Mechanics, Heat, Oscillations and Waves

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Lecture – 11
Vectors in 3 - Dimensional Space

Now, let us do something about a vector analysis, which is of great practical importance whenever you look at the vectors of different kinds, different vectors in a given problem and here is the general problem.

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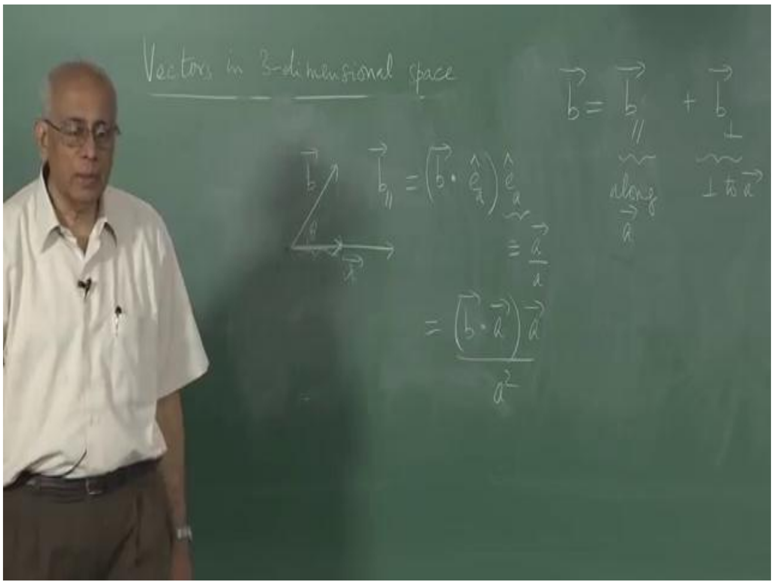

Suppose, I give you in three dimensional space a vector a and another vector b, arbitrary vector b in this fashion. I could ask the following question, how much of this b points along a, what is the projection of the vector b along a, what is the component of b along a; that is a question which arises in practices very, very often. Well, one way to do this would be to say, I write b in Cartesian coordinates, b x, e x, b y, e y plus b z, c z.

Similarly, for a and I take the overlap between the two which is b dot a and try to discover how much of b is along a, using the fact that e x dot e x is 0 and e x dot e x is 1, e x dot e y is 0 and so on, that is one crude way of doing this. But, there is a more elegant way of doing this and that is what I would like to show you today. If I project this vector on to this vector, it is equivalent to say I draw perpendicular from there and let me draw a better figure.

Here, if I project this b on to a; that is like saying draw perpendicular and this length here is the projection of b in the direction of a. Now, you can easily write down what that is, because the angle between these two vectors is theta. When, b dotted with the unit vector in the direction of a in this direction is of a, is of course the magnitude of this length here and the direction of this segment, line segment is along e a. So, this here is e prime, this is a scalar, this is a number and that multiplied by the unit vector will give me the actual vector.

So, that represents this vector in the tip ending here at this point. So, what I have done is to write this b vector as b along a. So, let me call it b parallel and the rest will be of course, perpendicular to a and let me call that b perpendicular. So, this is along a and this is perpendicular to a, that could be in a plane perpendicular to a anywhere, the perpendicular component could be anywhere, because b could point anywhere starting with this a here.

So, the target now is to write down formulas for b parallel and b perpendicular and b parallel is extremely simple to write down, this is equal to b parallel. Of course, we do not like me to write the unit vector in the direction of a, it is easy to see that this by definition is equal to the vector a divided by the magnitude of a. Because, the vector a is the magnitude of a multiplied by the unit vector in it is direction.

So, I can write b parallel to be equal to b dot a times a divided by a square, because there is one a coming from this unit vector and another a coming from that unit vector. So, I have a formula for b parallel, it is very straight forward. Now, what can b perpendicular be, well it is clear that b perpendicular is b minus b parallel.

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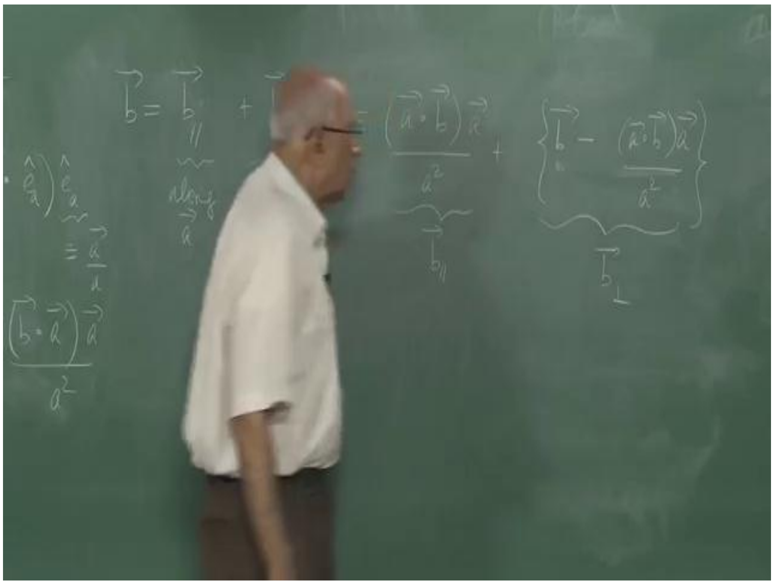

So, it is immediately obvious that I write an identity, this is identically equal to let me write it as a dot b, a over a square plus the vector b minus the same thing a square. So, this is b parallel and this part is b perpendicular and the question is, is there some simple formula for this portion is written in more compact way than in this rather in easy notations. It is just an identity, all we have done is to resolve this vector into two components, one of which is along a and the other is in a plane perpendicular to a, somewhere in this direction.

In this case, in this figure, it would something like that, that would be the perpendicular component and the two added up would give me this vector b. So, this is called the resolution of a given vector b, along some other vector a, any other given vector a. And what we seeking is a formula for this portion, which makes it look to very straight forward simple way of writing this resolution here.

Now, notice I am not chosen any specific coordinate frame here at all, because it is not necessary, we working entirely in terms of dot and cross products and that is enough. Now, what is the next step? Well, that requires a little bit of an identity and I am going to spend a little bit of time talking about this identity; that is the following.

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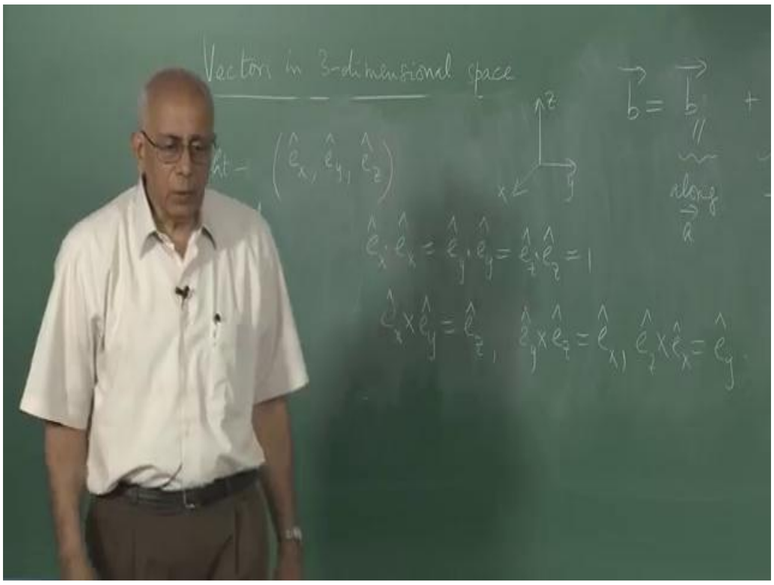

But before that, I need to do the following, I need to point out that in the Cartesian basis e x, e y and e z the unit vectors and these vectors satisfied the property e x dot e x is equal to e y dot e y is equal to e z dot e z equal to 1. And moreover, they form what is called a right handed system of a unit vectors, in the sense that e x cross e y is equal to e z, which says that here is x, here is y, here is z.

So, if you took the unit vector in this direction and you rotated it to 90 degrees up to e y, then the screwdriver, the right handed screwdriver would move in the direction of z. So, that is a right handed coordinate system, this forms a right handed triangle of unit vectors and so on. This is in slightly permutation e y cross e z equal to e x and e z cross e x equal to e y. I should mention here that in three dimensions, once you have x, y, z in cyclic permutation, then you can actually write down the value of any cross product of two vectors the way I write down for a and b.

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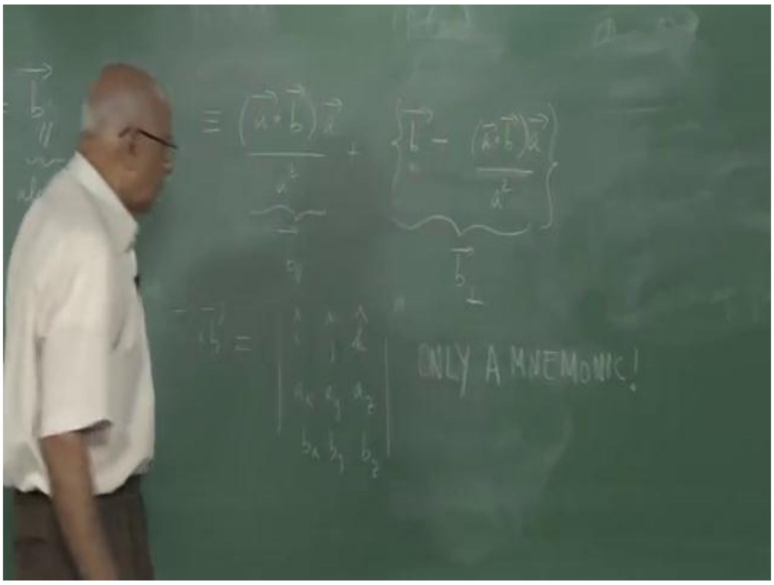

But, it is useful to have a mnemonic device for this and the one that is most common is to write, I think like and this is a standard thing in text books. If this has components a x, a y, a z and similarly for b, then this is written as the determinant i, j, k; a x, a y, a z; b x, b y, b z. I am going to put this in quotation marks, because it is not a great idea to write it in this fashion, it is very misreading. Because, you cannot have a determinant in which the first row consists of unit vectors and the other two rows consists of components of vectors.

In a determinant which is the proper determinant in the correctness of the word, the nature of all the elements should be exactly the same. You cannot have a determinant in which, one row has one kind of element and another row has another kind of element or anything like that mathematically speaking. So, this is only a mnemonic, it is not a rigorous formula at all.

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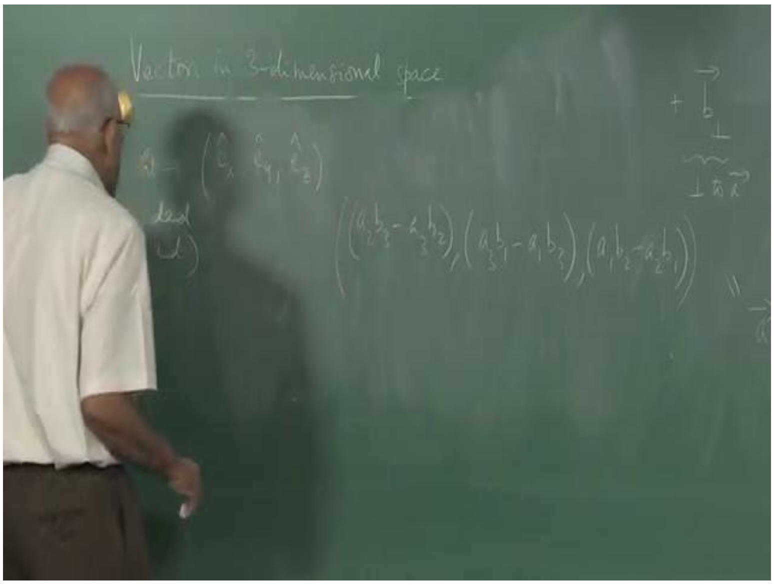

It is easy enough to write down what the cross product is, because I know that if I have two vectors a and b, then the x component of the cross product is a to b 3 minus a 3, b 2 and then, it is cyclic permutations of 1, 2, 3. So, it is a 3, b 1 minus a 1, b 3 and then, a 1, b 2 minus a 2, b 1; where I have call the vectors a and b or said the components are a 1, a 2, a 3 and b 1, b 2, b 3 instead of x, y, z.

So, it is easy enough to remember this and write it down and I do not find that this is particularly helpful, but this is written down in elementary text books very often. But, it is also misreading, because it is certainly not a properly determinant problem. Now, the next thing we need is the following.

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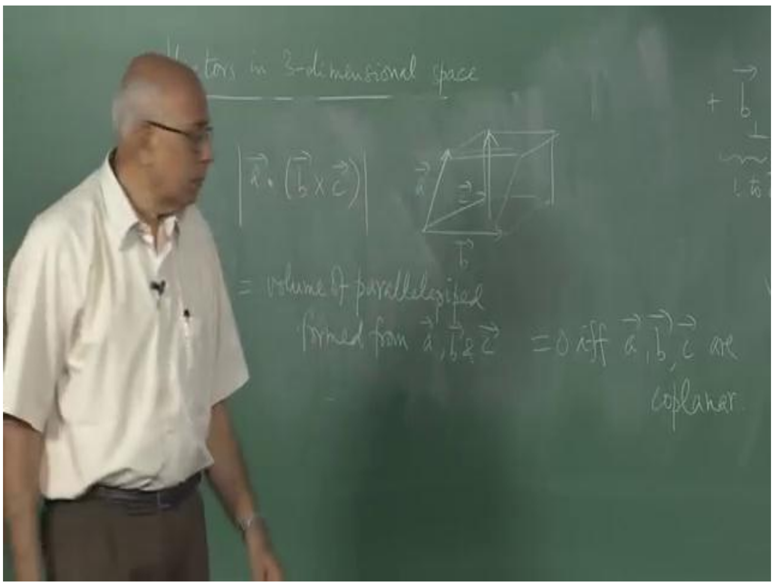

We have seen what the geometrical meaning of a cross b is in terms of the area of a parallelogram. Now, the questions we could ask them is, what is this quantity a dot b cross c? If you give me three vectors a, b and c in space, three dimensional space, what is the meaning of a dot b cross c? Well, that is not very have to figure out, because if these vectors are in this fashion v for example, so here is a, here is b and that vector c going into the board in a limit.

Then, b cross c as you know is the area of this parallelogram and you are dotting it with a, which means you taking the projection of a along the direction normal to this parallelogram. But, that in magnitude is precisely the volume of a parallelepiped form from these vector, not very good at drawing this, but it is precisely the volume of this parallelepiped from a, b and c.

So, the magnitude of this quantity, because the sign of this quantity, it is a scalar could be the plus or minus 1, but the magnitude of this quantity is equal to volume of parallelepiped form from a, b and c. And when would it be 0? It would clearly be 0, if this dot product vanished and when would it vanished, it would vanished if this vector was perpendicular to the vector b cross c.

But, what is that mean, b cross c is in a plane normal to the plane form by b and c and this vector is perpendicular to it, it means it is back in the plane form by b and c. So, it is equivalent to say that I have a parallelepiped in which the third vector, instead of sticking how to the plane form by the first two is in the plane itself. So, the volume vanishes, the volume is identically 0.

So, in the case in which a, b, c are coplanar, this is identically equal to 0, equal to 0 if and only if a, b, c are coplanar, that is a useful piece of information to know. We could also write down, what are the component, what is the actual magnitude of this quantity here, if you write in terms of the components of a, b and c.

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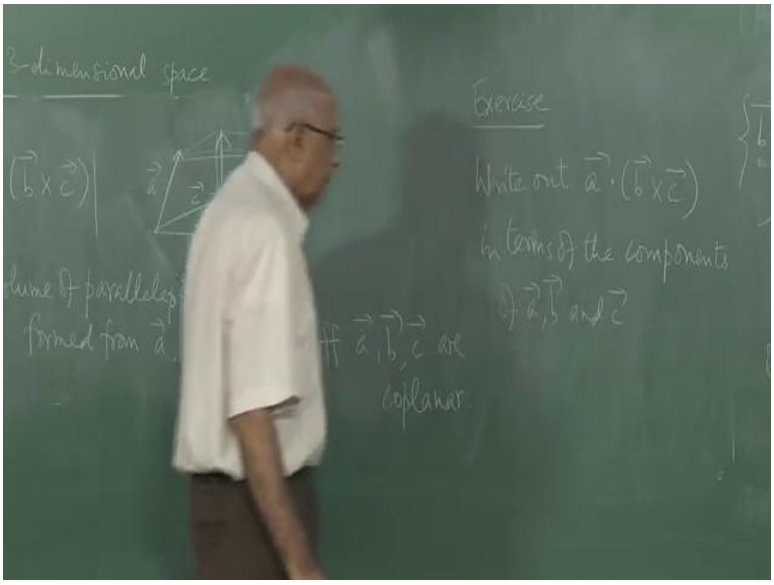

So, that is an exercise left to the student to you, it is an easy exercise. So, here it is exercise, write out a dot b cross c in terms of the components.

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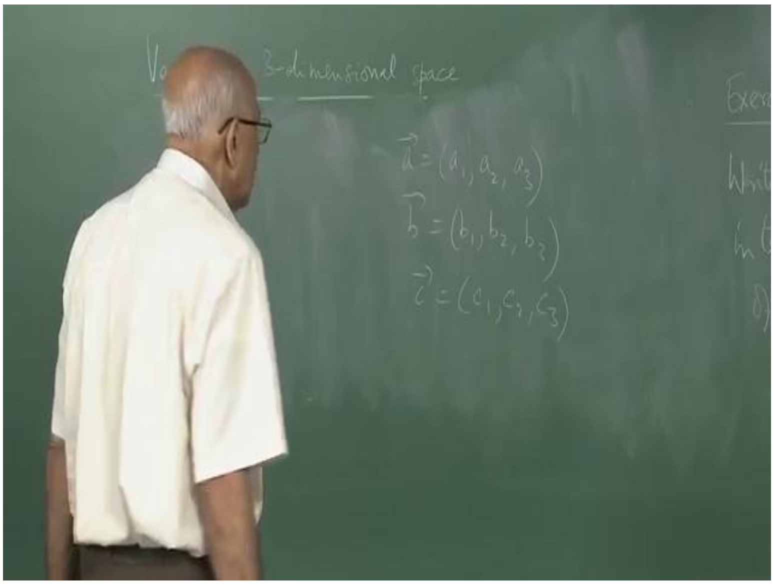

So, write it out in terms of the components of a, b and c, which means you take a to b. Let us write it as 1, 2, 3, a 1, a 2, a 3; b is equal to b 1, b 2, b 3 and c to be c 1, c 2, c 3. These are the Cartesian components of any three vectors a, b, c. In general, take it to the non planar in some arbitrary directions in space and then, write this combination out in terms of a, b and c, write down, what this volume of this parallelepiped is.

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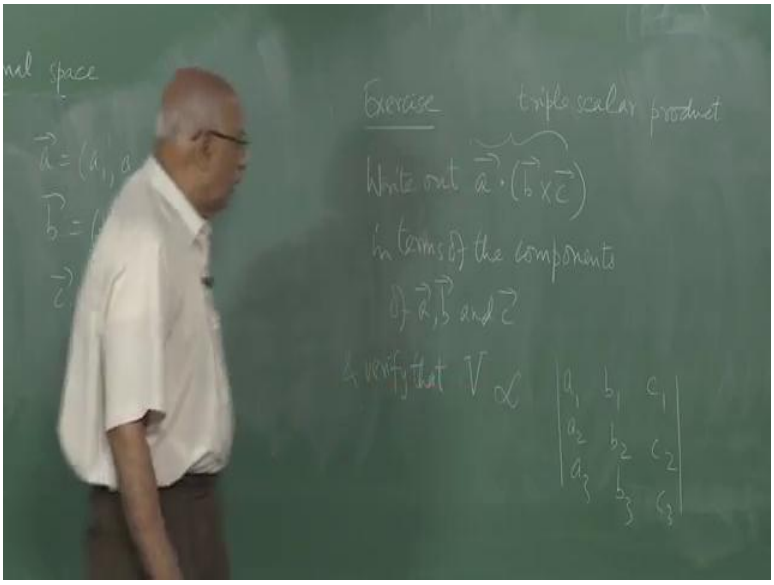

And verify the volume; let us call it the volume of this parallelepiped, that is what this quantity is, then magnitude that this volume is proportional to a certain determinant form from the components of a, b and c. So, verify that this is a proportional to a 1, a 2, a 3, b 1 or let us write it properly, this fashion into the matter a 1, a 2, a 3, b 1, b 2, b 3, c 1, c 2, c 3. So, this so called triple scalar product, triple scalar product has a very simple geometrical meaning.

It is related to the volume of the parallelepiped from this three c vectors and that volume in turn has a very simple expression in terms of the determinant form by writing the column vectors down as column vectors next to each other and then, taking the determinant of the result. So, all this a facts of important geometrical a significant and the lot of practical use as well, generalizations to high dimensions etcetera, etcetera exist.

So, this is something which is worth very fine that the so called triple scalar product is related to the volume and the volume can be return in form this. By the way, this is a reasonable thing, because you can already see in a dot b cross c, if you took the first component of it, the x component, so this is a 1 and b 1. The one component of this which is b to c 3 minus b 3, c 2 and indeed when you expand the determinant, you see you get a 1 times b 2, c 3 minus b 3, c 2. So, that is the first component already there.

Now, verify that the other two components add up to exactly this determinately up to a sign up to in overall sign.

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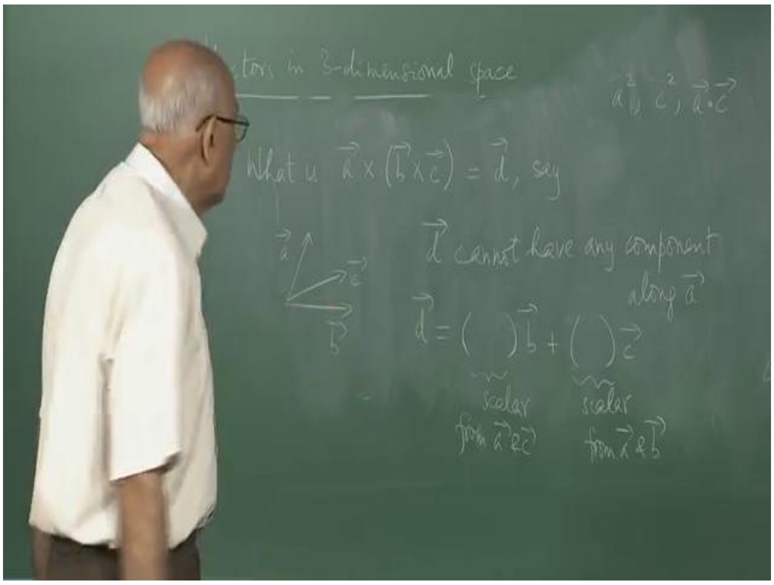

The next thing we need to look at is what happens, if I took the cross product of a cross product. So, what is a cross b cross c, again I emphasize that while this might look like just playing around with vectors and doing vector algebra, instead of doing series physics and like to say again and again that the physics is buried in it. We are going to use all these things in physics at various places. This is very basic mathematics and you need this for the proper understanding of physics.

So, what is this quantity equal to, well one way to do this would be very laborious, write down the components. So, write down first b cross c in Cartesian coordinates and then, write down a in Cartesian coordinates and use the formula for the cross product of two vectors and they should give you the answer completely. But, now I would like to show you that this no need to do that, you can actually use a much simpler trick, so write this down and it goes as follows.

Again, I draw a little picture and say, suppose this is b and that go inside is c and that is some vector a, write to know what is quantity is, it is a vector, the answer is a vector. Now, what I do know is that a cross any vector is perpendicular to the plane form by a and that vector. So, in particular it is perpendicular to a, therefore, this vector, whatever it is cannot have a component along a, because it is perpendicular to it, because of the cross product.

Let us therefore, call this vector d and the argument is the d cannot have any component in the direction of a, it cannot have any component in the direction of b cross c either.

But, they so little more complicated, then it say cannot have any component along this vector. Now, in three dimensional space, if I have three non coplanar, non co linear vectors, then the coordinates of any point can be written as some multiple of the unit vectors in the first direction.

The second direction and the third direction, independent of whether this directions of perpendicular to each other or not, I can use ably coordinates in a plane or in three dimensions. So, in general, I can write down the coordinate of any point, the position vector of any point, I can express as a linear combination of numbers times the vectors b c and a in general.

But, now I am saying that this vector d cannot have any component along a, therefore, it can only have components along b and c and therefore, this answer must be d must be of the form something times b plus something times c and these something must be scalars. Because, the vector part is already taking care of by b and c, so this is a scalar and this is a scalar.

Now, what can the scalars be, well the left hand side here has it a to first power and therefore, the right hand side must also linear in a, there is a b to the first part. So, it must be linear in b, it must be linear in c, but this already b here. Therefore, this term here must be a scalar formed out of the other two vectors a and b, a and c, this no other choice. So, this must be a scalar from a and c and this must be a scalar from a and b.

Let us look at this first, what are the possible scalars you can form two vectors a and c, well a dot a are a squared is one of them, c dot c or c squared is one of them and a dot c is one of them. That is it, there are no other scalar which a independent, which can be form from two vectors a and c. Everything else a function of these combinations c, but this side this quantity must be linear in a, if I multiply this left hand side, if I multiplied by a by 6, the answer must multiply by 6.

And the only happens if it is linear in a here and linear in a here completely. Therefore, this is rolled out, that is rolled out, no non-linear terms along, it is got to be a dot c. So, this must be a dot c and I put it in bracket to tell you that it is a scalar a dot c times b. But, there is nothing to stop this from being square root of phi times a dot c or 18 times a dot c we do not no.

So, this some constant lambda times a dot c, absolute constant, because independent of what a, b, c are this thing here must be the same number, it cannot depend on a, it cannot depend on c, it cannot depend on d. So, therefore, it is got to be a pure number, dimensional less number here.

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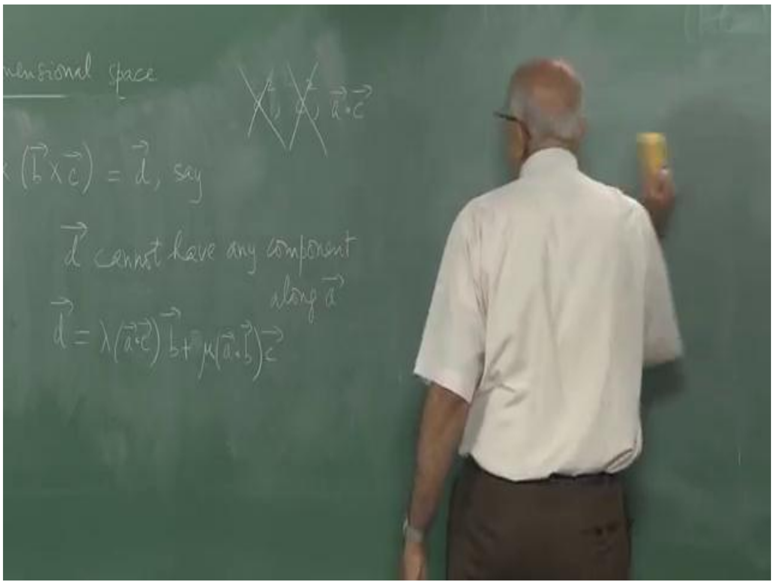

Similarly, the same argument says this must be a dot b and it must be multiply by some number mu. So, these are absolute constants pure numbers, nothing to do with a, b, c are anything like that, like 2 phi or 4, whatever. So, that is a formula for d, already we see the dependence on a, b, c is already explicit and then, there are two unknown numbers here.

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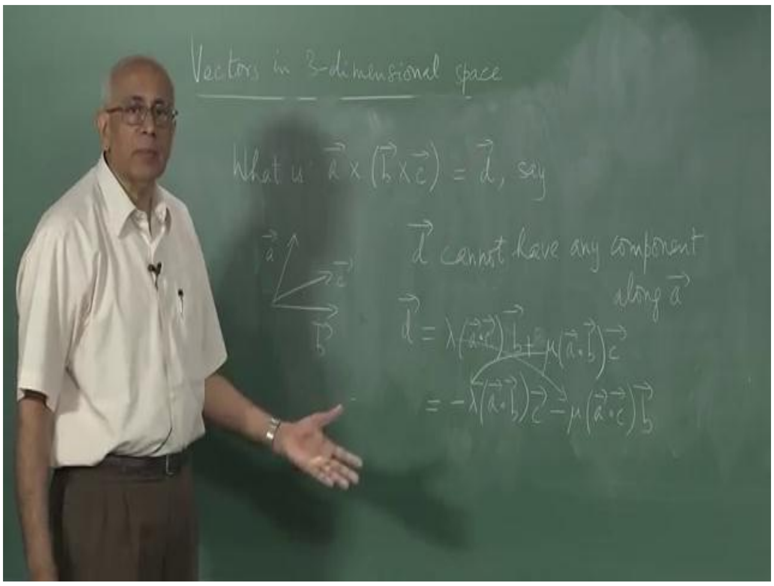

What do we do next, well we argued by saying that, the way we constructed the cross product of two vectors was such that while two vectors b dot c give a dot product, which is the same as taking it in the other order c dot b. But, b cross c is equal to minus c cross b, the cross product was anti symmetric under exchange of the two vectors and the product. Because, we wrote the components down we had things like b 1, c 2 minus b 2, c 1 and of course, if you interchange b and c minus sin appears immediately in each of the components.

So, this is an anti symmetric product is the anti symmetric. We explode that here and if I interchange c and b instead of b, I write c instead of c, I write b, then I get exactly the same formula here. But, wherever I got c, I must write as b, wherever I got b I must write c and put a minus sin. So, this must be equal to minus lambda times a dot b c minus mu times a dot c.

No, other possible, again I go the argument and I started with a cross c cross b by interchanging b and c and I got minus the original answer b should go to minus itself. But, b is given by this in the case of b cross c and it is given by this in the case of the interchange. Now, this is only possible as you can see these two must be equal to each other and this is only possible if mu is equal to minus lambda and lambda is equal to minus u, no other possible.

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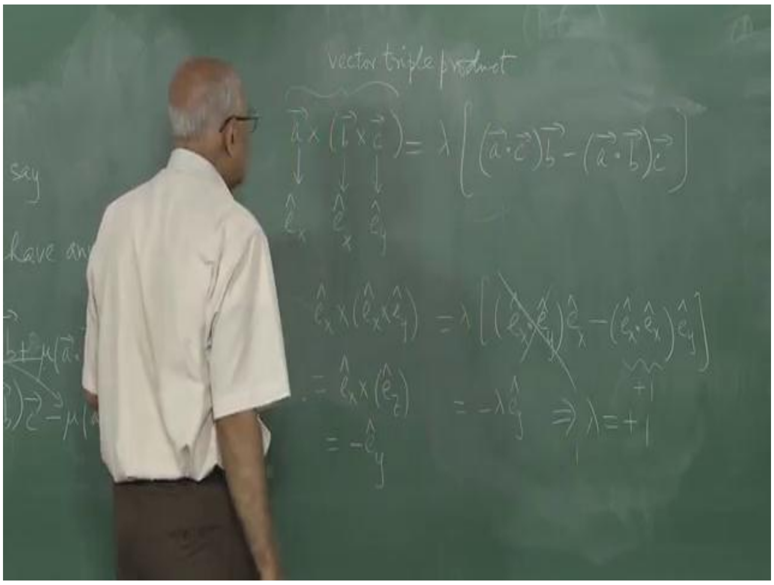

So, it says now almost home, it says that this quantity a cross b cross c must be equal to some constant lambda times a dot c b minus a dot c is no other possible, where lambda is an absolute number, pure number. The next job is to find out, what this number is, well for this now, since the does not depend on it a, b, c or I use the fact, that in the case of a simple choice like what if a, b, c, are Cartesian unit vectors, I know the e x cross e y is equal to e z and so on.

So, let we choose for instants without this is e x say and let us choose this to be e y and this back to be e x. So, a is e x b is e x, c is e y. Now, what is this reduce then, we know that e x cross e x cross e y is equal to e x cross e z, because that is what e x cross e y is for the right hand tried, which is equal to minus e y. So, on the left hand side in this choice, I get minus e y, but I apply this formula and I get lambda times a dot c. But, a was e x and c was e y and e dot e x dot e y, b was also these of x out here minus a dot b a dot b is e x dot e x and then, c was e y is a fashion.

But, e x and e y perpendicular to each other, so e x dot e y is 0 and e x is a unit vector. So, e x dot ex is plus 1. So, this side you get a minus lambda e y. So, you got the result that minus e y is minus lambda times e y and that is not a very hardly equation to solve this implies of course, that lambda is equal to 1 plus 1. And therefore, we have a general formula for this vector triple product by the way this thing is called a vector triple product.

And you can ready to write down, what this formula is lambda is got be 1; that is it; that is the formula; that is the formula for the vector triple product. And notice, I did not use any special coordinate system, I did not write down the components of a, b and c and so on. I use extremely general arguments to find out, that the answer can only be this, it cannot be anything else, this fashion.

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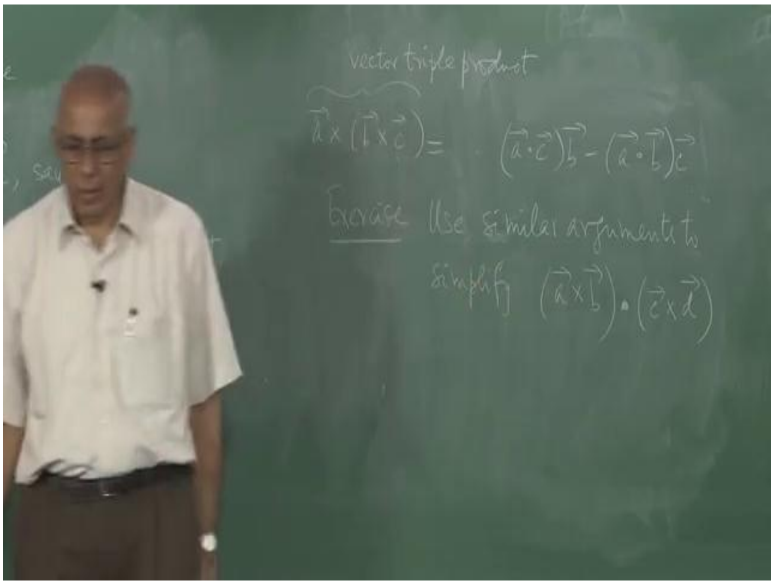

Exercise, use similar arguments to simplify a cross b dotted with c cross b, take four vectors in three dimensions a, b, c d arbitrary vectors, find the vector a cross b, find the vector c cross d and take the dot product between the two, it is a scalar of course. And find what this is, you get a formula similar to this expect is first scalar, this number here is a scalar and use exactly the same sort of argument.

The fact that the answer must be proportional to a, proportional to b, proportional to c, proportional to d, so it cannot be non-linear any of these vectors magnitudes and it is must be a scalar at the n. So, you need to know, what are the possible scalars you can form from a, b, c and d; other then the magnitudes e squared and the answer is of course, you can form a dot b, a dot c, a dot d in form b dot c b dot d and c dot d. So, you have six possibilities in the answer must be a combination of these possibilities.

So, use this is an interesting exercise, without going into components to simplify this combination. So, this is a little bit of vector analysis, what I will do next time is to go into dynamics. We going to talk about, what happens when you actually put in forces and so on and we will try to write down the equations of dynamic Newton's equations and vector form and see, how we can develop it further.