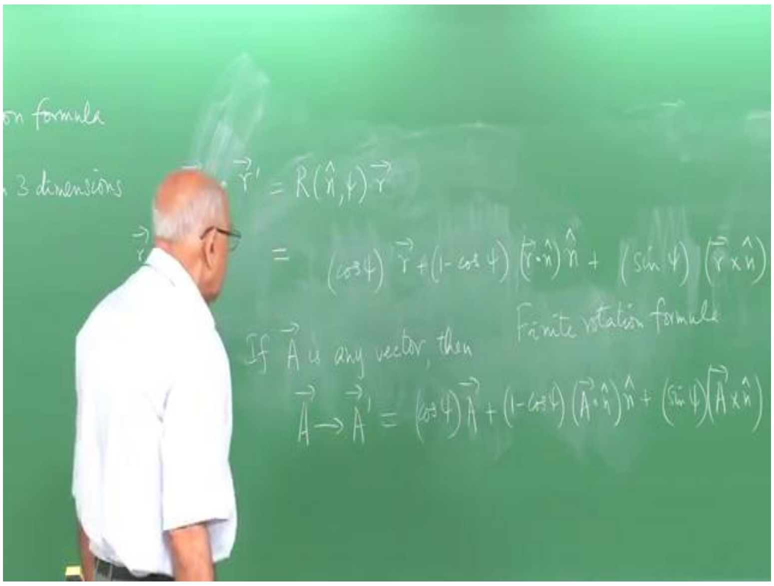

This turns out to be cosine psi times r plus 1 minus cosine psi times this quantity plus as we might guess by now sin psi times r cosine. This is called the finite rotation formula. It is a very compact and simple expression and telling you, how under an arbitrary rotation about the access n, through an angle psi. Any the coordinates of any point, the position vector of any point, transforms some new value, which is given by this formula here. Three terms, one propositional to r, one propositional to n, one ground that direction of r cross n.

Now of course, we know that the vector is a set of three quantities in three dimensions, which transforms precisely the way the coordinate transform. If A is any vector then under the rotation of the coordinate axis, the new components of A are related to the whole components of A, the precisely this formula. A goes to A prime, which is equal to cos psi times A plus 1 minus cos psi times A dot n along n plus sin psi times A cross n.

You have grantee that by the definition of the word of the vector, what is mean by a vector, a very useful and extremely practical formula, a very compact one which enable you to short circuit do not have to go through detail lodgments to find out, what a the new vectors are, new components are, you can use this formula. Now, once you have this, then many other things can possible.

If this is the position coordinate, then I can ask what happens to the velocity, what happens to the acceleration and so on under a rotation of the coordinate system. In particular and we do this subsequently, one can ask, if I gone rotating about this access n at a constant angular speed omega, then this psi increases with time like omega t, it is just starting from 0 it is equal to omega t increases with time.

So, I put that in here and then you see that the new coordinates a changing with time. Now, if you have this as the position of the new of a particle in the original coordinate system, one can now predict, what it will look do in rotating coordinate system. And then, one can look at what the acceleration does not and so on and will subsequently see that the whole idea of inertial forces or non inertial forces or Pseudo forces immerges from this formula here in a straight forward way.

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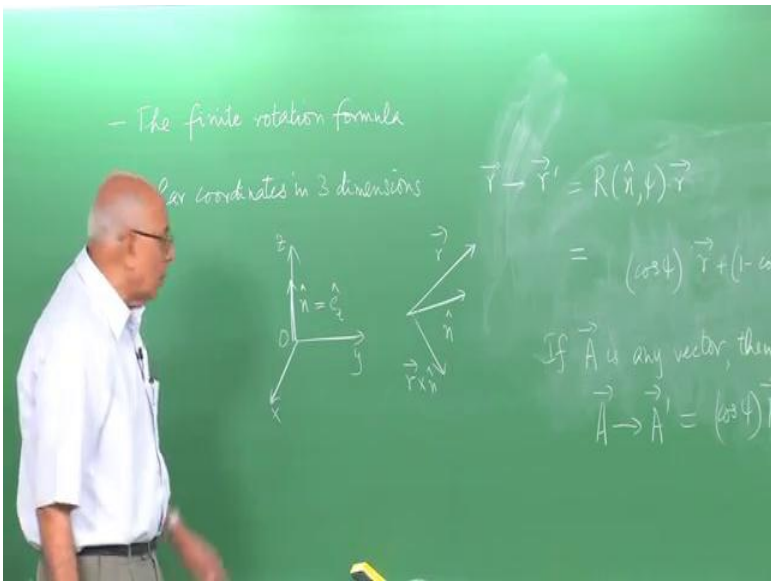

Mean, while before we do that, let me just point out, that this formula reduces in the special case, when you had a rotation in the z access along the z access. So, now, I am go to look at this situation x, y and z; that was the origin and I am going to look at a rotation along the z access itself, so that is the unit vector n, which is equal to the unit vector in the z direction. And ask, what happens to this formula, well it is immediately clear that we expect nothing along the z coordinate to change.

So, the z coordinate, the vertical coordinate of no particle should change at all and the x and y coordinates should get mixed up according to rotation in a plane and for that we already have formulas. So, is this true, this is coming out and the answer is yes.

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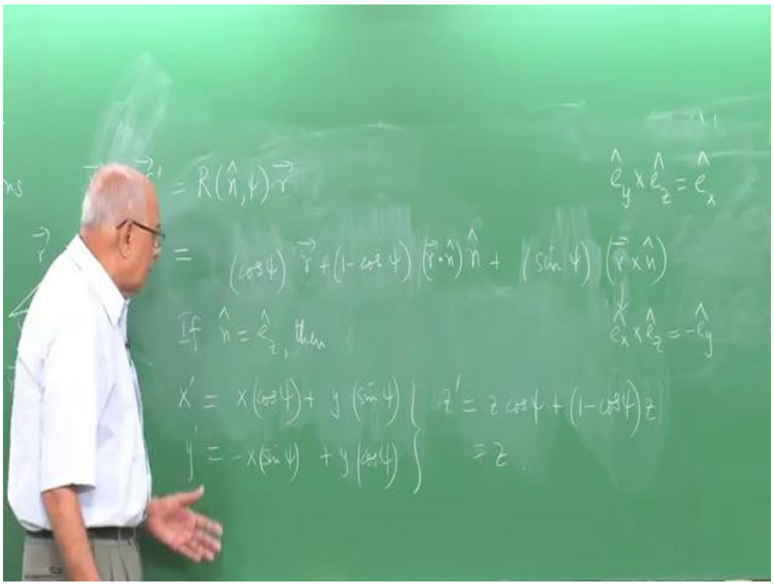

Because, if n equal to e z, then let us look at what is the new coordinates are and let us look at the x component of it. So, this is equal to x prime and that must be equal to the x component of r, which is x itself, x cos psi plus 1 minus cos psi r. Now, we looking at a x component, so this is prepositional to e x, got it with n is that is 0 plus sin psi that comes in plus sin psi and then, we looking at x component e x cross with e z out here goes to e y with the minus sign.

On the other hand the e y cross with e z is going to give you e x. So, can we do that, the y component here cross with e z gives you e x. So, I am using the fact that e y, so the y component of this r is of course y and then, there is a sin psi; that is it. On the other hand, the y prime component will have the y prime component here, so this is y cos psi have return this out e y dot e z is 0 once again and then, e y cross n is going to give me.

So, it is the x component that is going to give me a y and therefore, this is a minus and there is no z here, there is no z here. And what about z prime, again z cos psi plus this is z, e z dot e z, e z got e z is 1 and then, z times n out there that gives me z itself and then, e z cross e z is 0, so that does not contributes at all and this is equal to z. So, this checks out, it essentially tells us that if you rotate about z access, points the z coordinates of points do not change and the x and y coordinate get mixed up according to this plane trigonometry formula. So, the general formula reduces to this special case in that case we looked at earlier from simple geometry. But, this is the general formula here, a very useful one and we will have a occasion to come back to this subsequently.

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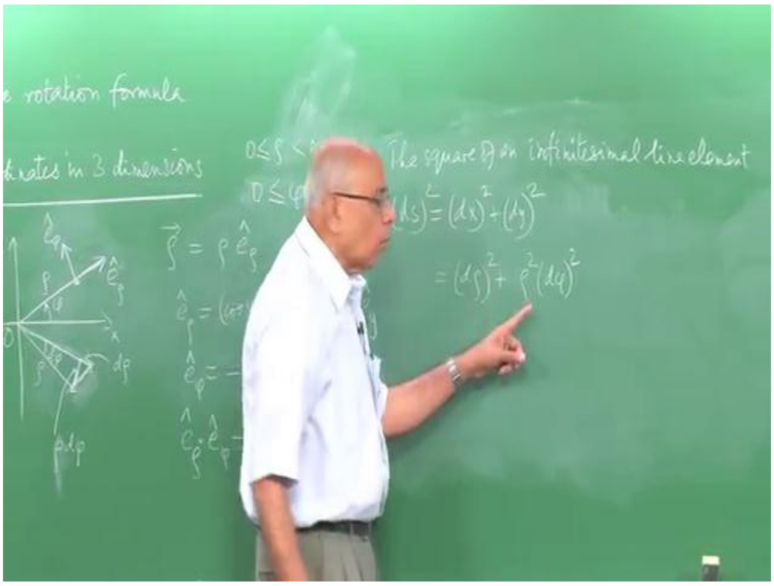

The next topic now is to look at polar coordinates in three dimensions. Again, the topic of a great important, so let us look at polar coordination in three dimensions. And just to refresh your memory, we looked out plane polar coordinates in the x, y plane with the origin here. And the statement was, if you gave me a arbitrary point there with coordinate x and y in Cartesian coordinates, I could look at it in terms of distance from the origin, which I call rho and the azimuthal angle phi angle made from the this access to this radius vector.

And we had the radius vector of any point was rho times e rho, the unit vector in the direction of increasing rho and we have a orthogonal vector in the direction of increasing phi. So, we had that thing there and the unit vector e rho, because equal to cos psi times e x plus sin psi times e y, whereas the vector e phi was equal to minus sin psi e x plus cos psi e rho. We also know that e rho dot e phi equal to 0, we are perpendicular to each other that every point.

We saw some length e rho is a function of psi of the position of the azimuthal angle of the point as is e phi, according to these formulas here, this is e y, e x and e y the same at all points, so that is e x, that is e y and it remains the same wherever you are in the plane. So, this much we are already seen from plane polar coordinates. What also remain was that in the infinite plane the ranges are 0 less then equal to rho the less than less then infinity and phi 0 less than equal to phi less than 2 phi in order to make phi single value, because 0 was identified with 2 phi.

One little point got left out and that is the distance between two points. So, if you have two points nest to each other, a point here and a point here and ask, what is the square of this distance in Cartesian coordinates, this is very simple. So, if I call this distance d s, that is d x and d y. So, it is immediately clear that d x squared by the Pythagoras theorem is d x squared plus d y whole squared.

But, you want to do it in terms of plane polar coordinates; you have to argue slightly differently; that is the rho for one point; that is the rho for other point, the neighboring point. There is a change in angle which is of the order of d phi, which is d phi here and this distance, if this is rho, when again to exactly the same thing as before. They draw circle of this kind of radius rho, then the reason there is a line element, this thing here this is square of this quantity is what you want to find. It is equal to some of this square plus this square.

But, this sum of this square here is purely due to the increase of rho, so that is d rho. On this other hand, this quantity here is rho times d phi, whereas that portion is d rho. So, therefore, the square of the line element can also we return as b rho squared plus rho squared d phi squared in plane polar coordinates. This square of an infinitesimal d s squared is given by these two quantities and dimensionally, you can see this is the all right, because this is length squared, length squared, length squared, length squared dimension less quantity here.

It is important to recognize that there is coordinate dependence in the line element here as soon as you go to any coordinate system other than Cartesian coordinates. This is absolutely crucial. Now, the extension of the this two third dimension can be done many ways, but the most trivial extension is to simply say this z coordinate is not effect at all.

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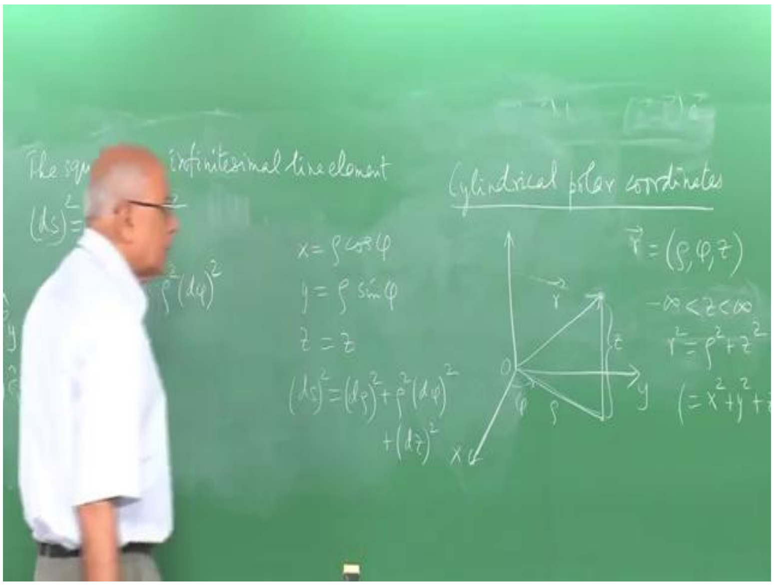

So, what one does is to introduce this cylindrical polar coordinates. Now, you do exactly what you did earlier for plane polar coordinates, so this is the x y plane on which use plane polar coordinates. And if I have a arbitrary point here with the radius vector r in three dimensions, then the cylindrical polar coordinates would corresponds to join the projection of this on to the x, y plane here; that distance is rho as before. This angle from the x access is the azimuthal angle phi as before and the third coordinate is this height here, which is the same as this height here and this is z.

So, position vector of any point is to describe in cylindrical coordinates by rho phi and z this try at of quantities. The ranges of rho and phi are exactly as over here and 0 minus infinity less than z less than infinity, z positive is above the x, y plane; z negative is below the x, y plane in this fashion. So, the connection to plane polar coordinates in that to the Cartesian coordinates is obvious is exactly as it is in plane polar coordinates in the sense that x is rho cos phi y equal to rho sin phi.

And of course, z in Cartesian is the same as z in cylindrical polar coordinates in this fashion. What is the line element now; that is as a remains, the line element is this square of two points, you can see by the Pythagoras theorem that r squared is rho squared plus z squared. So, you that r squared rho squared plus z squared and rho squared is x squared plus y squared. So, the geometry is extremely straight forward this square of the line element is d s squared equal to d rho squared plus rho squared d phi squared plus d z squared, this function. The unit vectors are interesting and this is what we will look at next.