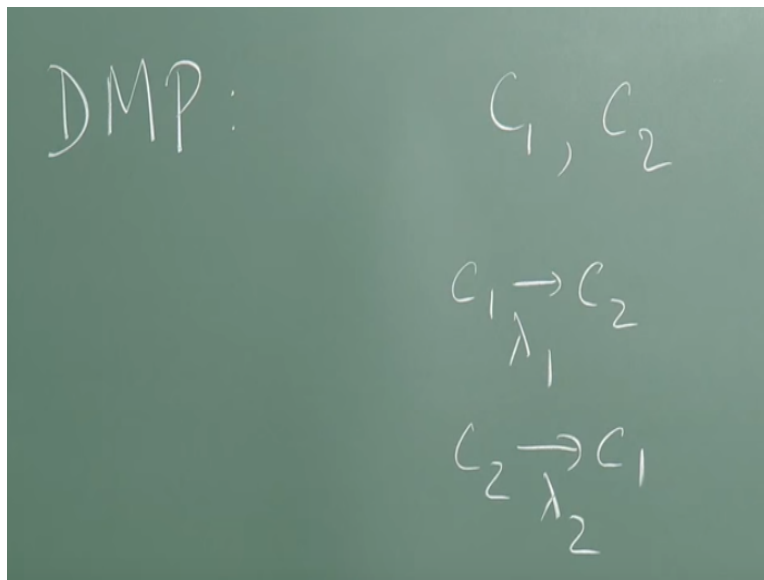


Physical Applications of Stochastic Processes
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology-Madras

Lecture - 08
Markov Processes (Part 2)

Alright, so we had started looking at the Dichotomous Markov process last time and I mentioned that the process was exponentially correlated. This is a two-state stationary Markov process in which the system flips between 2 values or 2 states with some mean rate λ_1 to go from 1 to 2 and λ_2 to go from 2 to 1 and we discovered that in general we could write down an explicit expression for the probability density, the conditional probabilities themselves for system starting either in state 1 or 2 and going to state 1 or 2 at the end.

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So if you recall this Dichotomous process which we call DMP for short systems flip between value c_1 and c_2 and c_1 to c_2 the rate was λ_1 and c_2 to c_1 the rate was λ_2 and we discovered in general that there is a correlation time which is $\lambda_2 \lambda_1^{-1}$ which is $1 / (\lambda_1 + \lambda_2)$ and that was just the harmonic mean of the mean residence times in state 1 and state 2.

I also mentioned that there are a huge number of potential applications for this very simple model of a random process and we will come across some of them as we go along 1 or 2 specific

applications. One for example could be the following. You could say what about a particle that is diffusing along a straight line in one dimension say and what it does is to move with a velocity plus c in one direction or minus c in the other direction.

Those could be the 2 states and it is randomly flipping back and forth between these 2. How far does it go? What is its mean displacement like, mean square like, and so on and so forth. So those are valid questions which have practical applications. They are models for various kinds of random process. Now, I mentioned something about the autocorrelation function and so let me explain what it is. In general I said it is the generalization of the variance of a stationary random process. So let us see what it actually tells us.

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The image shows a chalkboard with the following handwritten text and equations:

- At the top left, the title "Autocorrelation function" is written and underlined.
- To the right of the title, the equation $\langle X \rangle = \sum_j x_j P(j)$ is written.
- Below the title, the variable x_j is written.
- The main equation for the autocorrelation function is $\langle X(t) \rangle = \langle X(0)X(t) \rangle = \sum_j \sum_k x_j x_k P(k,t|j)P(j)$.
- Below that, the equation for the variance is $\langle \delta X(0) \delta X(t) \rangle = \sum_j \sum_k x_j x_k P(k,t|j)P(j) - \langle X \rangle^2$.

Autocorrelation function of a random process. So let us suppose that you have a random process which has values x_1, x_2, x_3 etc., so the typical value is some x_j and I call this the state j of this guy. Then I ask for what is x of 0, x of time t the average value averaged over all possible realizations of this random process okay. By definition this would be equal to suppose it starts with some value x sub j and we are asking what happens at time t .

And we ask for let us say the state at time t is k , you sum over all those possibility. So you sum over j , you sum over k and if the value in x of 0 is x_j then this is x_k and then you multiply this by the probability that you are in state k at time t given that you started in state j at 0 multiplied

of course by the probability a priori probability that you are in state j . That is the definition of this average as you can see okay and that is what the autocorrelation function is.

So we can call this c of t for the variable x is defined to be this quantity here. At $t = 0$ it of course reduces to the mean square value so this generalizes. It says on the average what is the degree of memory that is implicit in this, that involved in this random variable. What is this average, what is it reduced to. I expect this to start at some finite value the mean square value and decay to 0 as t goes to infinity either monotonically or in an oscillatory fashion.

This is what one would generally expect okay. This is provided the mean value is 0. As you know in the definition of the variance you also have to subtract the square of the mean. So the more general thing would be to say what is Δx of 0, Δx of t this quantity okay where this stands for the deviation of x from the mean value at $t = 0$ and at time t here and since the mean value is independent of time because it is a stationary random process this thing here turns out to be exactly the same as what we have here x_j, x_k minus the average value square.

That is not hard to show and what is average value square? Remember that this quantity is just summation over j, k $x_j x_k P$ of j, k . So this is what the autocorrelation is and the claim is this quantity tells us a great deal about the random process. Of course in a general random process this quantity stationary random process this depends only on the conditional density here. What started off as a two-time conditional density, that has now reduced to one-time argument because of stationarity and it does not say anything about the higher probabilities, the joint probabilities or the higher conditional probabilities.

But in a Markov process we have already seen that this says everything that has to be said about the process right. So in that sense for a Markov process, the autocorrelation function gives a great deal of information and one generally uses this as a characterizer of a random process to start with to measure how much memory it has etc. We will see examples of this. Now in the case of a Dichotomous Markov process, this quantity turned out to be a pure exponential, a single exponential.

Now in general of course you might have more than 1 relaxation time in a process. It need not even be an exponential. It could be a power-law decay or something like that. But very typically for this process you have an exponential decay here and we saw what this is. We could now, we can now, we wrote the answer down but we can actually put this in. We know this quantity here for a Dichotomous process. So we can put it in, do the calculation and work out what this autocorrelation comes out to be explicitly okay.

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$$C(t) = \frac{\lambda_1 \lambda_2 (c_1 - c_2)^2 e^{-2\lambda t}}{(\lambda_1 + \lambda_2)^2}$$

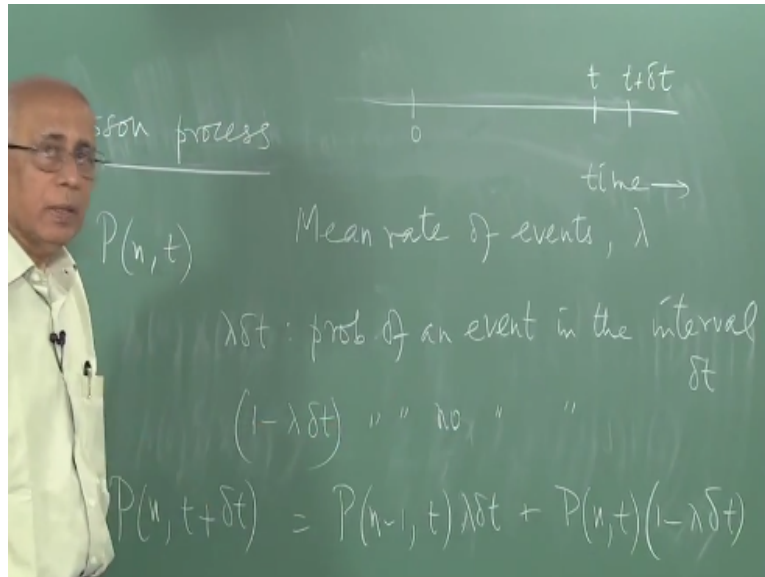
$(2\lambda)^{-1} = \text{correln. time}$

But I gave a simple argument to show that it is got to be c of t, has got to be of this form, this quantity for the Dichotomous Markov process. It turns out to be something like lambda 1, lambda 2 and then c 1 - c 2 whole square e to the - 2 lambda t / lambda 1 + lambda 2 whole square okay. So it is exponentially correlated here and this is what tells you, this is the reason for saying 2 lambda inverse equal to correlation time okay.

In this case it is quite straightforward to identify this quantity with the correlation time of this process. Now of course once things get a little more complicated, then evaluating a correlation time becomes nontrivial. Evaluating this quantity becomes a little less trivial and eventually we will talk about not this quantity itself but its fourier transform. So it is interesting to see what happens if you decompose it into fourier components and then ask how much is of the amplitude of this fourier transform is in a given frequency window okay.

And that will lead us to the concept of the power spectral density of a random process okay. We will come back to that aspect. Right now, what I would like to do is to go on to another process, a very famous one which occurs everywhere and this is the so called Poisson process.

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That is a process in continuous time okay. You have an integer value at random variable because a Poisson distribution as you know refers to a nonnegative integer value at random variable and here is a simple example of it. You ask you are waiting for a bus for instance and these buses appear at random where you are and you ask for the distribution for the number of buses in a given time interval and it is Poisson distributed if there is no correlation between different events okay.

Similarly, you have a sufficiently large sample of a radioactive nucleus and you ask I start the clock now and you ask what is the probability that you had n decays occurring in a given time interval t . This turns out to be Poisson distributed too okay. So let us ask what we know about the process and let us ask how to derive this expression here. So let me denote by $P(n, t)$; it is in continuous time, the probability that n events have occurred or whatever I'm interested in, n such events have occurred in the time interval t .

We are discussing a stationary Poisson process in other words. Again, the statistical properties are independent of time altogether okay. So I would like to have a certain equation for this in

continuous time and I argue as follows. I say that if I choose a sufficiently small increment of time Δt , I choose it to be small enough such that in that increment of time either one event occurs or no event occurs at all okay.

If the events occur with some mean rate of events is some λ inverse of some time, it is time scale. Then it implies that in a time interval Δt on the average $\lambda \Delta t$ events would occur and $1 - \lambda \Delta t$ is the probability that no event occurs at all. So either a single event occurs with probability $\lambda \Delta t$ in the interval Δt and $1 - \lambda \Delta t$ is the probability of no event in this time interval.

These are mutually exclusive events and there are no other possibilities. So for a given finite value of λ it is clear I can choose my Δt small enough that only 1 event or no event at all occurs in this interval Δt okay and that is the only assumption you need. The other assumption is, there is one more assumption which is that the events are not correlated to each other okay. They are completely independent of each other.

So if you took the time axis here and you put a cross every time an event occurred; these crosses could occur completely at random in an uncorrelated fashion okay such that the mean time between gaps, this mean time mean gap equal to λ inverse. That is the meaning of saying the mean rate is λ where mean residence time between the mean time elapsed between successive events is $1/\lambda$ okay.

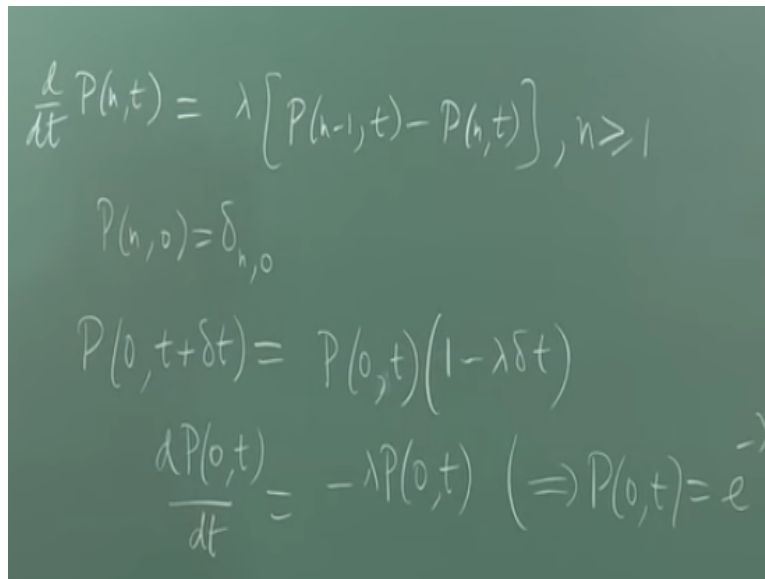
Then we need to write down an equation for $P(n, t)$ and the way one does it is to say alright let us look at $P(n, t) + \Delta t$. There are only 2 possibilities now. So I start with $t = 0$ here, time close in this direction and here is t , here is $t + \Delta t$. I ask for the probability that n events have occurred in the time interval from 0 to $t + \Delta t$ okay. Then there are 2 possibilities.

One of them is that all n have occurred here and nothing happens in between and the other possibility is that $n - 1$ have occurred here and one of them happens in this infinite decimal time interval. So the probability that nothing has happened here, no events have occurred means that sorry n minus events have occurred and the last one appears here is just this. So this has got to be

$P(n - 1)$ till time t and then you need one event to occur in that interval and the probability for that is $\lambda \delta t$.

The other event possibility is that all n have occurred in the time interval t and nothing happens in the remaining time and that is it. There are no other possibilities okay. Now of course the immediate thing to do is to move this P and t to the left hand side and divide through by δt .

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$$\frac{d}{dt} P(n, t) = \lambda [P(n-1, t) - P(n, t)], n \geq 1$$
$$P(n, 0) = \delta_{n,0}$$
$$P(0, t + \delta t) = P(0, t)(1 - \lambda \delta t)$$
$$\frac{dP(0, t)}{dt} = -\lambda P(0, t) \quad (\Rightarrow P(0, t) = e^{-\lambda t})$$

Which will immediately tell us that d over $d t$ of $P(n, t)$ is there is a λ here and there is a λ here too with a minus sign. So this is equal to λ times that is a rate so has got the right dimensions P of $n - 1 t - P$ of n, t and that is a differential equation for $P(n, t)$ okay. What is the initial condition that we need to impose? Of course no events have occurred at time $t = 0$. You are just starting okay. So this equation for what values of n is this valid?

One upwards right, n takes on is the number of events. So it is n greater than equal to 1 but you also need a probability equation for the probability that in a given interval no events have occurred at all. So you certainly need an equation for that and therefore you need to know what is $0 t$ plus δt equal to. This means nothing has happened right till t plus δt which means nothing should happen till time t and nothing should happen in the remaining time as well right.

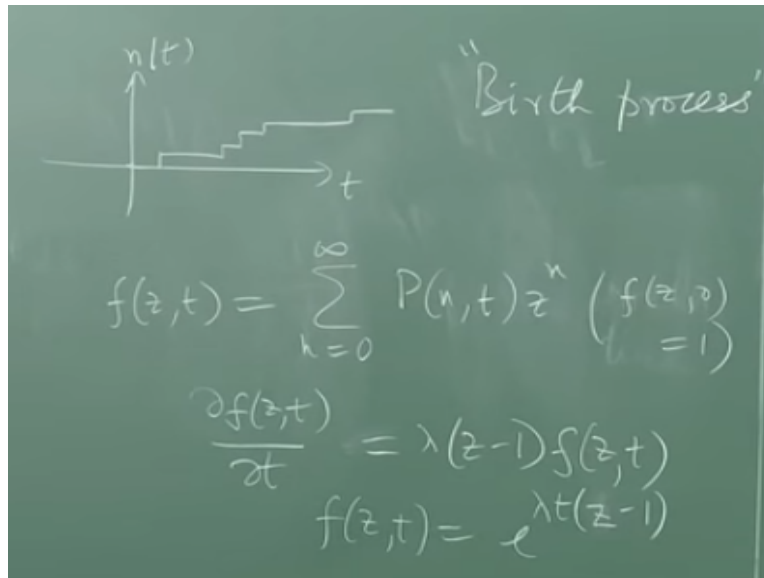
So this has got to be equal to $P(0,t)$ times $1 - \lambda \Delta t$. This is the probability that nothing happens in the interval Δt and this is probability that nothing happens till time t okay. So this says $\frac{dP}{dt}$ is equal to $-\lambda P(0,t)$ whereas this is true for all n greater than equal to 1. So it is clear that this probability can only decrease. You wait for a sufficiently long time, something is going to happen.

So the probability that nothing happens is constantly decreasing with time and the solution is immediate. The solution immediately says this is equal to implies that $P(0,t)$ equal to $e^{-\lambda t}$ times 1. That is the initial condition on $P(0, t)$ because $P(0,0)$ is 1. So this thing exponentially decreases but the rest of these fellows satisfy this set of coupled differential equations all the way to infinity and how does one solve it.

There are many ways of doing this. We could do a Laplace transform and solve it or better still find a generating function, that is the obvious thing to do. So let us write a generating function for it. Incidentally, what is the random variable out there in this problem. It is n , it is n of t and this n of t cannot decrease as a function of time. It could remain constant and then it will increase and increases by 1 each time. So it is like a step.

It is a birth process because it is constantly increasing. There is no death involved here okay. So this is like a step. It is an irregular step.

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So if I plot, typically if I plot as a function of time, I plot this random variable n of t , starts at 0 and then all of a sudden it becomes 1, remains like this and then maybe increases and then does this kind of t . It increases by 1 each time okay and it is a simple example, the simplest example of what is called a birth process. So let us define a generating function f of z , t to be a summation n equal to 0 to infinity P of n , t z to the power n .

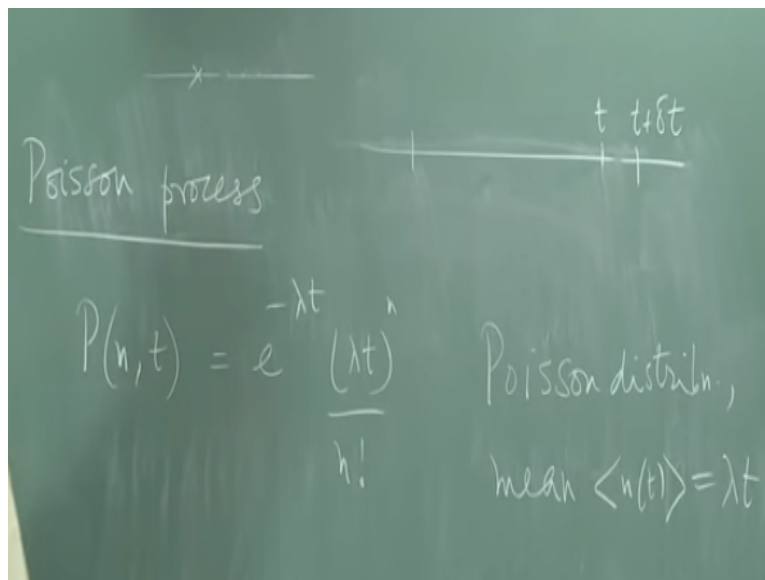
And all we need to do is to multiply this by z to the power n and sum and this by z to the power 0, this by z to the power 0 which is 1 and add it up along with the rest and therefore on the left hand side this differential equations therefore tell you that $\frac{\partial f(z,t)}{\partial t}$; I use a partial derivative because it is got another variable as well okay is equal to there is this λ here and all these terms are going to add up λ times this plus λz to the power n $P(n,t)$ is going to give you f itself and this is going to give us what.

You are multiplying this by z to the power n so take out a z and then it is z to $n - 1$ and you are summing from 1 to infinity. So shift the $n - 1$ n equal to 1 to infinity by 1. You sum from 0 to infinity but a z has come out okay. So it is immediately clear this thing is equal to λ and $z - 1$ times f of z , t . That just follows by multiplying both this equation by z to the n and summing up this whole set okay.

What is the boundary condition on this? What is the initial condition on this? Well, we know the initial condition on this is delta of n, 0. So f of z, 0 must be equal to 1 because z to the 0 is 1 okay and we need to solve this with that initial condition and the solution is immediate; z, t is e to the lambda t z - 1 and that is it. My handwriting is becoming execrable. My z's are looking like t's, the t's are looking like u's okay.

So that is the generating function and what do we need? The probability we need P(n, t) is the coefficient of z to the power n the expansion of this exponential. So it at once says that P(n, t) we have a solution for it.

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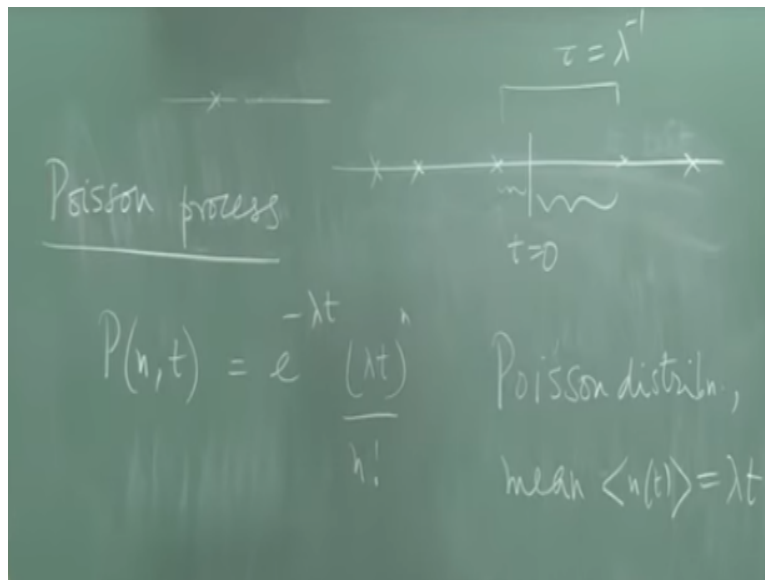
Which is P of n, t equal to e to the - lambda t and then there is a lambda t to the power n over n! if I expand this and that is a Poisson process. This is a Poisson distribution with the mean value given by lambda t okay n of t equal to lambda okay and that is it. Once you specify the mean value of a Poisson process you said everything that has to be said okay. All the variance is lambda, all the cumulants are lambda t.

So it is completely characterized by this single parameter lambda okay. Now what is the physical assumption that went into this whole business, the most crucial assumption; that these yes that these successive events are completely uncorrelated, independent of each other. So the probability that, you have a cross here, an event has occurred here; the probability that you have

another event here or here or here etc. is completely independent of what happened here, totally independent of this.

So in this process, stationary Poisson process is characteristic of something which is completely uncorrelated. If there is any degree of memory, anything other than this exponential form for $P(0, t)$ then immediately you have some memory in the problem constant and we will see examples of that. But otherwise this is certainly the simplest form you can think of. Now I ask the following question. Suppose this thing is this process has been going on for a long time. I have this radioactive sample and their decay is going on for a long time.

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So on the time axis every time there is a decay, there is a click and I put a cross on this, this thing here etc. What is the mean time between successive decays? Lambda inverse that is the mean time but I could ask for another question. I could say if this is an equilibrium what is called an equilibrium process going along for a long time a renewal process and I come along and start my experiment at some instant of time; so I start my clock at this time t equal 0.

I start looking at the system at this instant of time and then of course this successive things I record all the successive all the subsequent decays that happen. I could ask what is the mean forward recurrence time namely what is the mean value of this time. Having started the clock at

an arbitrary instant of time I could ask what is the mean value of this time. I could similarly ask what is the mean value of the time since the previous incident.

So this is called the mean backward recurrence time and this is the forward recurrence time. What would you say? How about this argument? I say look the time difference, the mean value of this gap here is τ say which is λ^{-1} and now I am putting a bar in between the 2 and I ask what is the mean value from the cross to this bar on the ones on the left and mean value from the bar to the cross on the right hand side.

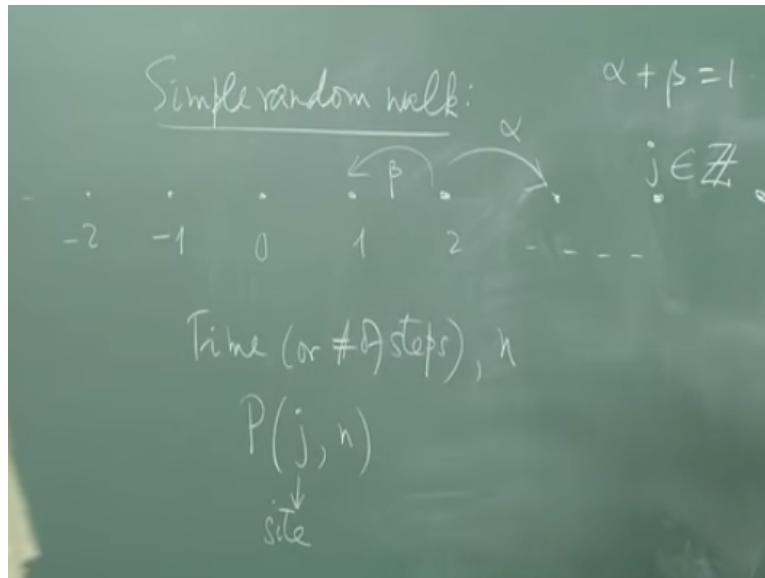
So the naive guess would be to say half τ and sort of split equally and so on and that would be wrong. That is wrong. The correct answer is each of these is again τ . Each of these is again τ because you see in some crude sense this system is completely uncorrelated. So it does not care whether it is a bar or a cross, it does not care at all. You could regard this as an event here.

So no matter where you start, the mean forward recurrence time is equal to the mean backward recurrence time as equal to the mean recurrence time itself always it is τ once again. So probability theory is full of these little paradoxes, apparent paradoxes and subtleties. But in this case it is very straightforward to actually show that this is the forward recurrence time okay.

Worth remembering because if you recall this old simple derivation of the conductivity or the mobility of charge carriers in metallic system, one of the quantities that appears in the formula for mobility is the mean time between collisions and so on. And what actually appears is the mean forward recurrence time and that is the same as the mean recurrence time. Otherwise, you miss the factor of 2 in the formula, okay. So, so much for the Poisson process.

We will look at the other processes which are governed by a Poisson process. In fact we are going to look at a whole family of random walks where something else happens at instance of time given by a Poisson process. So this thing if you like drives, what happens next okay. So let us look at an example of that and let us do this in several steps because it is a very basic process.

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And the simplest of these is the so called simple random walk. So let us do a simple random walk and let us begin in fact with an even simpler problem. Let us imagine we have an infinite lattice, linear lattice given by sites which are all distributed at regular intervals on the x axis. Let us call this site 0, this is site 1, this is 2, etc., etc. This is site - 1, - 2 on this side; an infinite linear lattice.

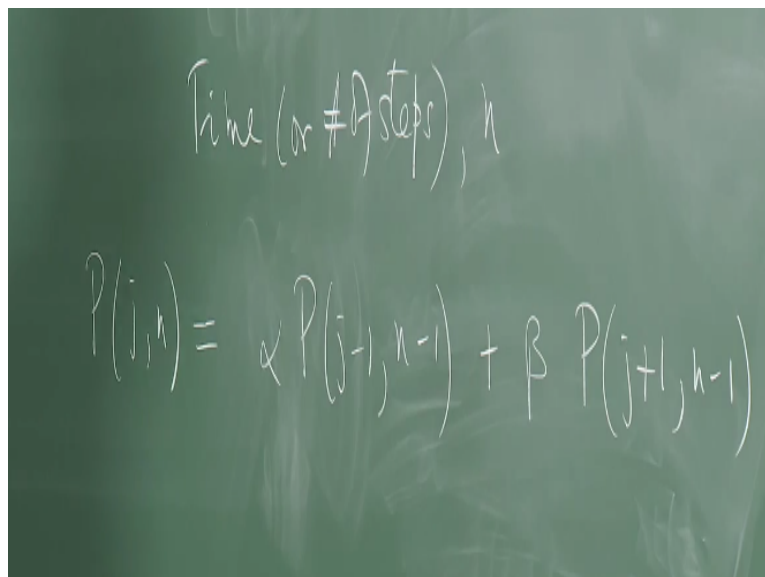
And the rule of the game is the following. I take a coin, I toss the coin and if it comes up heads I move one step to the right and if it comes up tails I move one step to the left. And I do this at the end of every second, so every time unit. So it is in discrete time and I ask where am I likely to be after n steps okay. So I would like to know in discrete time, time in discrete steps or number of steps is n. That is given to me and I take say 100 steps or 200 steps or whatever.

And I label these sites the general site by j, the index j and j is any integer and just to be specific I start at a site which I call 0 okay. As long as this is an infinite lattice, it does not matter which site you call 0; the starting site I call it 0 if they are translational invariance. All these assumptions you have to reexamine if you put other conditions. For example if I am in a finite medium with boundaries and so on then it makes a difference whether I start close to a boundary or whether I am in the middle and so on.

But otherwise on an infinite lattice it does not matter. So the site is labeled by j and I ask for the following quantity. I ask for P , the probability that I am at site j at time n and this is the number of steps. Having taken n steps, I ask what is the probability that I am at a site, any given site j and remember the probability of jumping to the right, let me call it some α and jumping to the left, let me call it β such that $\alpha + \beta$ equal to 1.

So I even include a bias in this walk and the problem is clear. With probability α I get heads, with probability β I get tails, it is a biased coin and I start the walk at $t = 0$ at 0 steps at some point which I label 0 and ask what is the probability of being at j okay. It is immediately clear that I can write a difference equation for this thing at once.

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The image shows a chalkboard with handwritten text. At the top, it says "Time (or # of steps), n ". Below that, the difference equation is written:
$$P(j, n) = \alpha P(j-1, n-1) + \beta P(j+1, n-1)$$

So we could write P of j n must be equal to, well at the $n - 1$ th step I must have either reached $j - 1$ or $j + 1$. Only then will I reach j at time n in the n th step. The probability that I have reached $j - 1$ in $n - 1$ steps is this quantity and then I need to take a jump to the right to reach the step the point j and the probability of that is α okay or there is a mutually exclusive event which is P I reach $j + 1$ at time $n - 1$ and then I take a step to the left and the probability is β here okay.

So that is my difference equation out here. Remember that j takes all integer values positive and negative and 0; n takes positive values. Starts at 0 and goes 1, 2, 3, 4, etc. And j is the random variable, we are asking for this distribution as a function in j out here for a given n . Now there

are many ways of solving this. You could write a generating function and do this and so on but actually we can write the solution down by inspection.

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The image shows a chalkboard with the following handwritten text:

- $|j| \leq n$, $n-j$ must be even
- $\frac{n+j}{2} = j + \frac{n-j}{2}$: right steps
- $\frac{n-j}{2}$: left steps
- $P(j, n) = \begin{cases} \binom{n}{\frac{n+j}{2}} \alpha^{\frac{n+j}{2}} \beta^{\frac{n-j}{2}} & |j| \leq n, n-j \text{ even} \\ 0 & \text{otherwise} \end{cases}$

First thing to notice is that it is clear that j must run from $-n$ to $+n$ because there is no way you are going to reach a point greater than n in magnitude on either side in n steps. So j is less than equal to n in modulus. So it is bounded on either side by the number of steps okay. The other point is that if n is an even number, you can only end up with an even site because to go to the site 3 for instance you cannot do it from 0 in an even number of steps.

You need an odd number of steps to do it. You go from 0 to 1, 1 to 2, 2 to 3 in an odd number of steps. Otherwise 5, you shoot, overshoot go to 4 you come back to 3 and so on. So it is immediately clear that $n - j$ equal to j modulus n must j must be even okay; n and j must have the same parity. If n is odd, j is odd. You can only end up in odd numbered site and if n is even you can only end up in an even numbered site okay.

Then the way to write the solution down is to argue as follows. Suppose here is j , some site j and I start at the origin I have to take at least j steps to reach that point right. That is for sure. So j steps to the right and that leaves n minus j steps and those steps have to cancel out in pairs. I overshoot, undershoot etc., and I cancel them out in pairs. So the number of steps to the right that I take has got to be $j + n - j$ over 2 right steps assuming that j is positive.

We will come back and see what happens when j is negative. So I need that many steps to the right and the remaining steps are to the left. So it says $n - j$ over 2 left steps. But this is $n + j$ over 2. So the probability $P(j, n)$ has got to be α to the power $n + j$ over 2 that many steps to the right and they are all independent of each other, the probability of each step to the right is α ; so you have this factor.

And then the rest have to be β to the $n - j$ over 2 and of course you do not care in what order you take these steps. Any sequence will do, will get you at the point j . So that tells you P of j, n equal to $\binom{n}{n + j \text{ over } 2}$ binomial coefficient times this and that is it. That is the answer. So this says $\text{mod } j$ less than equal to $n - j$ $n - j$ even. And in all other cases, the answer is 0, 0 otherwise okay. That is it.

That is the solution and I leave it to you to check that it actually satisfies this condition. These 2 binomial coefficients will add up to give you this quantity here for any $\alpha + \beta$ equal to 1 okay. Now I leave you again, leave it to you as an exercise to convince yourselves that if j were negative exactly the same formula is still applicable. It does not really matter okay. So this is the solution out here. What is the sample space of j ? What is the sample space of j here. We have to be a little cautious. Remember n is given to you.

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Handwritten mathematical derivation on a chalkboard:

$$-n, -(n-2), \dots, (n-2), n.$$

$$\text{Let } k = \frac{n+j}{2} \quad k = 0, 1, 2, \dots, n.$$

$$P(k, n) = \binom{n}{k} \alpha^k \beta^{n-k} \quad (\text{Bin-distrib.})$$

For a given n , what is the least value it could have, $-n$; it can be on the left hand side, it can be at $-n$ right. So the sample space of j is $-n$. Can it be $-n + 1$? No, no because for a given n it must have the same parity as n right. So this is $-(n - 2)$ dot, dot, dot, right. Those are the only possibilities for a given n . Whether it is even or odd we do not care but this is the set of values that j can have okay.

So it is almost like a binomial distribution if you like. What should we do to make it a binomial distribution? So you see immediately it is a binomial distribution. We would like to change the variable such that the variable runs from 0 to n over all integers from 0 to n . So the way to do it is to say alright I call a step to the right as success and a step to the left a failure for instance then I am counting the number of successes and that is $n + j$ over 2.

So let us introduce a random variable. Let k equal to $n + j$ over 2. This has got to be divisible by 2 because $n + j$ is even and what is the parameter space of sample space of k up to n ; k runs from 0 to n right. And then what is the probability that you have in n steps you have k successes. That is p^k of k , $n - k$ alpha to the k beta to the $n - k$ okay and that is of course the binomial distribution okay.

So we see that the simple random walk on a linear lattice, a biased random walk is essentially got a binomial distribution okay. It did not look like it when I used this variable the site index j when you had to be careful about the parity and so on but once I write it in that form it is exactly the binomial distribution okay. What is the average value of j . What do you think will be the average value of j . We can now compute it.

You can put a j here and you can sum it or you can do a generating function. You could put a generating function here, you put a generating function here; it does not matter. It is essentially the same thing. What do you think is the generating function here? Remember that this j runs from $-n$ to $+n$ in steps of 2 always.

So you could again say I take a z to the power j and I sum it up and things like that and you are going to get a z every time you move to the right, a $1/z$ every time you move to the left. So

it is of the form $\alpha z + \beta / z$ the whole thing to the power n . So what do you think is the average value?

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$$\langle j(n) \rangle = n(\alpha - \beta) \quad f(z) = \sum_{j=-n}^n P(j, n) z^j$$

$$\text{Var}(j) = 4n\alpha\beta \quad = (\alpha z + \beta z^{-1})^n$$

This is a function of n . It is an n step random walk. So it is a function of n and what is this equal to? You got to put that in here. Well, first of all if α is equal to β you expect it is not going to go anywhere. The average value is going to be 0 right? But now α is not equal to β and if α is bigger than β you expect it to be positive. If it is less than β you expect a drift to the left. So you would sort of naively expect that this thing is proportional to $\alpha - \beta$.

It essentially is because this difference $\alpha - \beta$ is acting like a drift, it is acting like a drift velocity. So that is like the mean drift velocity because in every time step you have a chance $\alpha - \beta$ of going either to one side or to the other side okay. And of course distance is time multiplied by the velocity, average velocity so time here is n so I expect this okay. Check this out.

Use the generating, by the way you can write f of z equal to summation j equal to $-n$ to n P of j , n z to the power j and with all those restrictions because I already said that unless j satisfies these conditions the P is 0. So formally you can just write it like this and this will turn out to be not surprisingly αz over $\beta z + \beta z$ inverse to the power n ; f of 1 is 1 the generating thing is normalized.

And when you differentiate this once you are going to get the average value and that is going to give you an $\alpha - \beta$ when you differentiate this. This fellow will go to $n - 1$ but it will become 1. So this is the answer out here. This acts like a drift velocity, this $\alpha - \beta$ and this has got significance because when we do diffusion in the presence of an external force like in sedimentation or something like that you will see that it is precisely modeled by the continuum version of this random walk here or I have a charged carrier and I have an electric field and I am asking for the mobility.

This is going to depend, this drift velocity is going to depend on this bias. What is going to be the variance of j as a function of n , the variance. There is a drift. It is very clear. So that is increasing your time and if you square the mean it is going like the square of the time, square of n . It is ballistic motion almost. On the other hand you are asking for the variance. So you are subtracting this portion out, this systematic drift out. And then what do you expect.

You expect it to behave like an unbiased random walk and in an unbiased walk, what is the mean square displacement proportional to as a function of time? It is linear in time. That is the famous diffusive process. So this will turn out to be $4 n \alpha \beta$ because if α is equal to β equal to half unbiased you expect it to be exactly in okay. That is the famous diffusive behaviour. The mean square displacement goes like exactly goes like goes linearly in time okay.

So check these out and again you can do that by generating function differentiating etc, etc. and verify that this is true. What we would like to do is to take this and do it in continuous time and now the rule is different. I am still on this lattice. I still toss a coin and with probability α moved to the right or β moved to the left, but I do so at random intervals governed by a Poisson process. So every time there is a cross I toss my coin.

So this means the instance of time at which I take the jumps are random completely. And the next step will be to say even that is continuous in the sense that I have a continuous diffusion process. This we will see when we go to the complete continued limit but right now we started

with a discrete lattice in space and discrete time and now I am making time continuous and saying that I have a mean rate of jumps governed by a Poisson process.

And I want to know what is the probability that I hit this point j at some instant of time, I am at j . So that is now a birth and death process because I look at negative j as death and increase in j as birth. And what is the equation going to look like.

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$$P(j, t + \delta t) = \lambda \alpha \delta t P(j-1, t) + \lambda \beta P(j+1, t) + (1 - \lambda \delta t) P(j, t)$$

$$\frac{dP(j, t)}{dt} = \lambda [\alpha P(j-1, t) - P(j, t) + \beta P(j+1, t)]$$

$$P(j, 0) = \delta_{j,0}$$

I have a $P(j, t)$ and I ask what is P of $j, t + \delta t$ equal to and what are all the possibilities. Here is the site j . So let us draw a picture. Here is the site j and this is the neighbouring site $j + 1$ and that is the site $j - 1$. So to be here at time $t + \delta t$ with a mean rate of jumps λ it is clear that I should have been here or here at time t and then in the interval time δt I either jump to the right or to the left right. So what is the equation what should I write here.

$P(j - 1, t)$ and then with probability $\lambda \delta t$ I jump to the right hand side and that probability is $\lambda \alpha \delta t$. Remember α is the a priori probability of getting a head in the coin plus $\lambda \beta P(j + 1, t)$ okay and $\alpha + \beta$ is 1 but we are not done yet. You might already reached the point j at time t and you do nothing in the time δt . So you must add to this $+ 1 - \lambda \delta t P(j, t)$.

So that is the equation, the master equation if you like for the probability $P(j, t)$ and all possibilities have been covered. Again, I take this term to the left hand side and I divide by Δt and take the limit. So I get dP of j, t over dt equal to and what is the equation that I get. There is a λ that comes out and then $\alpha P(j - 1, t) - P(j, t) + \beta$. That is the equation. What is the initial condition? Well I tell you I start at the origin at t equal to 0 right?

So what is the initial condition? A Kronecker delta; $\delta_{j, 0}$ okay. So there are these 2 gain terms but there is also this loss term here and the sample space of j is all integers minus infinity to infinity okay. Again, I write a generating function etc., etc., and solve this equation as before. But now, the generating function will be over all integers.

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The image shows a chalkboard with the following handwritten equations:

$$f(z, t) = \sum_{j=-\infty}^{\infty} P(j, t) z^j \quad \text{I.C. } f(z, 0) = 1$$

$$\frac{\partial f(z, t)}{\partial t} = \lambda \left(\alpha z + \frac{\beta}{z} - 1 \right) f(z, t)$$

$$\Rightarrow f(z, t) = e^{-\lambda t} e^{\frac{\lambda t}{z} (\alpha z + \frac{\beta}{z})}$$

$$\Rightarrow P(j, t) = e^{-\lambda t} \left(\frac{\alpha}{\beta} \right)^{j/2} I_j(2\lambda t \sqrt{\alpha \beta})$$

So I define f of z, t equal to summation j equal to $-\infty$ to ∞ P of j, t to the power j . So I multiply this by z to the j on both sides and sum over from $-\infty$ to ∞ . On the left hand side I get Δf z, t over Δt and that should be equal to on the right hand side. Well, this thing here if I multiply by z to the j just gives me an f straightaway with a λ . This gives me α times what? I got a z to the j but there is a $j - 1$ here.

So I pull out a z and then it is a summation from $-\infty$ to ∞ . Whether it is in j or $j - 1$ makes no difference. You still get the quantity f right. So it immediately says this is equal to λ times α times z because there was an extra z here which you needed and there was a

1 over z here. You need z to the j but you need a z to the j + 1. So you divide by z. You multiply by z to the 1 z and you divide it. So it is + beta over z - 1.

That is still going to be there times f okay. And what is the solution to this equation? Well, there is no t dependence anywhere. So it is just e to the power whatever is there times t, times initial condition but what is the f of z, 0; 1 because this P is delta of j, 0. So it is clear that the initial condition is 1 because this quantity is delta of j, 0 at t equal to 0. So this implies e to the - lambda t that comes out and then e to the lambda t okay.

And that is the exact solution for the generating function. What we need is the coefficient when you expand this in powers of z to the j in powers of z you need the coefficient of z to the power j but now remember j runs from - infinity to infinity. It is not a power series in positive powers of j. It is a Lawrence series clear. But we know by our experience now that this is the generating function for something or the other. For what? For the modified Bessel functions ya.

This is our famous formula which keeps appearing all over again and we will see why it is the modified Bessel function.

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The image shows a chalkboard with the following handwritten text:

$$e^{\frac{t}{2}\left(z + \frac{1}{z}\right)} = \sum_{j=-\infty}^{\infty} I_j(t) z^j$$

modified Bessel fn. of the 1st kind

So if you recall, if you recall e to the power t over 2 z + 1 over z equal to summation j equal to - infinity to infinity I j of t z to the power of j. This guy is the modified Bessel function of the first

kind. It is got a lot of fascinating properties. It is like an exponential. It is an entire function; nice, smooth analytic function of its argument. We know what its behaviour is for large values of the argument etc. We will write that down. We will write down some of those things.

But it is the generating function and therefore you can read it off. You can therefore read off what $P(j, t)$ is. This factor remains, $e^{-\lambda t}$. I need a $\sqrt{2}$ here. So I will put a $\sqrt{2} \lambda t$ and divide by a $\sqrt{2}$ so that gives me that half. But this has to be of the form something plus its reciprocal. So the trick is to take out the square root of $\alpha \beta$ right. So this immediately becomes α / β to the power $j / \sqrt{2}$ $e^{-\sqrt{2} \lambda t}$ $\sqrt{\alpha \beta}$ and that is the solution.

It is a very weird kind of distribution. It is not a binomial. It is not negative binomial. Does it remind you of anything? It reminds you of the Skellam distribution, the distribution of the difference of 2 Poisson random variables right and indeed it is so because you see we are now saying that you have a Poisson process with rate $\lambda \alpha$ and that is when the jumps occur. But the jumps are completely uncorrelated with each other. So you really have 2 Poisson processes mixed in.

One corresponding jumps to the right with average rate $\lambda \alpha$ and jumps to the left with rate $\lambda \beta$ and now if you say I am going to be at the point j it is the number of steps to the right minus the number of steps to the left. So it is the difference, this j is the difference of 2 Poisson processes with different rates. But we know what that is. We know what that distribution looks like.

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$$e^{\mu(z-1) + \nu\left(\frac{1}{z}-1\right)} \quad \begin{array}{l} \lambda \rightarrow \infty \\ a \rightarrow 0 \end{array}$$

$$\mu = \lambda \alpha t, \quad \nu = \lambda \beta t$$

$$f(z, t) = e^{\lambda \alpha t(z-1) + \lambda \beta t\left(\frac{1}{z}-1\right)}$$

$$= e^{-\lambda t} e^{\lambda t\left(\alpha z + \frac{\beta}{z}\right)}$$

$$\langle j(t) \rangle = ? \quad \text{Var}(j(t)) = ?$$

We know that if you have a single Poisson process with mean value μ then the generating function is this. But now I got 2 different processes with rates μ and ν and I want the difference between these 2 guys right. So it immediately says that it is this + ν times 1 over $z - 1$ because there is a - sign when you write the generating function and becomes z to the - n in that case.

But all we got to do is to write μ is $\lambda \alpha t$ and ν equal to $\lambda \beta t$ and that is it. That is the generating function. So I know that the generating function f of z, t in this case has got to be e to the $\lambda \alpha t z - 1$ + $\lambda \beta t \frac{1}{z} - 1$ but $\alpha + \beta$ is equal to 1 . So this is precisely e to the $-\lambda t$ e to the whatever was written there $\lambda t \alpha z$ plus β over z .

That is precisely this quantity and then of course you use the generating function formula for the modified Bessel function to write the solution down explicitly okay. I urge you to find out if this actually satisfies the difference equation. So go back and ask does this satisfy that differential equation? It is an interesting exercise to check out that the differential equation is satisfied okay but it is a unique solution in this case. Now, we can ask interesting questions.

Actually, we can either work with that Bessel function which is a little messy unless you know all the properties of it or better still work with a generating function. Because if you want to find

mean values and so on it is sufficient to work with a generating function. So compute now what this quantity is. It is now a function of t and then the variance this equal to and you can do so very easily by looking at just this guy here okay.

So take successive derivatives etc. and confirm that indeed the mean value of j of t depends on α minus β and the variance is linear in the time out here, again characteristic of diffusion. So this is an example of a birth and death process okay. Again, it is a very simple process because we have said that the rate is exactly the same everywhere and we have said only nearest neighbour jumps are allowed.

It is possible that you may have random walk in which you jump not just nearest neighbour but with some probability beyond the nearest neighbour. In fact, it could be a range, a whole range of values with decreasing probabilities. Those are much harder problems to solve because you would not get difference equations with constant coefficients. You are going to get coefficients which are functions once again and those are much trickier to solve.

But this is the simplest random walk here. So what have we done so far? We looked at the discrete space, discrete time random walk with simple random walk, biased random walk and got an exact solution for the probability. Then we looked at the discrete space but continuous time random walk and got another exact solution. It remains to make the space also continuous. All we got to do is to put a lattice constant a and then I make a go to 0 in such a way that I get a finite limit. The time already has become continuous here.

So what you need to do is to say the step length is going to become infinite decimal, the rate of taking steps is going to become infinite or the mean time between steps is going to go to 0. So we are going to take the limit λ tending to infinity and the lattice constant a tending to 0 in such a way that this goes into the diffusion process continuous diffusion and then we will work backwards and show it satisfies a diffusion equation okay.

Now what sort of limit would you expect this to be? Would you say λ times a should be finite? No, because you see it is a diffusive process. So in some vague sense, and some very

general sense the square of the distance goes like the time. So you got to be a little careful. You want λ goes to infinity, a goes to 0 such that λ times a square is finite and then you get a finite answer a diffusion constant.

Because what is happening otherwise is that you see if you say a system has the square of the distance is proportional to the time it means in an infinite decimal time Δt and infinite decimal distance Δx the ratio Δx over Δt must be infinite because it is Δx whole square/ Δt that is finite. So this means Δx over Δt must be itself infinite; equivalent to saying that a diffusing particle formally has an infinite velocity and we will see what the meaning of that statement is.

What is meant by saying that Brownian motion has infinite velocity okay. We will look at it from various angles but it is already emerging. Those things are emerging here. So although this is a very simple process this guy it has very intricate properties and we will look at some of these properties as we go along. But before that we will look at other random processes related to this thing here. Let me stop here and we will take it up on Monday.