Physical Applications of Stochastic Processes
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Lecture - 09
Markov Processes (Part 3)

Okay, we would like to, I would like to make some more comments about the solution to the random walk problem the linear random walk on a linear lattice with a bias. There are certain points about this solution which are general in nature and I would like to start with that and then go on to some more properties of general birth-and-death processes okay together with hopefully some physical examples of what is happening.

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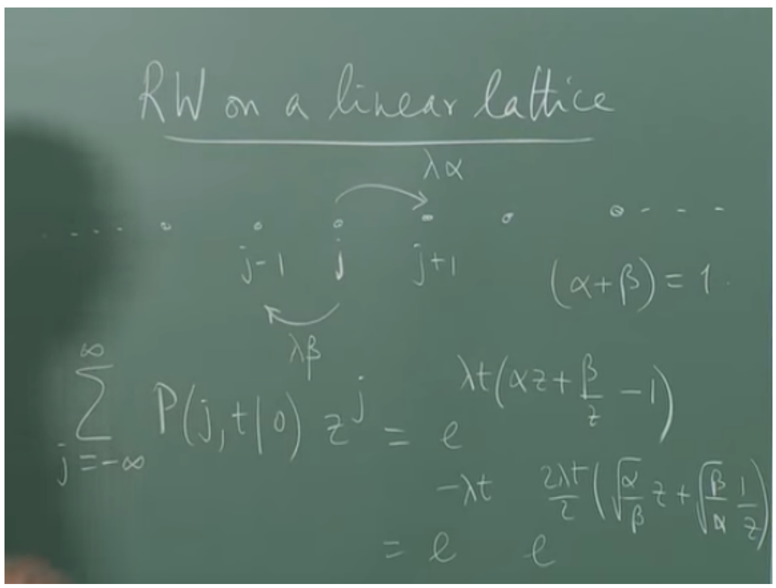

Now, if you recall our random walk problem, so this was random walk on a linear lattice. If you recall we had these sites on the lattice. This was site j and this was j + 1, j - 1 and so on all the way on either side over the full set of integers and the rule of the game was that from every site j you jumped to the site J + 1 with a mean rate which was lambda times alpha and you jumped from j to j - 1 with a mean rate which was lambda times beta such that alpha plus beta is 1.

And then we discovered that if you start at some arbitrary origin, say 0, the probability P(j, t) starting from the origin, this had an explicit solution and the generating function for this was z to the j summation j equal to minus infinity to infinity was a quantity we could evaluate explicitly and this was e to the power minus well e to the power lambda t times alpha z + beta over z - 1 and that was it. We interpreted this in 2 different ways.

One of them was to write a birth-and-death equation for it and solve it using a generating function or to say that this process, this random process j is the difference of 2 Poisson processes with rates lambda alpha and lambda beta and that immediately gave us this as a generating function okay. Now from this the mean values could be written down. The solution itself could be written down etc.

And just to recall to your mind what happen this was e to the - lambda t and then we could write the rest as e to the 2 lambda t over 2 square root of alpha over beta times z plus square root of beta over alpha 1 over z and that helped us immediately identify P(j, t) as a certain Bessel function right.

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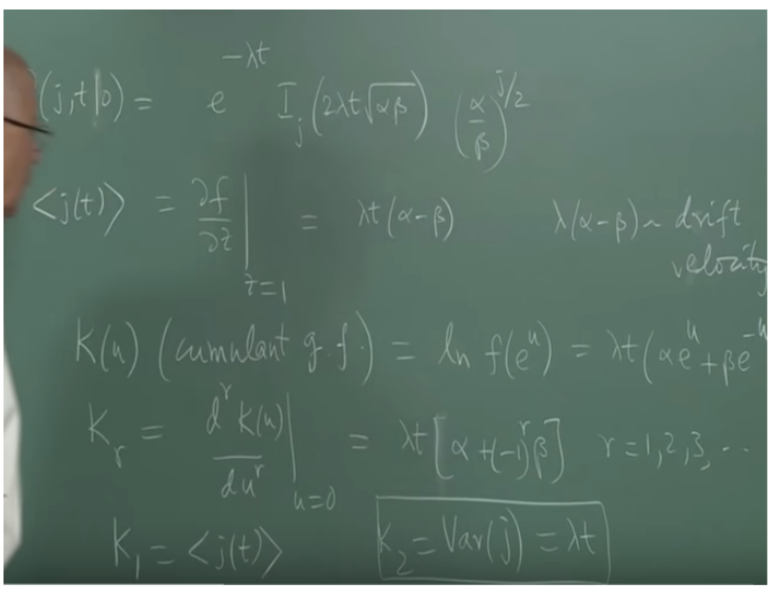

So this immediately gave us the solution P(j, t) starting from the origin to be equal to e to the minus lambda t and then the Bessel function I j of 2 lambda t times square root of alpha beta and then there was this factor alpha over beta square root to the power j or this to the power j over 2. So that was the explicit solution okay right to this probability for this probability okay. Now from this we can read off all the moments etc., etc. and interpret them in some sense.

For instance the mean value j of t, this was equal to if I call this thing the generating function f of z, t equal to this. This is the generating function. The mean value here is just delta f over delta z at $z = 1$ and that follows directly from here. You differentiate this, you pulled down a lambda t and then an alpha - beta if you put $z = 1$ this goes away the exponent goes away and you have this is equal to lambda t times alpha - beta.

So you see what is happening is that this bias alpha - beta this differential preference to the right or the left depending on alpha being bigger or smaller than beta acts like a velocity. So this is like the mean position of displacement after time t is proportional to the time and of course it must be multiplied by a velocity and this drift velocity is essentially lambda times alpha - beta. So lambda times alpha minus beta is like a drift velocity okay.

You can now compute the mean, the mean square and so on and we know that the variance are actually proportional to lambda t. So one should do this a little more elegantly instead of going on differentiating this function here, let us just find the cumulant generating function. That will solve the problem in one shot. Now the cumulant generating function if you recall is the log of the moment generating function and the moment generating function K of u is the cumulant generating function equal to the log of the moment generating function m of u.

But that was the same as f of e to the power u okay and that is equal to well all you got to do is to take logs out here. So this is equal to lambda t times alpha e to the u + beta e to the - u - 1 and that is it okay. Therefore it immediately follows that the rth cumulant K r is just d r K of u over d u to power r evaluated at $u = 0$ okay and what does that give us? All you got to do is to differentiate this fellow.

And each time you differentiate e to the u you get the same thing e to the u and you differentiate this you get a - or + sign depending on whether it is even or odd. So this becomes equal to lambda t times alpha plus - 1 to the power r beta that is it. Now the first moment, first cumulant is of course a mean value itself. So it is clear that this K 1 = j(t) and that is of course alpha - beta which we already derived right.

But what is interesting is that the variance is independent of this bias. So the drift tis cancelled out by the fact that you subtract the square of the mean when you compute the variance from the mean square and therefore K 2 equal to the variance of j. This is equal to lambda t. This is a crucial result. It is typical of diffusive behaviour. We know that the root mean square displacement must go like the square root of the time and that is exactly what is happening here.

When you compute the variance you subtract out the systematic drift and then you end up with precisely diffusive behaviour here okay. We can of course examine and ask whether P(j, t) itself has a limiting distribution or not. Remember we are in an infinite lattice. Now what would you physically expect. I will expect if I start with probability 1 of being at the origin at t = 0, as time goes along you are spreading infinite amount of space.

You spread out on both sides. So although the particle is somewhere on this line the probability of being at each particular point should be vanishingly small. So you would expect it should go to 0 and there should be no stationary distribution in this problem.

So in a birth-and-death process when you have an infinite lattice when this variable is not is unbounded from both sides then the probability stationary distribution will not exist unless you put in some external condition like you put in a current of particles from one side and let them jump out from the other side or something like that. But that is an external condition. Without that in general it will turn out that this probability should vanish as t tends to infinity. But that is that happens here. We can check that out.

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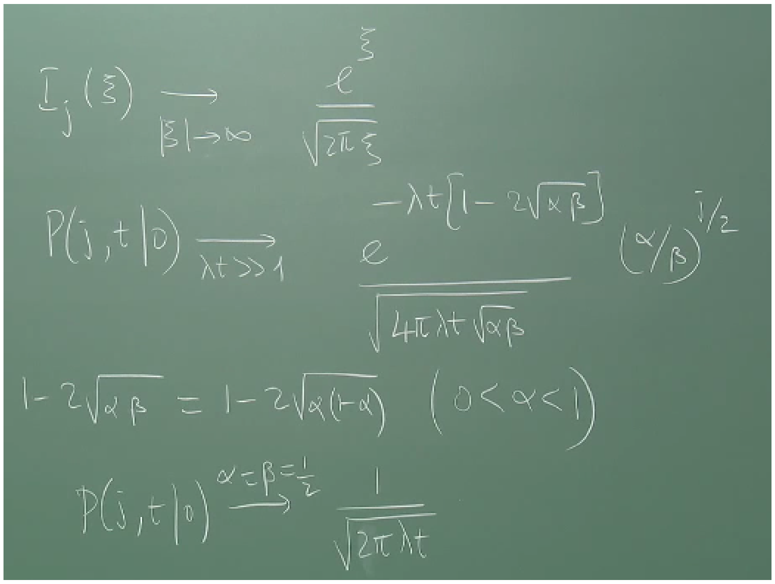

Because it turns out that I, j of whatever is inside I, j of let us call it whatever variable you want to call it psi whatever it is, this tends as mod psi tends to infinity, this fellow tends to e to the power psi over square root of 2 pi psi. That is the leading behaviour of the modified Bessel function and it is independent of j, the order j, provided the order itself does not become infinite okay.

So if you apply that here it is immediately clear that P of j, t on 0 tends for long times which means lambda t much greater than 1 tending to infinity in fact because the time scale in the problem is set by lambda inverse. That is important to remember. This tends to e to the - lambda t times 1 minus twice square root alpha beta and then alpha or beta to the j over 2 divided by square root of whatever, square root of 2 pi so that is 4 pi lambda t root of alpha beta.

And this quantity vanishes exponentially rapidly in t as long as this is bigger than that and remember that this quantity is 1 - 2 root alpha beta equal to 1 - 2 root alpha times 1 - alpha where 0 is less than alpha is less than 1. And when does this quantity become large, the largest? At alpha equal to half because otherwise on both sides, it is a parabola, it comes down on both sides at alpha equal to half it is maximum and then of course the 1 cancels out.

So this quantity dies down exponentially as long as the coin is biased as long as the random walk is biased, either way we do not care but this exponentially vanish. On the other hand when alpha so P of j, t 0 tends in the case alpha equal to beta equal to half it goes like a power law. So now this factor cancels out and it goes like 1 over square root of 4 pi 2 2 pi lambda t. Goes like 1 over t to the half. That is typical of diffusion in one dimension linear problem.

Incidentally, now that we have written this down, we can actually write down the solution to the random walk problem on an arbitrary dimensional lattice, square lattice or a hyper cubic lattice or a hyper cubic lattice. There is no problem at all in writing this down because one can do this by inspection because all the jumps are independent of each other and then you can see what is going to happen. So this generalizes right away. Then we come back to this.

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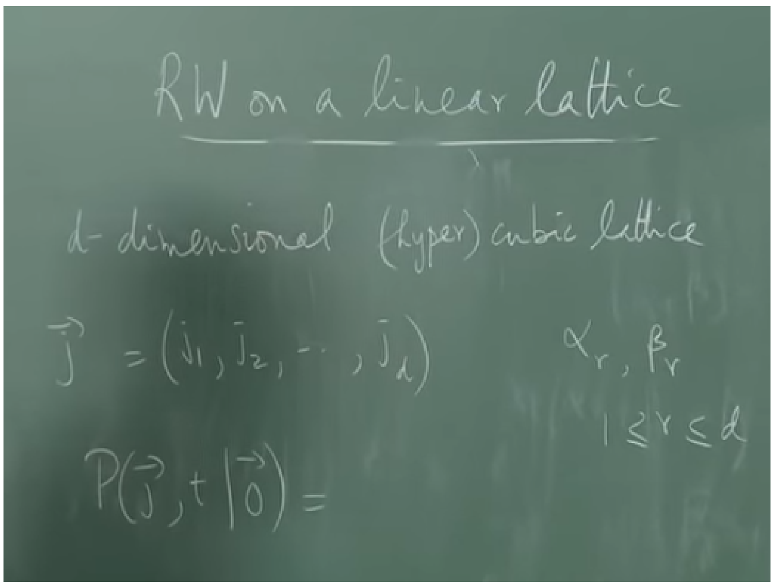

So suppose I have a d-dimensional cubic lattice or hyper cubic and let us suppose that the coordinates of any point are labeled by j 1, j 2, j d. Now let us call this the vector j okay. So the components are all integers and they label the lattice point right. I start with some arbitrary origin. It is an infinite dimensional lattice and then suppose I put biases on both sides which you can always do so to move to the right there is a probability to move to the left there is a different probability to move up or down or move in or out and so on right.

So you can associate probabilities alpha, alpha r, beta r where r runs from 1 to d and the sum of all these alphas and beta is equal to 1. They are all positive numbers. Then we can write down what the solution is for this quantity P(j, t) given that you started at the origin. What would this be? What would this be because the total rate of jumps out of a site is lambda and that is now split. Earlier it was split between lambda alpha and lambda beta on this side.

Now it is split between lambda alpha 1, beta 1, alpha 2, beta 2 and so on. The whole thing adds up to 1. So what do you think will happen? What do you think will be the generalization of this? Now it is not the difference of 2 Poisson processes. There are several processes going on at the same time right and they are all independent of each other. So one can actually write this down by inspection. What do you think it will be?

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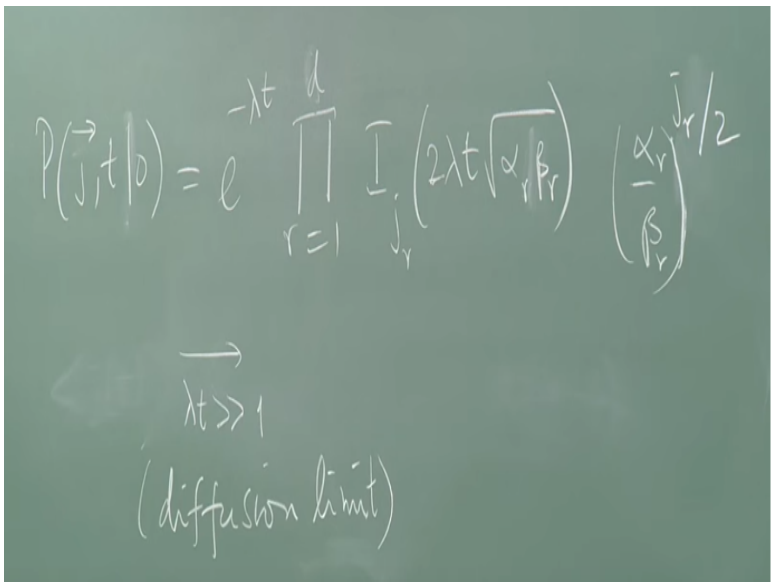

Well yes you certainly would, let us write it down here, this is j. You certainly have this factor as before. This comes remember it came from the generating function because there was a, you subtracted in the master equation, you subtracted – P(j, t) with alpha + beta added to 1. So that produces this e to the - lambda t and then the rest of it is sitting here. So what you would have is simply the same thing as this except this would be alpha r over beta r j sub r over 2.

And this would be j subscript r alpha r beta r here e to the - lambda t, a product from r equal to 1 to j to d and that is the solution. It immediately follows that this is the solution in the d-dimensional hyper cubic lattice okay. So once again one can ask what happens if t tends to infinity lambda t tends to much greater than 1. So this tends lambda t much greater than 1. They call this the diffusion limit, very long time limit.

What would this go to? All you have to do is to use this fact up there and then whenever there is a bias there is going to be an exponentially damped factor. What happens if there is no bias at all? What would you expect?

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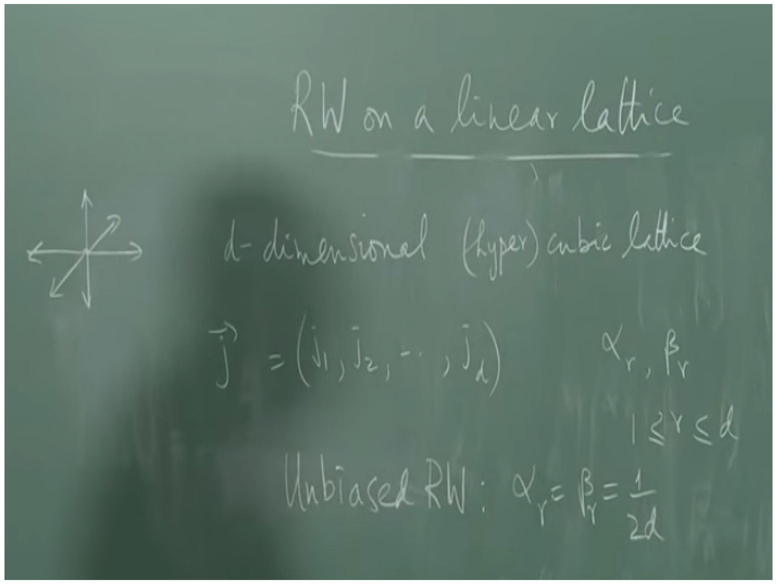

If there is no bias, unbiased random walk what would this correspond to for each alpha and beta on a hyper cubic lattice? Ya, so you have for example from this point you have up you can go down you can go this way you can go this way you can go into the board you can come out of the board and so on. So there are 6 nearest neighbours in 3 dimensions. In d-dimensions there are 2 d nearest neighbours right.

So for an unbiased random walk alpha r equal to beta r equal to 1 over 2 d. So when you sum over all these 2 d numbers you get unity okay. Then what would this become? These factors disappear. All these factors disappear. This factor in the exponential there cancels out exactly and you are left with the product of these fellows an alpha r beta r 1 over 2 d. So there is a 1 over 2 d multiplying this gives you some constant etc.

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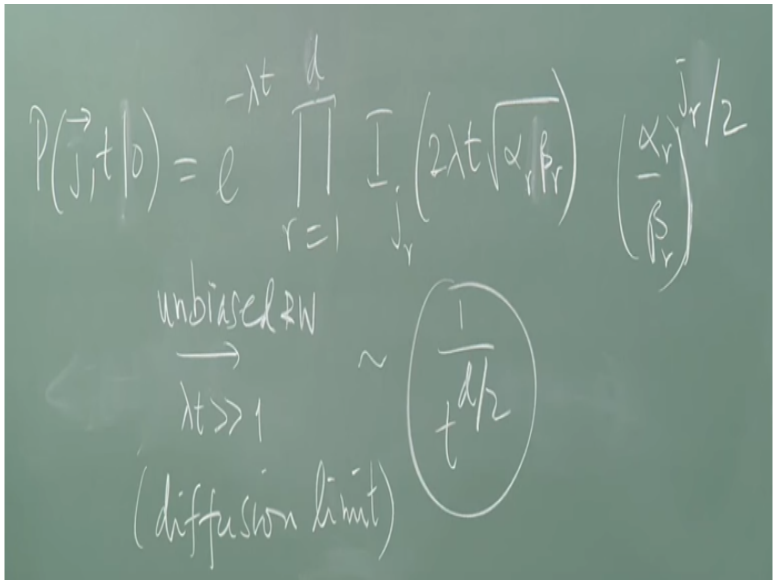

But the crucial point is unbiased random walk. The crucial thing is the t dependence and what does the t dependence do? There is this factor for each of the Bessel functions right and there are d such Bessel functions. So this immediately becomes proportional to 1 over t to the d over t okay. So it is a power law decay but it is proportional to 1 over t to the half d over 2 where d is the dimensionality of the space.

When we solve the diffusion equation in continuous time and space then again you will discover this Gaussian solution has a decay which is exponential which in general is 1 over t to the for an unbiased diffusive process it is 1 over t to the dimension over 2 in this fashion. So for 1 dimension it is 1 over square root of t which is what we found this now okay. So you can begin you can see that there is going to be a close connection between this and the diffusion process and we will explore that when we go to the continuous space case.

But already from the random walk on a lattice one can deduce these facts. Now of course this depends you would say on a in higher dimensions it would depend on the lattice, the nature of the lattice and what are the nearest jumps etc., etc. But actually this, this feature is very general although this solution is very specific to that kind of lattice this is very general and it actually follows from much more general considerations.

So we would not even get into that right now except to say that I can go to a much more general kind of a random walk. I can say that I am not on a lattice. I take steps in different directions of different lengths. They could be distributed. They could themselves be drawn from some distribution. As long as the step length has a is distributed in such a way that it has a finite variance, finite first two moments this kind of feature will always persist, these general features will persist. So it is independent of dimensionality.

It is independent of the nature of the walk etc. as long as the walk is Markovian there is independent steps. The steps are not correlated with each other. If on the other hand you have a real material in which you have some correlation then this whole thing goes out of the window. If for example you jump from left to right and then there is a predilection to jump back to the vacant site back again then of course you have correlation between different steps.

All these things have to be modified or if you have situation in which you remember what happened earlier and you never visit the sites you already visited once then those have a lot of memory and they go completely outside the purview of these Markovian random walks okay. So Markov assumption is very crucial here. Ya, yes okay. Now let us try to generalize this a little bit and put it in a slightly more general framework.

And that is the framework of what I call birth and death processes. They have other names as well. In the semi-conductor industry business they call generation recombination processes and so on and so forth. There are different names in different areas. We will stick to this birth and death business. Now what was the fundamental assumption? The assumption here was that on a linear lattice you jumped either one step to the right or one step to the left or in state space if j labels the states of the system they are integer valued and then j changes by + 1 or - 1. This is all.

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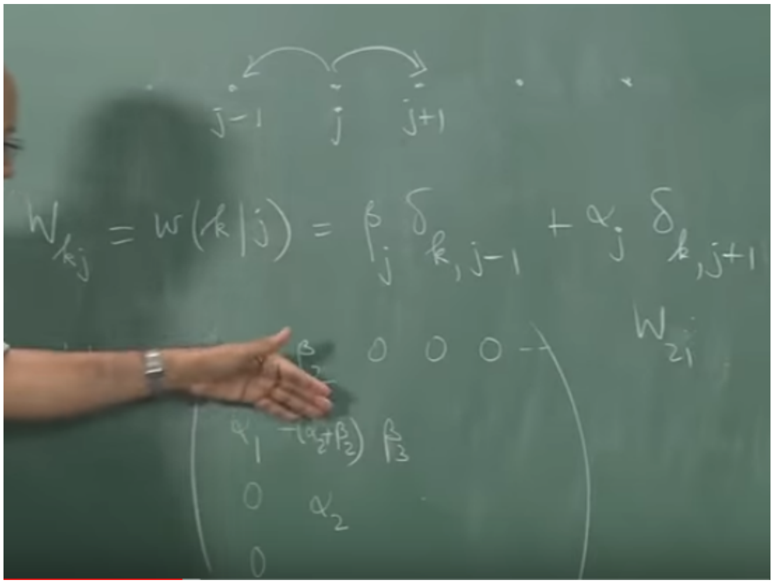

If you generalize this a little bit then once again not with any reference to a lattice or anything like that but if I just write down the states of this system, some random variable which takes integer values and you are at some point here let us call it n not let us call the let us call these sites by j once again. So here is site j and if the process is such that you change this, the transitions occur the w matrix I am going to write down will only connect neighbouring states, then I want to model for this quantity W k j if you recall was w k j the transition rate to go from an initial state j to an final state k.

If this thing only connects neighbouring states then there has got to be something which is proportional to a Kronecker delta such that k can only be j - 1 or j + 1. So there is a transition probability this way and one that way from j and all the other transition probability has 0. Then what is the most general thing you can have. You could have this to go such that the final state is out here with some rate.

And in the random walk problem we said that is a constant rate lambda times beta or apart from that lambda factor let me just call it beta but this could be dependent on where you are. So as you move down these states the rate at which it makes transitions you still assume it goes to neighbouring states on either side but that could depend the rate could depend on where you are in which case this would be some beta j times this plus the final state could be j plus 1.

You could jump out there and in reference to the random walk problem where we used alpha and beta for the biased factors let us continue to call it alpha but it could depend on j in this fashion okay. Now there are 3 possibilities for the state space. Either the state space which takes on integer values runs from 0 to n or 1 to n some finite set or it is infinite on one side and goes 0, 1, 2, 3 all the way to infinity or it runs over the full set of integers.

These 3 are 3 different classes of problems altogether because when we write equations down with finite boundaries then I have to change the rate equation at the boundaries to make sure that the space is confined to that range. Just as when we did the Poisson process I said you have the rate at which the probability of 0 decays in a time t changes that decreases with time and then 1, 2, 3 etc., etc. They are all functions of time.

This is a Poisson distribution but the rate equation for dp 1 over dt was different from that for dp 0 over dt. We'll write the rate equation down once again right. So you have to be careful about the boundaries and then you can put all sorts of boundary conditions on it but for the moment let us take j to be a general number integer. Then what does the W matrix look like once you have this. Suppose j runs from 0, 1, 2, 3 in that case you'd have a finite matrix of this kind if it is bounded from the other side too.

Otherwise it is infinite that way and what does it typically look like. Let us look at W 1, 2 for instance. So j so in this case k is 1 and j is 2. So this is 1 sorry this is beta 2 and then when k is 1 it fires. So you certainly have something here and then you have beta 2 in this case and this term contributes 0 okay. All the others have 0, 0, 0, blah blah etc. and what happens to 2, 1. So if you look at W 2 1, this term does not contribute but this contributes.

K is 2 and if j is 1 it contributes a beta and what and alpha and what does it contribute? It contributes alpha 1 and everything else is 0s down here and when it comes here you have a beta 3 contribution because that could have jumped to the left on this side and when you come here this could have an alpha 2 contribution on this side here. And what would the diagonal element be. It would of course be we know this is equal to alpha 2 + beta 2 etc.

This alone because I started with 1, 2, 3, 4 labeling things with 1, 2, 3, 4 there is no state 0 here in this problem this alone would be - alpha 1. That would ensure that this is a stochastic matrix. The sum of each row, each column must be equal to 0 you know that. But the general structure of this matrix is a triangular matrix with something on the diagonal elements, negative elements and then the super diagonal has all these data factors and the sub diagonal has all the alpha factors.

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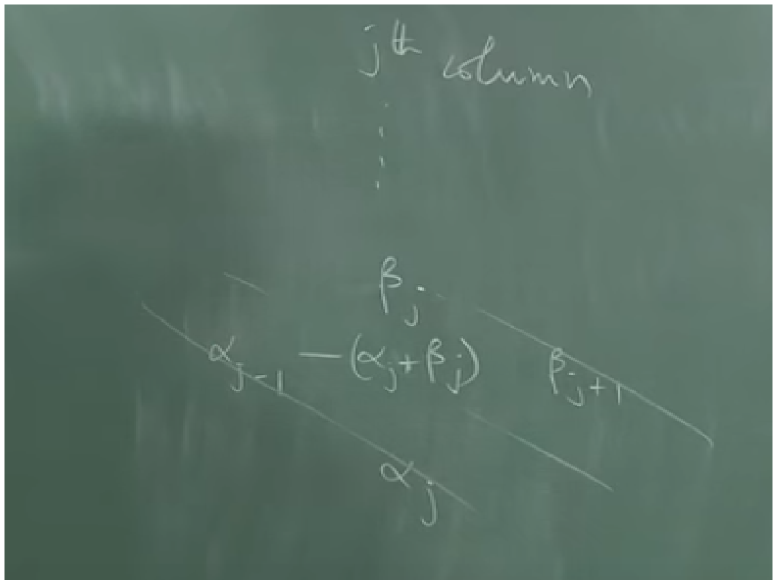

So typically at some point in the middle a the jth column jth column everything is 0 till you hit a beta j and then a - alpha j plus+ beta j and then you hit an alpha j and on the right hand side you have a beta j + 1 the risk diagonal keeps going, this diagonal keeps going and on the left hand side you have an alpha j - 1 etc. That is the general structure of this matrix okay and if there are n points to this matrix then you got to be careful at those ends etc.

And what would the rate equation correspond to in this case, what is the rate thing? Now let us get over get rid of this j because that had this connotation of a lattice point but the examples we looked at earlier like the Poisson process etc. we looked at a random variable n some number n. So what would the rate equation be. So let us write that because we are going to look at cases where n is running from 0 to capital N or 0 to infinity or minus infinity to infinity.

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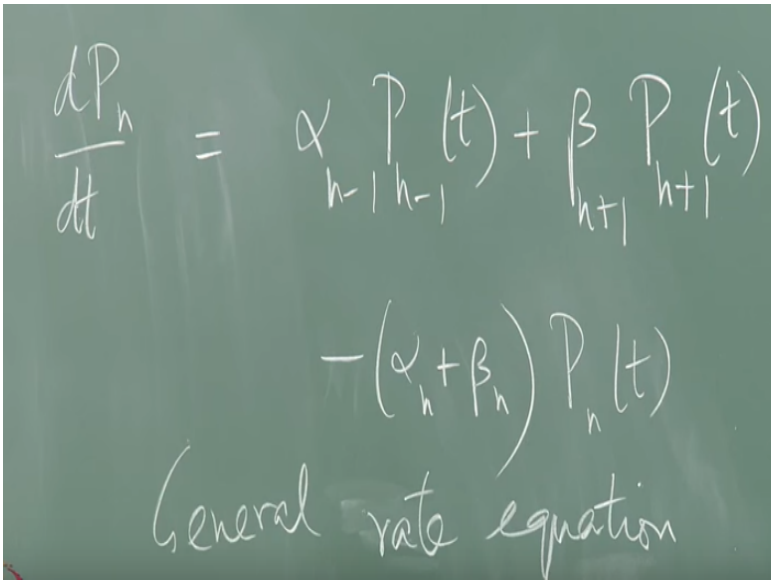

And what does that look like corresponding to this W the rate equation dP over dt dP n over dt as a function of t what is that going to be equal to. Well, it is W times this P you write this P down in this fashion and what is the first term that is going to appear. Again, you can see what is going to happen. The growth of any particular, now let us call this n, the growth of the probability here depends on fellow is jumping from in from here and people jumping in from here right.

And whatever is jumping in from here has got to be a beta but dependent on the site n + 1 and what is jumping in from here is alpha dependent on the site n - 1 right. So it is clearly alpha n - 1 P n - 1 of t + beta n + 1 P n + 1 of t - alpha n + beta n, those would be these 2 processes which go out so this is alpha n a beta n here. This fellow was an alpha n - 1 and this was a beta n + 1, so jumping in and these 2 are the ones that jump out and that is multiplied by P n of t.

That is the general equation which you have for a one-step birth and death process in which the elementary transition probabilities only connect you from the state n to the state n plus minus 1 okay and these are general functions; these rates are general function okay. Apart from the fact that they are positive or nonnegative you cannot say very much more okay. So this is the general equation, general master equation or rate equation.

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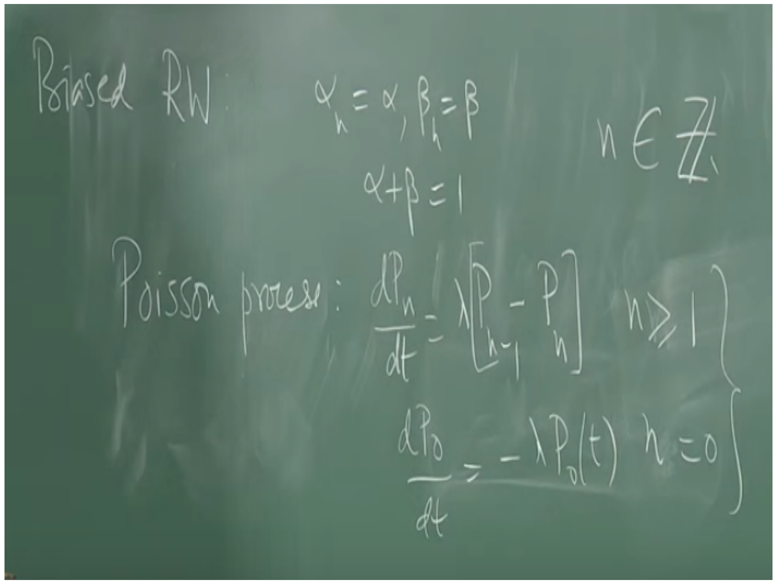

And we looked at special cases of course, we looked at special cases of it so random walk, biased random walk, alpha n equal to alpha, beta n equal to beta, alpha plus beta equal to 1 right. What about the Poisson process. What happened to the Poisson process and incidentally n was the element of the set of integers. What about Poisson? What did that correspond to? We had a certain rate lambda but n now ran from 0 upwards.

And the Poisson process equation was for n = 0, 1, 2, etc. to infinity and in this case alpha equal to beta equal to lambda but the process was dP n over dt equal to lambda times P n - 1. So you would have finished n - 1 jumps in time t and then in the time interval delta t after that you did one more with probability lambda delta t but you might have already reached n and you are now jumping to n + 1 decay so therefore it is a lost term.

So this equation was this minus P n and that was true for n greater than equal to 1 and for n = 0 you had d P 0 over dt was minus lambda P 0 of t. It is clear this is a special case of this very general thing with suitable choices of alphas and betas right. This is a pure birth process because the number n kept increasing with time. It did not ever decrease beyond a certain value. So the leftwards probabilities were all 0 and you only had increase on one side always.

Incidentally, this was a stationary Poisson process. Stationary in the sense that the statistical properties of the system don't change with time which means this rate lambda does not change with time. What would have happened, by the way we know the solution to that right.

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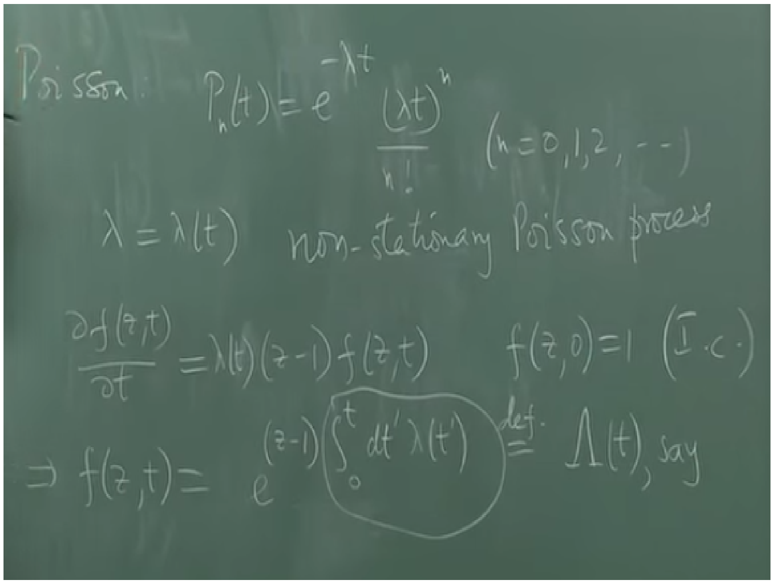

We know we know completely for the Poisson process, we know that P n of t is e to the - lambda t lambda t to the n over n! n equal to 0, 1, 2, blah blah. We know we can write down the generating function for this and compute what the coefficient of z to the n is and that is the end of it out there. We did that explicitly. By the way what would happen if this Poisson process was non stationary.

Suppose it turns out that for some reason lambda equal to lambda of t and you have a non-stationary but Poisson process. What would have happened to the solution? This equation is still true because all you are saying, all you are saying in these in this entire birth and death business is you are saying that in a time interval delta t sufficiently small time interval delta t either there is a transition to the left or the right or no transition at all.

It does not say there cannot be transitions to further points further states. It just says those are of order delta t whole square and so on and so forth whereas the transition to the nearest neighbours is proportional to delta t itself and then we got that differential equation. So now the question is what happens if it is non-stationary. You have a nice function lambda of t but it systematically changes with time in some deterministic fashion.

What would happen to the solution? I would get exactly the same rate equations as before except that this becomes lambda of t on this side and the initial condition is once again a t = 0 and is 0 okay. What would have happened to that solution? Well, the way we got that solution was we multiplied this by z to the power n and summed over n to write the generating function down right and we got an equation which said essentially delta f of z, t over delta t was equal to lambda times z - 1 f of z, t.

We got a z because there is an n minus 1 here. So when you multiply by z to the n you got to remove a z and write it as z to the n minus 1 and sum okay. So that is how the z came out. Now what is going to happen is this. Once again the boundary condition is the initial condition is the same as before. We know that only P 0 has a value at t = 0 that is 1 so f of z, 0 equal to 1. This is the initial condition. What is the solution to this equation with that initial condition?

This is not a constant. This is not a constant so how do you what is the solution? Ya. It is e to the power z - 1 integral from 0 to t dt prime lambda of t and that is it okay. You have to integrate both sides and separation of variables immediately gives you this solution okay. That is the general solution. In the case when lambda is constant that exponent becomes lambda t as you can see okay and it satisfies the boundary conditions and got a unique solution.

So if you define this quantity, you define this to be some lambda of t calling that and we are ready to write down what the probability density probability itself is. Once again it is just the coefficient of z to the power n.

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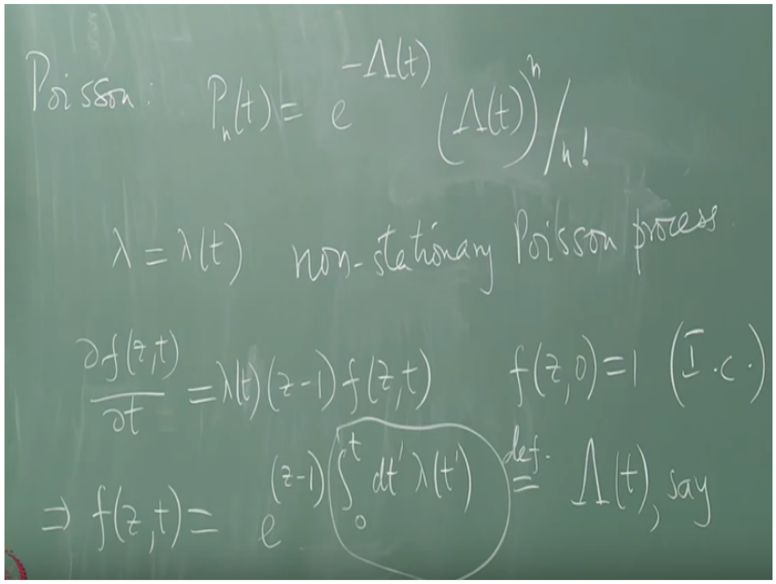

So you will get exactly the same solution except that you now have e to the minus lambda of t lambda of t to the power n over n! and that is it okay. So this particular making it non-stationary is quite revealed, we get the solution immediately okay. This occurs in the problem of what is called time dependent short noise okay but it is a trivial extension of the original Poisson process okay. What is much harder to do is solving the general equation, this equation here.

This is not at all a trivial matter even though it is a one-step process in which you only go from n to n - 1 or n + 1 even then if these coefficients, these coefficients are sufficiently independent in a complicated way then writing down an explicit solution is out of the question for the set of coupled differential equations okay. Constant, no problem. That was a biased random walk problem when alphas and betas are independent.

Well, the next thing you would guess after that which is solvable would be to say suppose these are dependent on n linearly okay that will be the next natural thing to do linearly. For instance there is a physical problem in which you have a quantum mechanical harmonic oscillator say. The levels are all quantized and let us suppose it is put bathed in radiation of the right frequency so that the oscillator can absorb a photon go to one state higher than it or it can emit a photon of the same frequency and go to a state lower right.

And then you ask what is the population probability of population of the state n of this oscillator. That problem would be exactly this kind of rate equation in which the alphas and betas are linearly dependent on n. In fact that is how Einstein actually derived that is how Planck originally got the Planck law.

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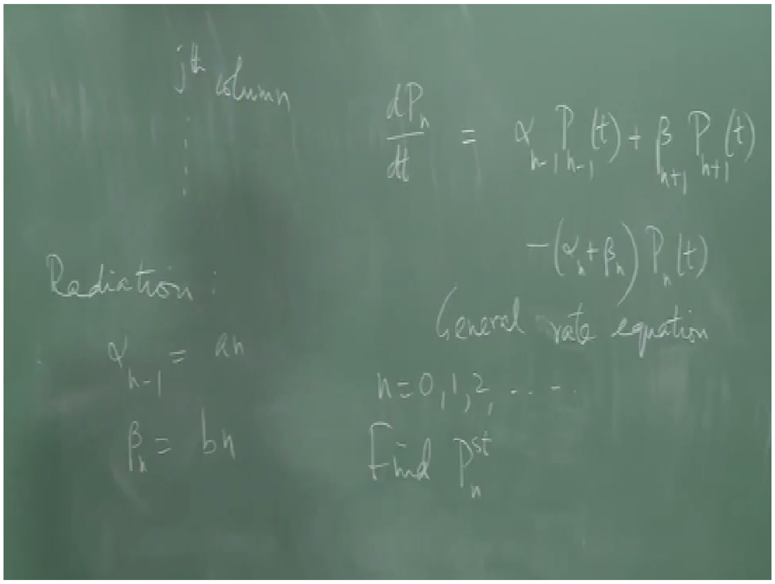

In that case what happened was for the radiation problem it turns out that alpha n + 1 equal to some constant times n and beta n equal to some of the constant b times n; alpha n - 1 was proportional to n and beta n was proportional to n once again with some constants a and b okay. I am going to leave it as an exercise for you to solve this set of differential equations given this. Not all together trivial but at least find the stationary distribution.

In this problem n runs 0, 1, 2, 3 up to infinity. They label the levels of the harmonic oscillator right. So n is and find P n stationary okay. I will give the answer too. It is a geometric distribution which depends on the ratio a over b okay. So try to derive that. What you have to do is to put this in and then equate this put P stationary here everywhere no t dependence and then equate it to 0 okay and then you try to find out what the stationary distribution is in this case okay.

But actually when you have a linear process you can go a little further and that is interesting because this rate equation can give you some physical information and here is how it happens.

Let us suppose for ease of algebra, let us suppose for a moment that n runs over all the integers so that I do not have to worry about these boundary points. Otherwise I have to be very careful about the boundary conditions and put in appropriate conditions each time.

I might for example if n runs from 0 to infinity I will have to define alpha - beta - 1 as identically 0, put in a formal beta and define it to be 0 and so on. But let us just say that n runs over all integers.

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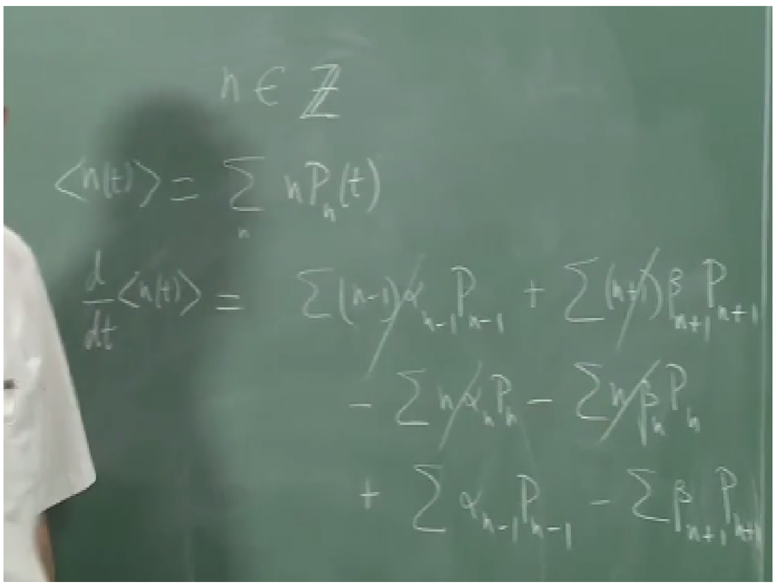

Suppose n element of 0. I look at this and I ask what is the average value of this random variable n at any time t. So what I need to do is to compute n of t, this is equal to summation over n, n times P n of t over all the integers. Since I am summing over all integers whether the summation index is n or n + 1 or n - 1 does not matter. It is over the same set. I am going to freely shift things okay. Then what does, what happens to this. What is d over dt of n of t equal to.

I multiply both sides by n and sum over n over all the integers. Then the first term gives me an n but I make it an n - 1 and remember to add that 1 later on. So it is going to be a summation n - 1 alpha n - 1 P n - 1 plus summation, now in this fellow here I multiply by n so I make it n + 1 and subtract the n later on. So summation n + 1 beta n + 1 P n + 1 minus summation n alpha n P n minus summation n beta n P n and then let us be careful.

I multiplied by n. I wrote it as n - 1. So I got to add. So there is a plus. I add 1. So this is summation alpha n - 1 P n - 1 and then I have to subtract this fellow no, no, no. I had an n, I wrote it as n - 1 + 1. I had an n, I wrote it as n + 1 so I subtract 1. So minus summation beta n + 1 P n + 1 okay.

And this is over all n but I can shift the summation index to n - 1 and it is over the same range and this term cancels this and this term cancels this completely right. So we have an interesting result which says I do not care what these coefficients are but if they are linear whatever they be I do not care.

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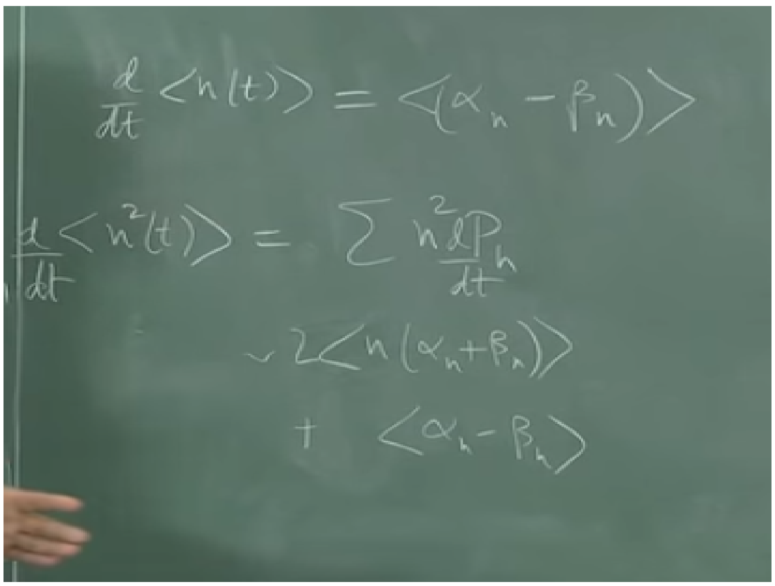

There is an equation which says d over dt the average value of n of t is equal to this quantity and that quantity but what is this equal to? Again, I translate the summation index to n but I am got the weighted average of alpha sub n whatever function of n that be. So this is equal to alpha sub n + beta sub n okay. So I have an equation actually. Pardon me, a minus okay. That equation could be nonlinear in n. In general it could be complicated etc.

So it is not trivial to solve it but you have an explicit equation for the average value. In terms of possibly higher moments in general but you have an equation. Similarly, if I multiply by n square and compute what is n square of t here? Again, this thing is equal to n square P n and the trick is exactly the same as before. Write it as n - 1 square and n + 1 square, shift it and that will cancel against these fellows here.

But there was a 2n + 1 which you added. You must subtract those fellows out, add or subtract those guys out but remember each time whenever these fellows are n - 1 you better match that here. So in the linear terms you got to be careful once again subtract etc. and you get a closed equation once again, an equation in terms of average values. So in this case, I do not remember what it is but one can work this out.

On the right hand side you are going to get things like average value of n times alpha n + beta n and then there is possibly an alpha n - beta n etc. plus something times this maybe there is a 2 here. There was a 2 remember each time, etc. So these factors can be worked out. But this is the sort of equation that you would get okay. Now tell me if the process is linear in n what is the special feature that emerges at once. Suppose the alphas and betas are at best linear in n.

Each of them is some constant times n plus some other constant. What would what happens then to these 2 equations. Sorry, this is d over dt. What happens then? What is the special feature that emerges? Well, this fellow is also linear in n. So at best it is going to involve this guy. So you get a closed equation for it right away which you can in principle solve okay and then what happens here? Again, the worst you get is a quadratic in n, the average value.

So this set of equations is closed completely and you can solve it. Even if you are not able to solve the full time dependent problem for P(n, t) you can still solve the equation explicitly for the average value as a function of t and the variance completely provided the process is linear provided the coefficients are linear. So when I say the whole differential equation these are linear equations in the P's, no doubt about that but when I say linear, nonlinear etc.

I mean these coefficients. If they are linear in n or nonlinear in n you have a big difference. If they are linear then not only you get a closed set of equations for the mean and the variance but you can also solve these equations explicitly but if its nonlinear then it depends changes from problem to problem okay. We will, I will give some physical examples of this next time but you have already seen that the simplest of cases like the random walk problem when everything that is mapped on to the random walk problem these coefficients are constant.

And then when it is linear you have the radiation as an example of this. Chemical reactions would have similar kinds of rate equations etc. okay. A very interesting question is, is the existence of a stationary distribution and the rule was and we saw already explicitly that in the random walk problem there is no stationary distribution because they ran from minus infinity to infinity.

But when you have finite boundaries you physically expect there would be a stationary distribution unless these boundaries are absorbing boundaries. They cause the system to disappear and we will give examples of that as we go along okay. So let me stop here today and take it up tomorrow.