Mechanics, Heat, Oscillations and Waves

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Module – 10
Vectors in 3 - Dimensional Space

In the last lecture, we went through some properties of vectors in two dimensions, vectors which lived on a plane, the infinity Euclidean plane. And we looked at how unit vectors changed, when you went from the Cartesian basis to basis in which you had plane polar coordinates and you looked a little bit at kinematics in a plane. Today we are going to start on vectors in three dimensions, after all we leave in three dimensional space and there are a large number of interesting properties of vectors which are physically very important.

And therefore we are going to devote a little bit of time and attention to studying these properties and analyzing, what vectors do in three dimensions in different ways. Now, the first point is that to define a vector in three dimensional Euclidean space; we use the same strategy as we used in earlier cases in the plane for instance.

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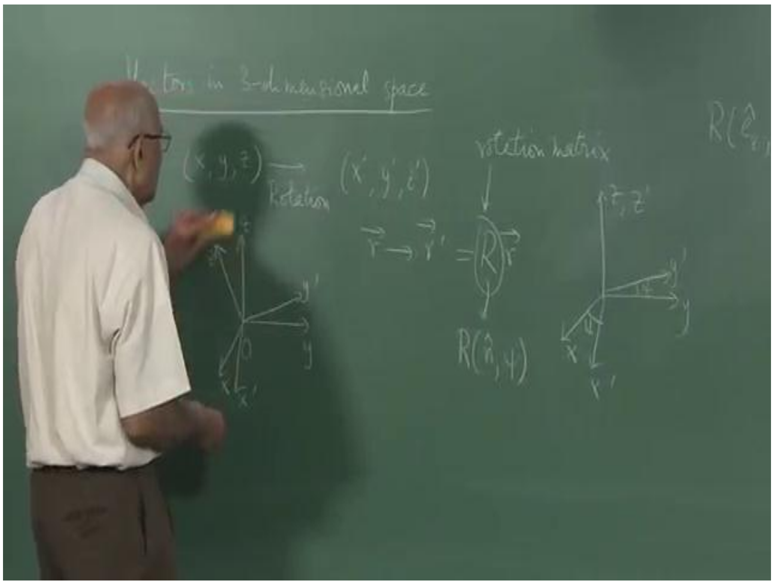

By saying that under a coordinate transformation of the Cartesian coordinates x, y, z, under the rotation of the coordinate axis, these coordinates went to some new coordinates x prime, y prime, z prime, where these three quantities are given as specific linear combinations of x, y and z, the original coordinates. When I make a rotation of the coordinate axis about the origin so here is my coordinate system, the original one with x, y and z.

And the idea is you rotate this coordinate system to look at, to make it oriented in some other direction and in the new coordinates system, a given physical point which has coordinates x, y, z in the original system has new coordinates x prime, y prime, z prime, which are linear combinations of x, y and z. So, the idea is that if you had another coordinate system, which perhaps looks like this and that was x prime and this is y prime and this is z prime.

When this is a formula which connects these three new coordinates to the old coordinates and in symbols, this is written in the following way, this goes any position vector R goes to r prime, which is equal to some rotation matrix acting on r. When I write this R as a column vector, 3 by 1 column vectors; then this R is a 3 by 3 matrix, which multiplies this 3 by 1 column vector to produce a 3 by 1 column vector in the prime coordinates.

Now, this rotation matrix is more complicated than the two dimensional rotation matrix, which we wrote down, where only one angle was involved, there was a theta and there was the cos theta sin theta minus sin theta cos theta was the matrix and that was it. But, now we have much more complicated situation, because to specify the new coordinates or the rotation, I rotate about some direction in space through a certain angle.

So, really speaking, this rotation matrix in general, you need to specify an axis of rotation, a unit vector which tells you the direction of the axis and you need an angle psi through which you rotate. So, you have two parameters here to tell you, what the components of this unit vector are, because it is the unit vector, the third component can be written in terms of the first two, and then you need an angle.

So, physically what it says is, if you have a direction in space as given by this piece of chalk, then you can rotate about that direction in a plain perpendicular to it through an angle psi. And that gives me the rotation itself and this is specified by some 3 by 3 matrix, which is fairly intricate and I would not want to write this down. But, the end result will be that x prime and y prime and z prime will be a linear combinations, homogenous linear combinations of x, y and z in general.

A very simple case would be, where I actually have a coordinate system here is x, here is y, here is z and I rotate about a z axis through a certain angles psi. So, all that happens is that, this x becomes x prime through this angles psi, the y goes in and becomes through this same angle psi, that is y prime and z prime is the same as z.

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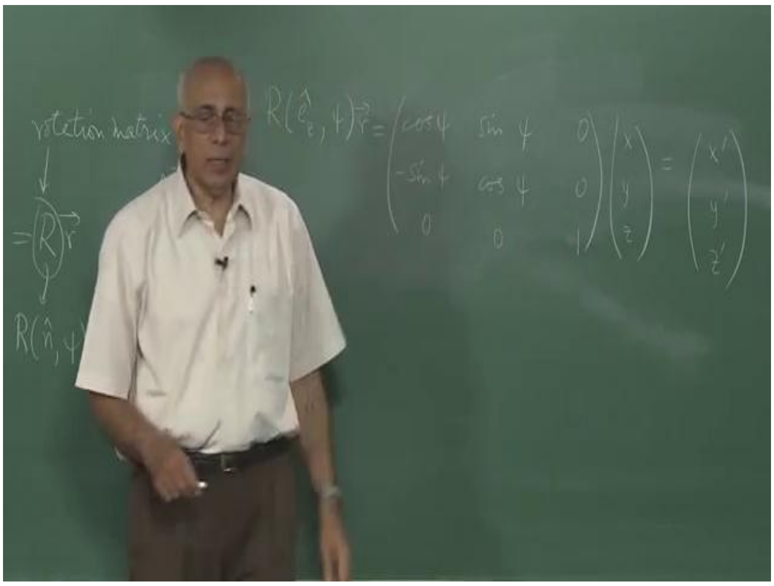

Whereas, particularly simple case of a rotation and we can write the matrix down in that case, because then R the unit vector is the z direction is z and psi and this will of course, be cos psi, sin psi and 0 here minus sin psi, cos psi and 0 here, 0, 0 and 1 here. Because, now it is very clear that all that happens is since that z coordinate is unaffected, the x and y coordinates would act as if or transform as if. There is a rotation in the x, y plane through an angle psi for which we know the formulas.

So, this matrix when it acts on x, y and z, which is the 3 by 1 column vector representing a point in three dimensional space produces for you. So, this thing acting on r is equal to this, this produces for you the new coordinates x prime, y prime and z prime. And of course, you see immediately that x prime is x cos psi plus y sin psi plus 0, y prime is minus x sin psi plus y cos psi plus 0 and z prime is equal to 0 plus 0 plus z.

So, it is clear that you have preserve the fact that z prime and z do not change at all, because if you rotating about this z axis, then the z coordinate of any point is not change at all. So, that is the matrix in this very simple case, but in the more complicated case, where this is an arbitrary direction, the formula gets much more complicated. We do not need it at the moment at all.

So, much for what happens to a coordinate point, now what is the vector? A vector is a point, which transforms by definition is the vector is the set of three numbers, the three components which transform under a rotation in exactly the same way as the coordinates do.

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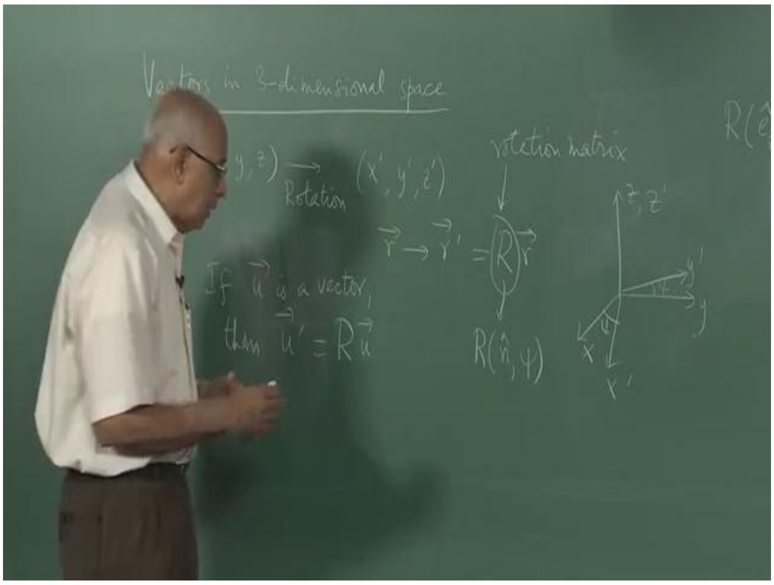

So, if this quantity, if u is a vector, then under a rotation u prime must be equal to the same matrix on the components of u. The same as governs the transformation of the coordinates and I am not going to write it down explicitly. So, this is u x prime is a linear combination of u x u y u z and so on and exactly the same way as x prime, y prime, z prime a linear combinations of x, y, z. That is now by the correct definition of what the vectors is in three dimensions.

So, once you have that in place, then we can create the dot product between two vectors and we can do a large number of other things, but since we are going to change basis and so on. Let me write down what the Cartesian basis is and keep that in mind, we will go on to other coordinate systems as well.

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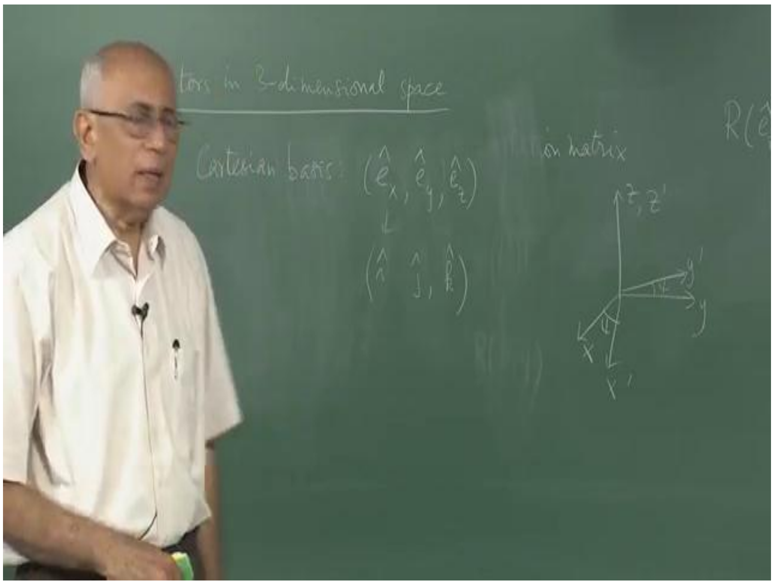

So, the Cartesian basis consists of this triple at of vectors this e z, which in elementary notation is sometimes written as i, j, k. I prefer for reasons I explained already, I prefer this notation, simply because we will go to other frame other coordinate systems in which case, I would like to be able to find out to look at a glance tell, which unit vector I am talking about. Therefore, I preferred to write it, the Cartesian basis in this form.

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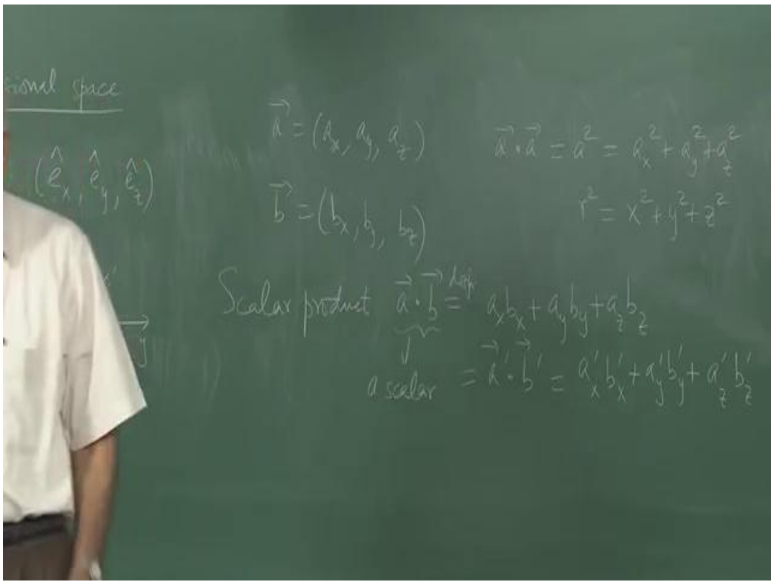

Now, we also defined, what is called the scalar product or dot product between two vectors? And that was the scalar because of the following reason. So, if you have a vector a, which is got components a x, a y, a z and a vector b, which has components b x, b y and b z. Then, this so called dot product or scalar product of a and b, written as a dot b is by definition equal to a x, b x plus a y, b y plus a z, b z and this is the definition, it is defined in this fashion.

Now, exactly as we did in two dimensions, we could in principle write down, what are the prime values a x prime b x prime c x prime etcetera and a y prime b y prime etcetera. And we can verify that this is equal to a prime dot b prime, which is a x prime b x prime plus a y b y prime. There are more elegant ways of verifying it using just the properties of this matrix or the rotation matrix are, but I am not going to go into that here, it is a matter of little bit of algebra. But, it is an easy to show that this quantity a not b does not change under a rotation.

And therefore since it is the same in the unprimed and primed coordinates, it is a scalar; this thing here is the scalar quantity. In particular, if b is z equal to a, then this number a dot a for any vector is also written as a square and that is of course by definition a x squared plus a y squared plus a x squared and that is the scalar. This quantity here, the magnitude of the square of the magnitude of the vector is a scalar.

A special case of that is when you have a x is the same as the x, a y is the same as y and so on and then of course, r squared is x squared plus y squared plus z squared; that is the scalar. So, the distance from the origin to any point does not change the coordinate systems is rotated about the origin. A little generalization of that, the distance between any two points in this space does not change, if the coordinate system was rotated about the origin or any point for matrix.

So, these are all scalars, these quantities are scalars and we have the concept of a scalar product. But, in three dimensions something else happens, something very special happens and that is the following. But, before that we will answer this question of what happens, if instead of a proper rotation about the origin, I will look at an in proper transformation similar to the reflection, I looked at in two dimensions.

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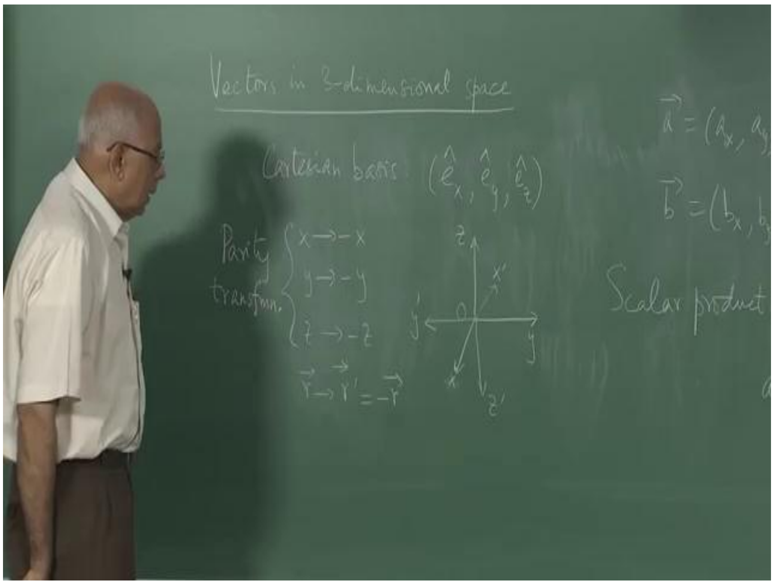

So, suppose I look at a transformation and which x goes to minus x, y goes to minus y and z goes to minus z. This is like saying that I take a coordinate system of this kind x, y and z, the right hand it coordinate system, this is the origin. And now, I will look at x comes ticking out of the board, so I go in and that is my x prime. This y is flipped in direction and that is my y prime and the z is flip in direction and that is my z prime.

What about that coordinate system, now it is a matter for you to verify that, if I take on the right hand and I call this x, y and z, there is no rotation that I can do, which will take me to this new coordinate system x prime, y prime and z prime. So, this will the original x, x prime is gone inside, y prime is on the left and z prime is pointing down. There is no way in which you can reach that coordinate system from the original coordinate system by a rotation; this is called a parity transformation.

When you change r goes to in the new r prime, which is equal to minus r prime. So, a vector changes sin under a parity transformation, this thing is called parity. No, once this happens in general vector, if it behaves exactly like that coordinates under these transformations will, therefore change sin each components will change sin under coordinate transformations. On the other hand what happens to the scalar product, well if a goes to minus a and b goes to minus b, a dot b remains unchange; it is exactly the same under the parity transformation, which is why it is called the scalar.

Had it change sin, I would have called it a Pseudo scalar, but it does not change sin, so this thing is need a scalar. So, now, let us ask, what are the possible vectors, what are these things look like, well let us write down the physical table here of physical quantities, which are vectors in the sense and which will know from other means from other information or like vectors.

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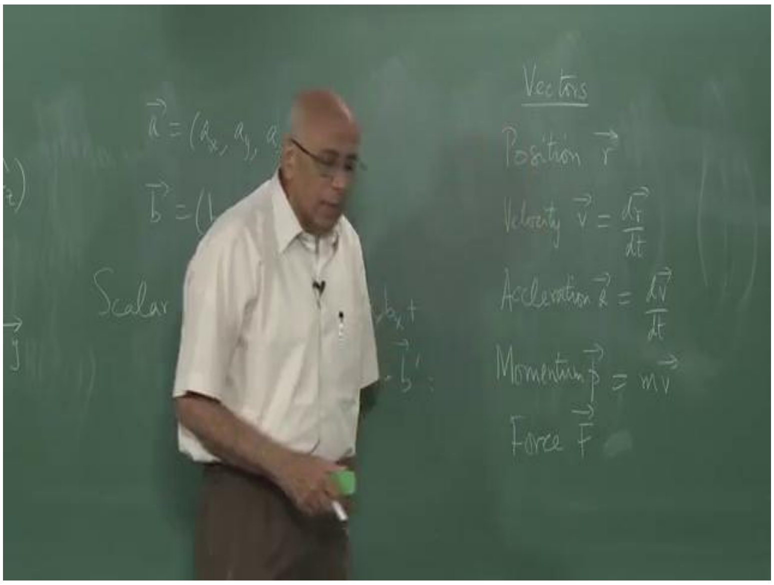

So, here is in set of vectors, position r, position of a point in space or a particle in space is a vector. Then, the velocity, let us look at some physical quantity, velocity v, this is equal to d r over d t, the rate of change of this position with respect to time and there is an r here, so it is obviously, a vector once again. And it is also a vector in the same sense under parity that the position is, because t does not get affected by changing x to minus x y to minus y etcetera, so that is a proper vector.

What about acceleration a, this is the second derivative, what is the derivative of the velocity and that of course, this the second derivative of the position first derivative of the velocity that two is the proper vector. It will change sin under parity transformation and otherwise behave under a rotation, exactly as a vector direction as the position does. What about a quantity like momentum, linear momentum p, well in the naive definition of a momentum for a particle, this is equal to m times the velocity and that quantity m is a constant.

So, the scalar, and therefore this is also a vector behaves also exactly like the velocity does under rotations under parity transformation and so on. Force, what would that do, well we need an equation which connects force to some other things, we already know about and we know Newton's third law, which tells us the mass times the acceleration is equal to the force. So, since we have a connection of that kind, we say this quantity is like a vector that is the scalar. So, this quantity is the vector, a proper vector. So, certainly this is a proper vector and so on.

We can create a whole list of these vectors and say that they all would have to transformation properties, which are known. Once, you know the transformation property of the coordinates under rotations, but now comes a curiosity, which is valid only in three dimensions, a peculiarity of three dimensions. Rather, similar to what we found in two dimensions, but three has the owns special nature and that is the following.

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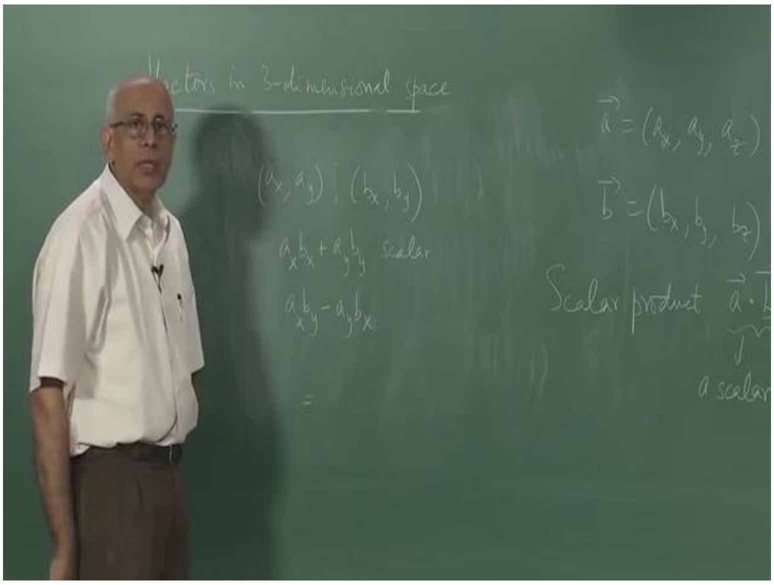

You discover that if you took two vectors of this kind, then just as in the case of two dimensions, recall that we had a x, a y and b x, b y for two vectors, and then I said look a x, b x plus a y, b y was the scalar in two dimensions. And then pointed out that a x, b y minus a y, b x behave under a rotation of the coordinate axis about the origin in x, y plane like a scalar, did not change. But, under a reflection x to minus x or y to minus y, it change sin and this became a Pseudo scalar, this anti symmetric combination became a Pseudo scalar.

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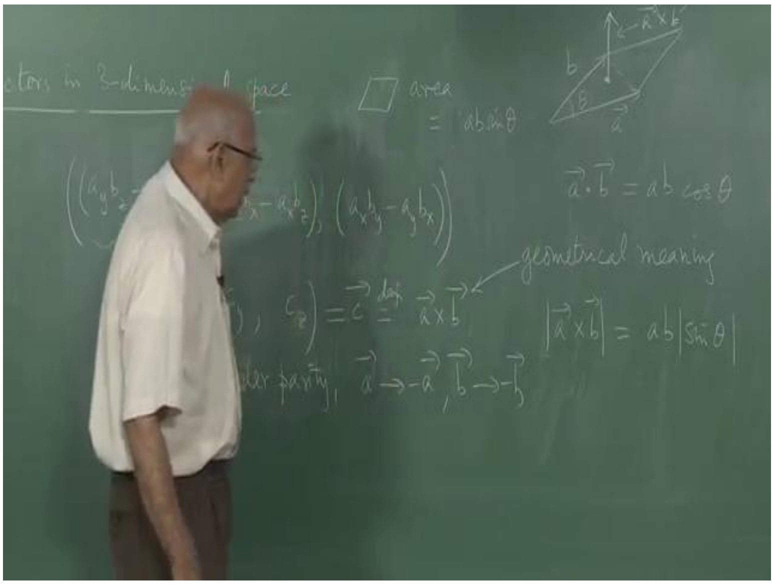

And now in three dimensions, we have a more general possibility. So, in addition to this scalar product, which is a x, b x plus a y, b y plus a z, b z. We also have the possibility of combinations like a y, b z minus a z, b y, but we have two more such combinations possible and that is a z, b x minus a x, b z. The third possibility is a x, b y minus a y, b x and those are the only possibilities for this anti symmetric combinations of one component of a with another component of b and then in the reverse order, this guy.

So, we have there such quantities which are the analog of a x, b y minus a y, b x in two dimensions. We have now because there are three components possible, we can choose them two order time and this is what you get, I choose x, y, I get this, I choose y, z, I get this, actual z, x, I get this. It turns out, that when you compute what these quantities do, it turns out that this set of quantities transforms exactly like a vector as under rotations.

So, this means that, if you call for the moment is quantity here to be are component of a vector c subscript x, c subscript y, c subscript z. These three quantities qualified to be a vector in the sense that they transform exactly as the coordinate x, y, z to under a rotation of the coordinate axis about the origin. Therefore, these three are components of a vector from by taking the vector a and b combining them in this by linear fashion to produces a new vector, whose components of precisely these combinations here.

So, the x component is y z minus z y, the y component is z x minus x z and the z component is x y minus y x in cyclic permutation. This quantity is called by definition the vector product of a and b. This one component of a in every term, this one component of b in every term and it is called the vector product here or cross product for this cross. In contrast to dot product, this produces the scalar; the cross product produces a vector in three dimensions.

In two dimensions, it did not the analog, just produces a Pseudo scalar, but here it actually has produce a vector under rotations. Now, one could ask just as if I took two arbitrary vectors a and b and there is a certain angle theta between them, I also know that a dot b instead of writing it in Cartesian components, I could also write this as a magnitude of a times the magnitude of b times in the cosine of the angle between them. In exactly the same way, can I write some formula for a and b given two arbitrary vectors a and b of that fashion on that kind in three dimensions and the answer is yes.

But, now you have a vector and what is the meaning of this is geometrical meaning, so I write this geometrical meaning in this turns out to be the following from a and b, you can form a parallelogram, let us draw this again. So, here is a, here is b and you can extends this and form a parallelogram and it turns out that the area of this parallelogram is given by the magnitude of this vector. And the direction of this vector happens to be the direction normal to this parallelogram.

You are already aware of the fact that, if took this triangle the diagonal here, this area is equal to that area and this angle is theta, when the area of that triangle is equal to half a times b times sin theta. We know that from elementary geometry and of course, the area of parallelogram is twice that, so it goes away here and the parallelogram area is equal to a b sin theta and that is exactly what the module is of a cross b is.

On the other hand, the direction of a cross b is perpendicular to the plane of this parallelogram in this direction and that direction is found by taking a and rotating from a to b into through an angle theta in the direction of a right hand is true. The direction in which the right handed screw driver moves up is the direction of this vector a cross b. So, this is the vector a cross b,, therefore a cross b equal to the magnitude of a cross b equal to a b sin theta. The magnitude of a sin theta here and the direction of a cross b is along the normal to this parallelogram in this sense in which you right handed screw driver moves, when you rotate from a towards b.

So, a is coming out here and I rotate into b, when screw driver moves up and that is the direction of the vector a cross b. So, we have a prescription, the geometrical meaning which will tell us not only the magnitude of a cross b, but also the direction of a cross b in this fashion. Now, this construction is peculiar to three dimensions, because in two dimensions, the analog is to do Pseudo scalar.

But, in three dimensions it has produces a vector and it terms out in higher dimensions, it does not produces a vector in those dimensions, it produces something called a ten serve, which we are not talking about here. But, the fact is that in three and only three dimensions is the cross product of two vectors going to produce another vector and that is the very profile observation of very important one for all physical problems and many physical problems.

Now, that we are know that is there any different at all between this kind of vector and the kind of vector that the coordinates are any of these vectors here is there any difference at all. Well, yes indeed there is, because look at what happens on a parity, if a and b are proper vectors under parity, a goes to minus a, b goes to minus b, a dot b, it does not change sin and it is the scalar, we already established that. But, a cross b is going to remain unchanged in sin and yet it is the vector.

We already said under rotations a cross b behaves exactly like the position does are the velocity does and so on, and therefore it is the proper vector. So, if a and b are proper vectors, a cross b is a proper vector under rotations, but on the parity it does not change sin, whereas, all vectors change sin. Therefore, we have two kinds of vectors in three dimensions. We have those we change sin under parity and those which do not change sin under parity all these are vectors.

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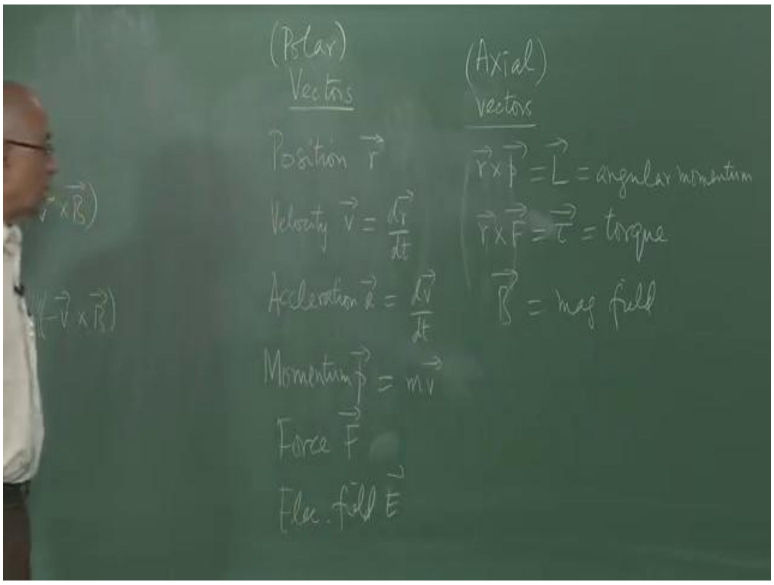

There also known as polar vectors and the class of vectors, we change sin, they are called Pseudo vectors or axial vectors and these are the cross products of any two ordinary vectors. For instance r cross p equal to the angular momentum that is an axial vector, it behaves exactly like a vector does under rotations of the coordinate axis. But, the moment you have a parity transformation, it does not change sin, unlike what are vector really does.

What about r cos v, wells going to exactly do exactly what this does, because v and p just before by this constant multiples here and so on. So, whenever you take two vectors from this column here and take the cross product of the two, you are going to get an axial vector. For instance if you did r cross F, this is equal to tau and it is equal to the torque. So, the torque is an axial vector the angular momentum is an axial vector, the cross product of any vectors here is an axial vector.

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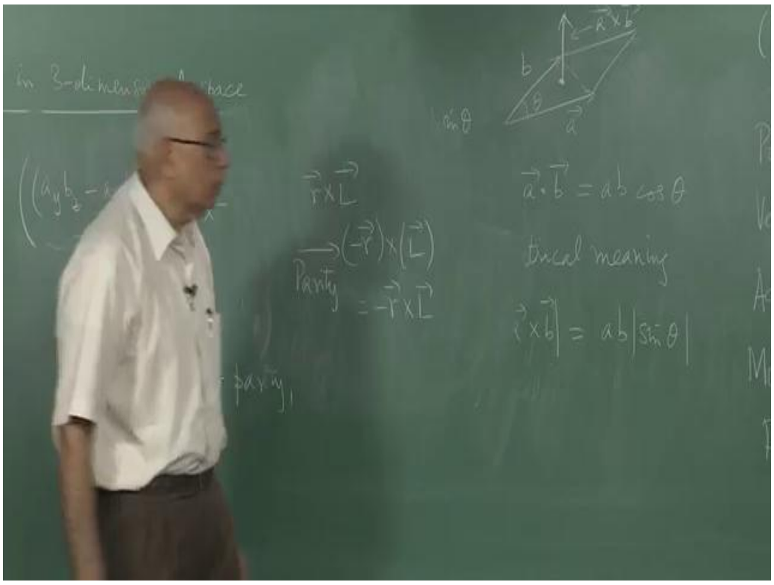

What happens if I take L an axial vector and take the cross product to r, will now that easy to write down, what is going to happen, because if you look at r cross L under parity is goes to minus r cross and it remains L. So, that is equal to minus r cross L, and therefore it is back to be thing a polar vector once again. So, if you took the cross product of a vector from here and a vector from here an axial with a polar vector, the back to a vector, polar vector.

If you took the cross product of two axial vectors, what you thing will happen, if I took a L cross tau for a example, L cross any other vector here, which is an axial vector, neither of the actual vector changes sin, so it remains a axial vector in this case. But, the moment you take something to here and something from there and take a cross product to back here in this, back to the ordinary vector.

Well, we can write down using exactly the sort of input we wrote namely using Newton's law, we wrote down for instance, what the electric feel, what the angular momentum does, what the topped as and so on, the force does.

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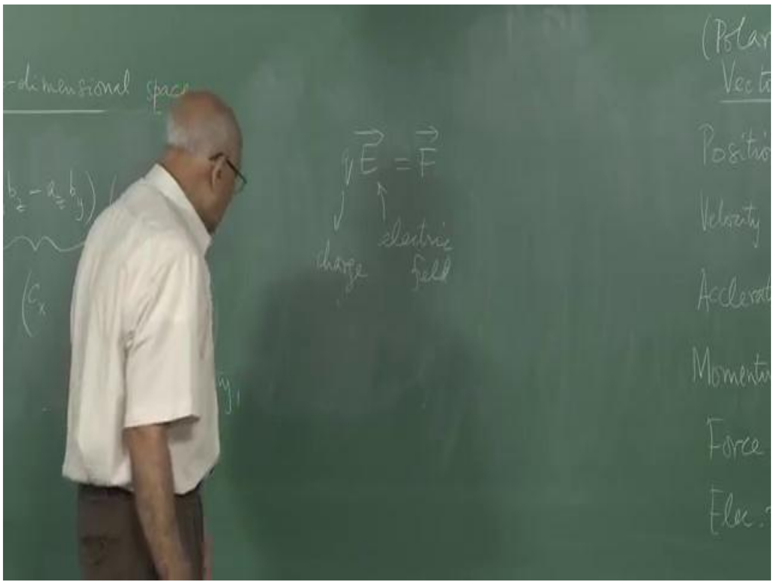

We can now write down, what the electric feel would do for instance, I know that if you have a charge q and I apply an electric feel E, then the force on this electric feel is q times E. So, if this is the point charge and this is the electric field apply, I know that the force on it is the charge multiplied by the electric feel intensity. Since, this is the scalar and I know that is the polar vector, this is the polar vector. So, I had my list electric feel.

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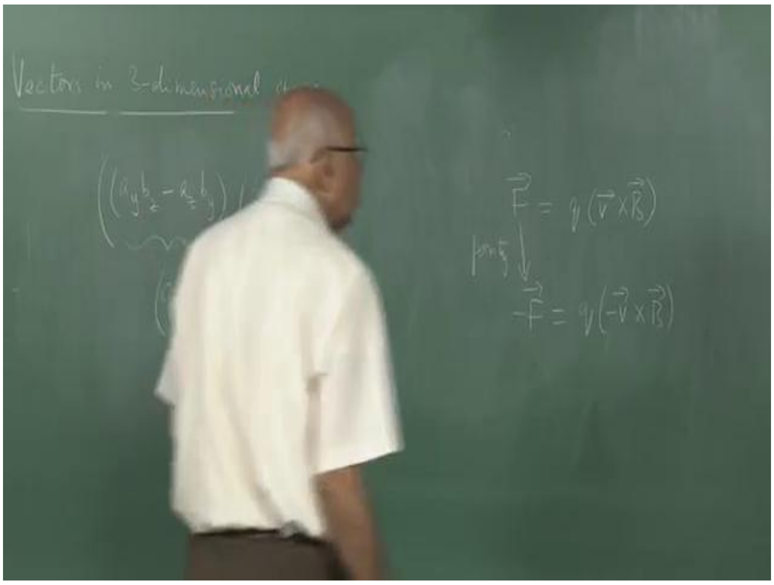

What about the magnetic feel, well that is the little more complicated, when I applied a magnetic feel b on a charge q and the charges in motion, then the force the magnetic force on this charge is q times the cross product of v with the feel B. That turns out to be the force is called the Lawrence force on a charge, which is moving in a magnetic feel which is applied. I know that this a polar vector; that is the scalar; this is a polar vector that, therefore has got to be an actual vector, because this quantity will change sin and a parity.

It will become minus F under this law is to remain the same, whether it change coordinate system from a right handed to left handed coordinate system are not. This has to become q times minus v cross v the minus sin have to cancel out to give me exactly the same law in the old and new frames, which tells me that B cannot change sin. So, the conclusion is that the magnetic feel B is an actual vector and so on.

So, we can actually fill up this list from one to the other, we logically reduces using a law each time, which brings in a new physical quantity. We can classify whether the new vector that we are dealing with is an axial vector or a polar vector. But, remember that as for the rotations are concerned there is no difference between the polar vector and an axial vector. It is only when you bring in a improper transformation such as reflection or parity that there are changes of sin. But, it is a very valuable thing to keep in mind, because it is again serves a consistency and tells you, there are certain differences between quantities one kind of the other.

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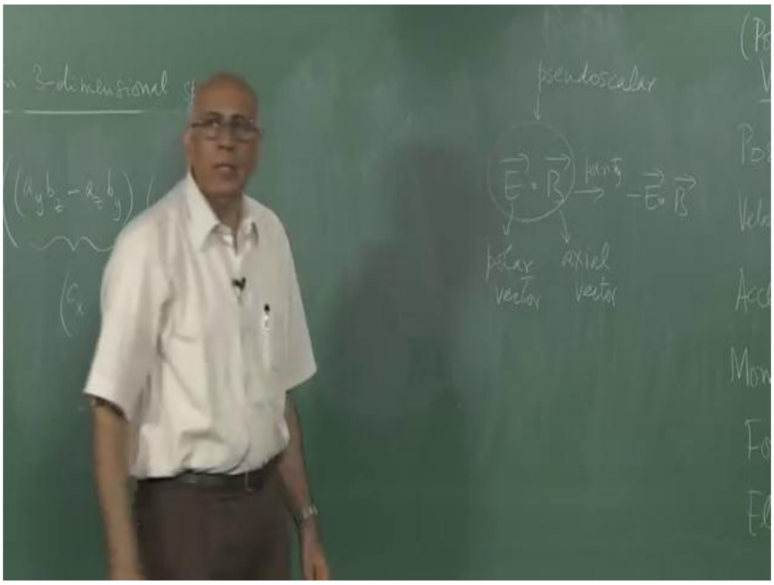

By the way, if you took the dot product of something here and there, what you thing is going to happen, if I took for instance the electric feel E and did a dot product with B; that we know has to be a scalar, it is a scalar product. This however, is a polar vector, this is an axial vector and this combination under parity becomes minus E dotted to plus p. And therefore changes sin and what are we called a scalar quantity, we change a sin under a reflection, under a parity transformation, we called it as a Pseudo scalar.

So, this thing here it is not an ordinary scalar, it is a Pseudo scalar. So, in this fashion, we can actually write down the transformation property is varies physical quantities. Once, you have a small dictionary which tells you what some basic quantities transform like. So, we will continue with this and we will continue to keep this in the back of our mind that took hence a vectors and three dimensions and the cross product of two ordinary vectors, polar vectors is actually an axial vector, which is like a different kind of vector.

But, there are physical quantities which are ether polo vectors are actual vectors and they have to own importance. So, this is useful information to know, we still have some way to go and understanding vectors and three dimensions. The next task will be to see, what is some other basic formulas are.