Physical Applications of Stochastic Processes
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Lecture - 07
Markov Processes (Part 1)

So last time you recall, we were talking about the master equation for Markov processes and just to refresh you, your memory we said

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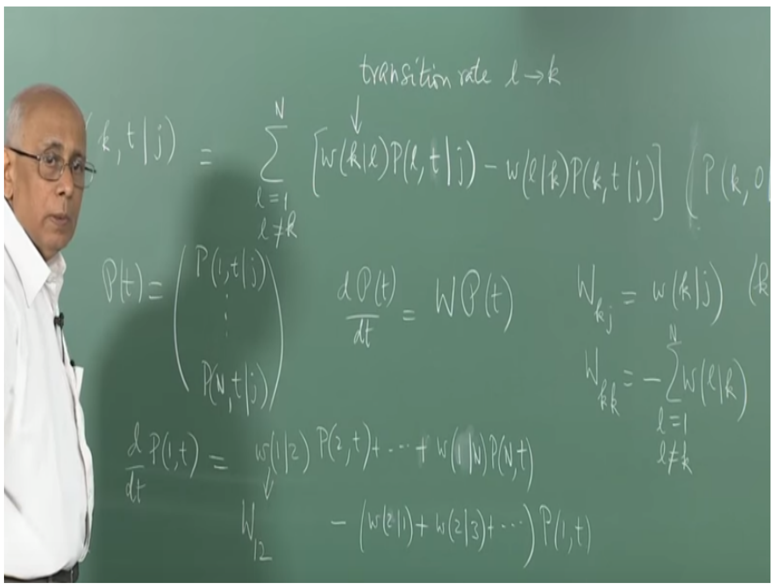

Consider a conditional probability density that the system is in state k at time t given that it was a j at time 0 and this quantity this set of quantity set of quantities obeyed a set of coupled differential equations which was of the form d over dt of this was equal to on the right hand side you had a summation over all possible intermediate states so l = 1 to N w k l P of l, t, j minus w of l k P of k, t given j okay.

And these quantities were the transition rates to go from the state l to the state k and there was a constraint here which said that l not equal to k and this is what I call the master equation. It is a couple set of first order differential equations for this quantity and of course the initial condition could be anything you want you care to specify since we are talking about conditional probabilities the initial condition is P of k, 0, j equal to delta k j.

So the task is to solve this set of equations given that initial condition and then you have a whole host of probabilities okay. Now notice that I wrote this rewrote this as I wrote a column vector so say let P(t) be a column vector with elements P(1, t) for each j etc., up to P of N, t, j in this fashion and then this equation became dP of t over dt equal to a matrix w acting on P(t).

And this matrix W as you can easily verify has the following elements W k j equal to W k j for all k not equal to j all off diagonal elements were of this form and the diagonal elements W kk was equal to minus the sum of all the other elements in that row okay. So this is equal to W for a given k if k is 1 for example it is W 2 1, 3 1 etc., etc. So it is W l k summed over l equal to 1 to N, l not equal to k. So we have this picture of a matrix where each column, each row adds up to 0.

So the determinant of W is 0. Now W has real elements. All the off diagonal elements are either 0 or positive and the diagonal elements are all negative. Just go back to this equation. Let us write this equation for instance for the k's when k is 1 okay and let us suppress the index j though it is not necessary here. So essentially we have a equation like d over dt P of 1, t equal to on the right hand side you have to sum over all k, 2, 3, 4, etc., etc.

So it is equal to W 1 2 P of 2, t plus dot, dot, dot, plus W N sorry I put k = 1. So 1 N P of N, t minus this quantity here which is just a sum over l leaving out the index 1. So this is equal to minus W 2, 1 plus W 2, 3 plus etc. times P(1, t) in this fashion and this is what I call W 12 capital W 12. So it is clear that all the off diagonal elements of this big matrix W are precisely W 1 2, W 2 3 and so on so forth multiplying P 2, P 3 etc. column vector and what multiplies P 1 because after all what does this equation imply.

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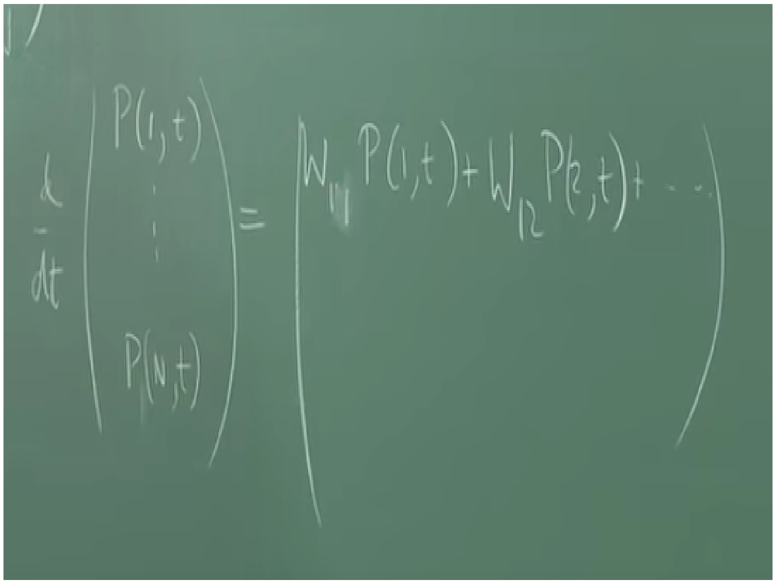

It says P of 1, t dot dot dot P of N, t the d over dt of that equal to W times P. So that is W 1 1 P of 1, t + W 1 2 P of 2, t plus dot dot dot dot etc. on the first column and then similarly for the other guys. So d P 1 by dt is this fellow here and W 1 1 as you can see is minus of all these just writing this out explicitly okay. So the important thing to remember is that the off diagonal elements of W are the transition probabilities transition rates and they are nonnegative.

They are either 0 or positive okay. If 2 states are not directly connected then that particular W happens to be 0 okay. We will be making some assumptions here which I have not technical assumptions about the nature of this W matrix. We will talk about it later on when I give you special cases of it. There are there could be situations where, we are talking about the generic case in other words we are talking about a Markov process in which you can reach any state from any state no matter where you start.

There are no subsets of states which are you know closed among themselves and so on. So we are taking the general case where all the W's exist and then you have transitions possible. Now you must remember that it is entirely possible that you may not be able to go from say state 6 to state 8 directly. There may be a 0 transition probability but given enough time you may go from 6 to some other state 5 and then 5 to 8 and so on.

So that possibility exists of course and that is really what happens most of the time because we will look at cases where you may be able to jump only from a given state to neighbouring states on either side and yet over a period of time the system wherever you start will reach any other state okay. Okay so we have this W matrix and then we wrote down a formal solution to it and that was just the exponential of this matrix W.

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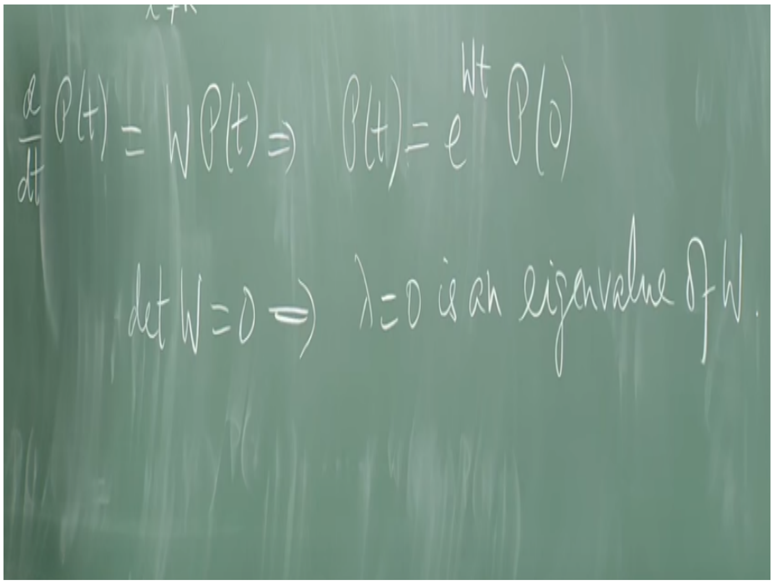

So we had d over dt P(t) equal to W times P(t) and this implied that P(t) was e to the Wt is equal to e to the Wt times P(0) whatever that is and in the case we are talking about this P(0) is a column vector in which the elements are all 0 except for the jth row which has got 1 because you are starting with the state j okay. So the task reduces to finding this quantity here.

And the statement made was what sort of time dependence can we generically expect and the answer is you expect things to decay in time to some equilibrium distribution and we already know what that distribution is. We have some idea of what it is because remember that determinant W equal to 0 implies lambda equal to 0 is an eigenvalue of W. You can actually prove more than this. You can show that this 0 is a simple eigenvalue.

That is not a repeated eigenvalue generically. Moreover that it has a right eigenvector which will be the stationary probability distribution and in the simplest cases we are looking at it is a unique vector. What is important is when you are going to identify the column the elements of this eigenvector as corresponding to probabilities we got to make sure that the elements cannot be negative okay.

So we need a guarantee that this eigenvector has elements which are not only real but also have are also nonnegative. There are theorems which assure us of that. So one of them was the statement I made that the eigenvalues of this W can never have positive real part.

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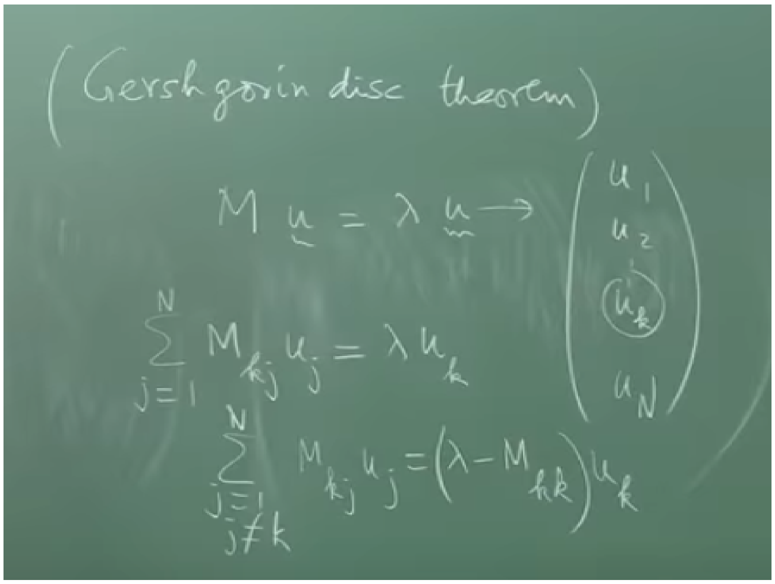

And the reason I said was due to this disc theorem which said that if you give me a general matrix m and this was the famous Gershgorin disc or circle theorem and what it says is if you give me an N by N matrix M then take its diagonal elements and mark those points in the complex plane and take the sum of the moduli of all the off diagonal elements in that row add them all up and draw a circle about the center point of that radius and the element the eigenvalues are guaranteed to remain inside these or on these discs.

That was the statement. This is not a hard theorem to prove because look at it here is a simple way of doing this. Suppose lambda is an eigenvalue. Then it says some M times some eigenvector u equal to lambda times u. Now let us look at a particular eigenvalue lambda and let us look at a corresponding eigenvalue lambda and corresponding eigenvector u. Then suppose this u is of the form u 1, u 2 etc.

And let us say the kth element in it is the largest in magnitude okay. So this fellow has elements u 1, u 2 up to u N and let us suppose that some element k this fellow has the largest magnitude in magnitude over all these of all these elements. Then if you write this element down it is clear that M k j, u j must be equal to lambda times u k. I write this down for that element out here and there is a summation over j running from 1 to n in this fashion.

So let us take out the j equal to k element and put it on the right hand side and then it says summation j equal to 1 to N, j not equal to k M k j mu j equal to lambda - M kk on u k okay. Now take modulus on both sides.

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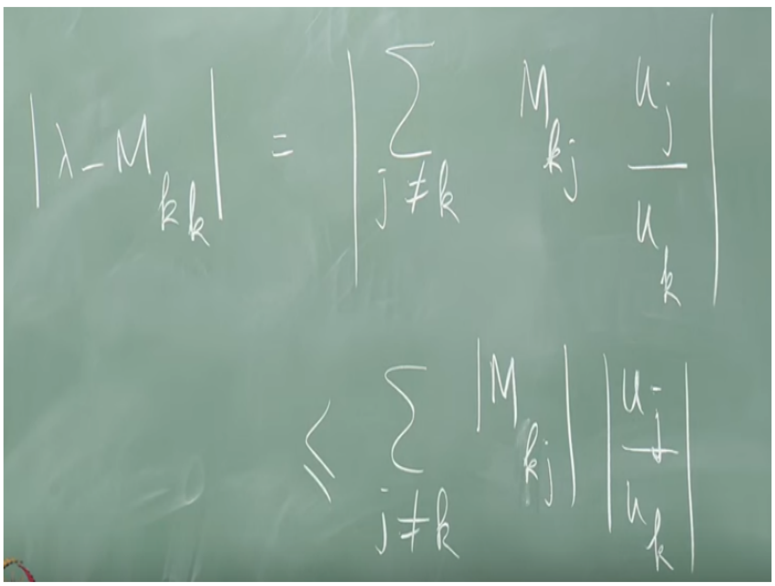

Then it immediately tells you that modulus of lambda - M kk times modulus of u k I will transfer it to the right hand side equal to summation j not equal to k summed over j M k j u j over u k modulus. But of course this is less than or equal to summation j not equal to k modulus M k j times the modulus of this; because if I take the modulus inside the summation I get maximize the sum itself.

But these numbers are guaranteed to be less than or equal to 1 because we took this to be the largest right. So this is less than equal to summation j not equal to k modulus M k j okay and that is the disc theorem because it says this eigenvalue is inside or on the circle of radius M kk whose radius is given by this out here. So this region is a disc and you are guaranteed the eigenvalue is inside or on that disc okay.

So all the eigenvalues of this M are either in or on the union of all these discs on the Gershgorin discs okay. There are further refinements to this. For example if one of the disc is disjoint you can show that that disc has at least one eigenvalue etc., definitely has an eigenvalue and so on okay. Now what we need is just one part of this theorem applied to this matrix W.

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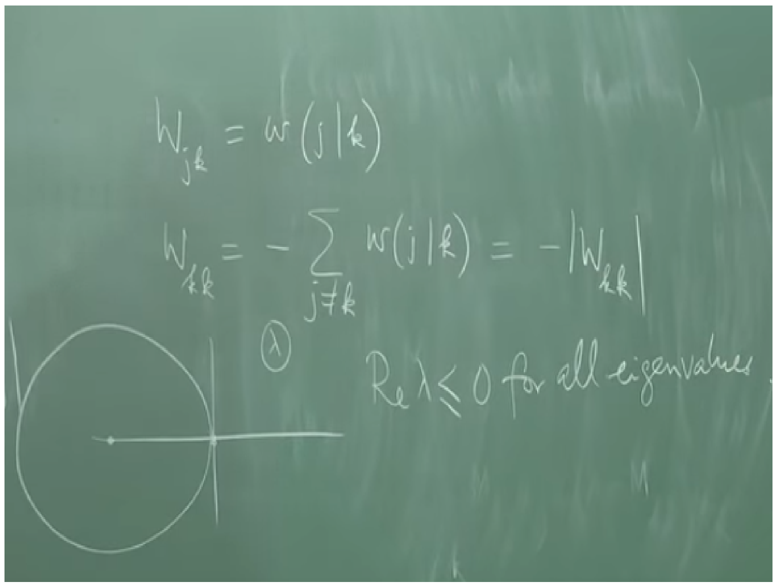

Remember that we had this W whose off diagonal elements j k were all w j k. These guys were all positive numbers and the diagonal W kk was equal to minus summation j not equal to k w j k. So that immediately tells us that 0 is an eigenvalue in the eigenvalue plane and all these numbers the diagonal elements are all at negative values because these are nonnegative numbers inside here.

So this is equal to minus modulus W k k and so you immediately know that the centers are all sitting here and this look like this, each disc looks like that. It touches 0. Therefore the real part of lambda is less than equal to 0 for all eigenvalues. So the eigenvalues are such that e to the minus lambda t must be k to 0 except for the 0 eigenvalue when it remains at 0 there is no t dependence at all okay.

Therefore W is a relaxation matrix, things relaxed to the equilibrium state okay and what is that given by. Well that stationary distribution is not difficult to write down.

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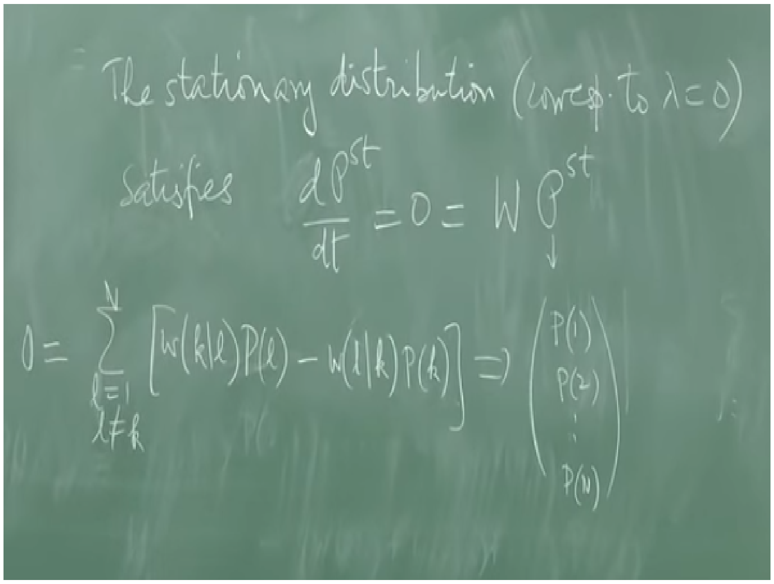

So the stationary distribution corresponding to lambda equal to 0 satisfies W d of course it satisfies dP stationary over dt equal to 0 which must be equal to W times P stationary. So it is the right eigenvector corresponding to the 0 eigenvalue of W. Since each column of W adds up to 0 it is immediately clear that the uniform eigen uniform row vector is an eigenvector, the left eigenvector; 1, 1, 1, 1 etc.

But this is not a symmetric matrix. So the left and right eigenvectors need not be the same in general and the right eigenvector is indeed the equilibrium the stationary distribution okay is that clear. Now how do we find it? Well, you need to put in this condition and discover what it is. So let us put that in and if I write out what W is and let us call this P of P stationary P 1, P 2 up to P N. It is stationary. There is no time dependence. So I would not put in a t index at all.

There is no t dependence. It is the single time probability independent of t right and what is that given by. Well, we have summation l running from 1 to N, l not equal to k, W k l P of l - W l k P of k must be equal to 0 for the stationary distribution. Again it is a couple set of equation, simultaneous equations, linear in all these fellows. Is there a guarantee that this will have a nontrivial solution?

It is a set of homogeneous equations and when does a set of simultaneous homogeneous equations have a nontrivial solution, when the determinant is 0 which indeed it is. We know determinant W is 0 right? So it is got a solution right. It is got a unique solution and you need to find that by solving this set of equations here whatever it be okay. So this problem reduces to an algebraic problem; nothing more than that.

So this thing will lead you to implies you can find this quantity explicitly okay. Now there are some important cases where you can write the solution down explicitly. Remember it is the sum that is 0, summed over l. But you could ask what happens if each of these terms is 0 inside the sum okay. Then you get a very special kind of distribution okay which corresponds to saying that term by term this sum is 0 and that is called detailed balance so since it is so important in physical applications, let me write it down.

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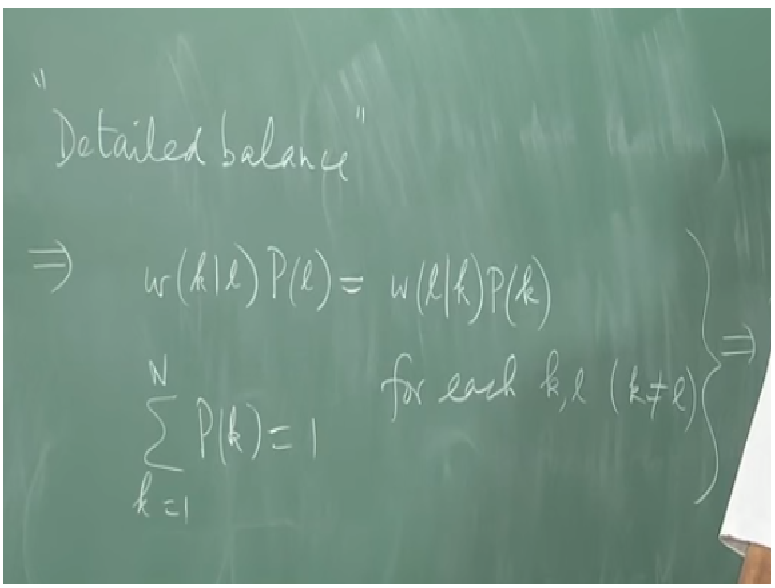

And this is called detailed balance implies that this quantity w k l P of l equal to w l k P of k for each k l k not equal to e. For every pair this is true, pairwise and of course if that condition is true then immediately you know that this that is going to give you the stationary distribution without doing much solution at all. Now what is this actually saying? It says you wait for a long time, get to the stationary distribution, let P(k) be the probability that you are in state k then that multiplied by the transition probability rate that you go from k to l must be same thing in the reverse order. So here is the initial state, here is the final state.

If you interchange the two you get just this guy here. So the flow both ways pairwise is exactly the same, the weighted flow of probability mass. Now this would be a curiosity, just a curiosity except for the fact that systems in thermodynamic equilibrium physical system satisfy detailed balance provided the underlying dynamics has a property called time reversal invariance. I would not get further into the details of this at the moment but just to tell you that it is a very important special case.

There are lots of physical systems under very general conditions which satisfy detailed balance in which case this thing is immediately true and what will that tell us. It gives us the ratios of all these probabilities. So you choose any one of them say P 1 then you can find P 2, P 3 and so on in terms of this P 1 and how would you determine P 1 itself? You normalize the probability. Remember you have to normalize right. So remember that you definitely need this.

Summation k equal to 1 to N P of k must be equal to 1. So we know all the ratios P 2 over P 1, P 3 over P 1 up to P N over P 1 and we know P 1 itself from that set of equations provided you have a normalization condition. There is got to be one inhomogeneous condition which is what this is and with this normalization you can find all of these fellows. I leave it to you as an exercise to show that in the general case for arbitrary N when you have this detailed balance condition valid then this will imply

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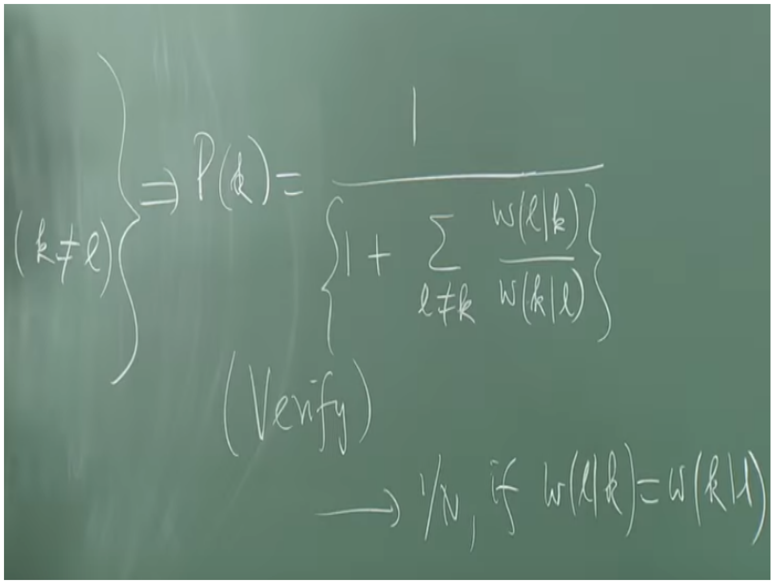

This will imply that the solution P(k) looks like this. It looks like 1 over and this is a well-known form so it is worth memorizing it, 1 plus a summation l not equal to k 1 to N the ratio, I get to verify this okay. So there is a very simple formula essentially an algebraic formula for the stationary probability distribution of this Markov process, stationary Markov process in the case when detailed balance is valid okay and then it is just that simple algebraic formula out there. What happens if the rates are all the same?

The rate from l to k is the same as the rate from k to l. What do you think would happen? Now you are saying not only detailed balance you are saying that look this rate is equal to that rate. That tells us this must be equal to that the probability's distribution must be a uniform distribution. There are n states. You are in the steady state and each of them is equally probable. So what is the probability of any one of them; 1 over n.

And indeed that is true because if these rates are all equal this cancels, gives you 1. There is a summation here except for 1 index k so this is n minus 1, you add it to 1 and you get a 1 over N okay. So this goes over okay but when you do not have that, when you do not have that extra condition that the rate, transition rates are also equal then of course you have a nontrivial solution in terms of all the transition rates okay.

Now let us look at the simplest case. The simplest possible case is when you have just 2 states possible and this is such a famous case and is applied in so many places, it occurs in so many places that **it is** got a special name to itself.

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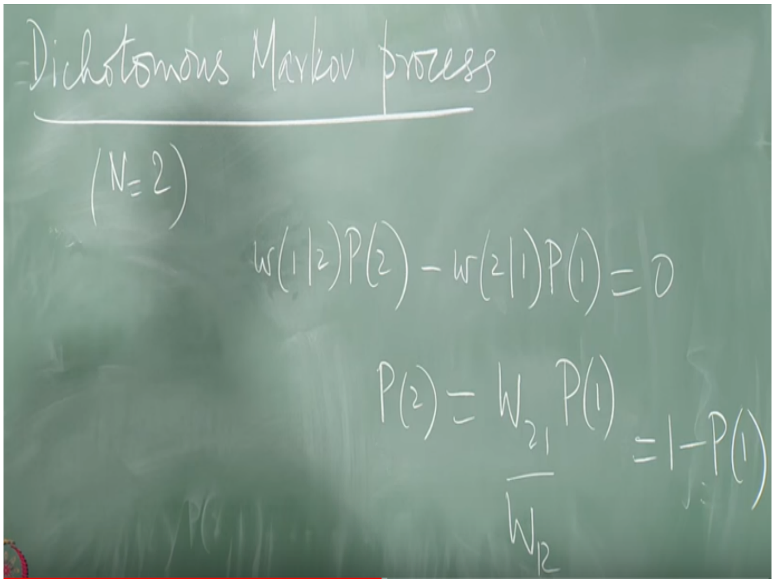

It is called the Dichotomous stationary, Dichotomous Markov process, corresponds to N = 2, just 2 possible states. We can write down what the solution is in this case because the stationary distribution is utterly trivial in this case. You have summation over l and for any given k it runs over only one other value for 1 or 2 right. So l, k etc., run over values 1 to 2 and you have w for the stationary distribution you have w say 1, 2 P(2) - w 2, 1 P(1) should be equal to 0 because there is nothing else to sum over.

If I said k equal to 1, I have this and I sum over l there is only one other value 2. And similarly you have w well you have essentially this equation and nothing more okay. So this is what we call W 12 and let me call this P(2) equal to W 21 P 1 here. So P 2 is this fellow, P 2 is this and P 1 plus P 2 must be equal to 1. So this must be equal to 1 - P of 1 and what does that lead us to. It gives us explicit solution.

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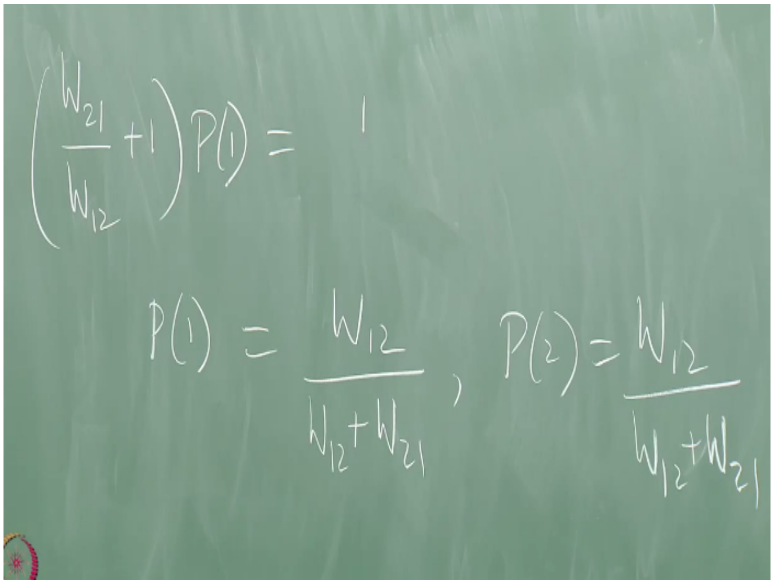

It says W 21 over W 12 plus 1 P(1) = 1 or P(1) = W 12. W 12 is the rate, transition rate to go from 2 to 1 okay and similarly P(2) = W by symmetry, that is it. Those are exact solutions for the stationary distribution okay. Of course we also have to write the time dependent solution for P of t. We have just solved the problem W on P stationary equal to 0 and we have this.

But now think about it a little bit and you see immediately how physical this solution is because what we have is a process and we can now draw this. Let us draw this.

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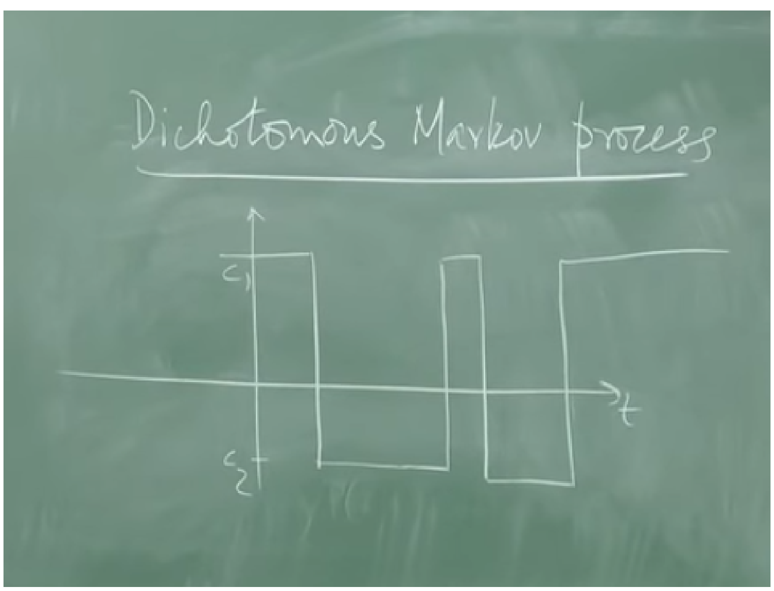

Is a process which has a function of time. It takes on 2 states possible. Now we need a symbol to say what the values of this random variable are which can take 2 possible values. Let us call those values 2 constants c 1 and c 2 for example. So let us say c 1 is sitting somewhere here and c 2 is sitting somewhere here. Then what it does is it starts in say state 1, it goes on for long time. So at any arbitrary instant of time you can put the origin of time and then it abruptly makes a jump to c 2 and then goes on for a while it make c1 and then it does this etc.

So this is c 1 and that is c 2, 2 states. It is a two-state process and in each state the value of the variable is either c 1 or c 2 okay and it keeps switching back and forth randomly between these 2 states. So you can now imagine, you can easily imagine how many possible situations this will model immediately. Anything which has got 2 possible states either an on and an off or a passive state and an active state you name it. There are huge number of applications where this model will work, this simplest possible model. Now what is the rate or average rate at which it switches from one to the other.

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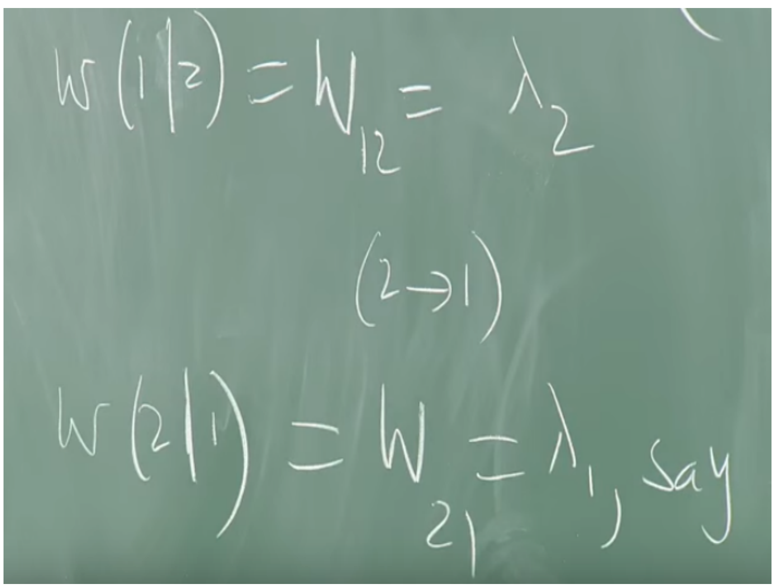

So we have already said that, we've said that W 1 2 is what we call W 12 is the rate at which 2 to 1 switching is happening. So let us call this lambda 2, the rate at which the system switch a state. If it is in state 2, the rate at which it switches to state 1 and similarly 2 1 is the rate at which it switches from 1 to 2 equal to lambda 1 say okay.

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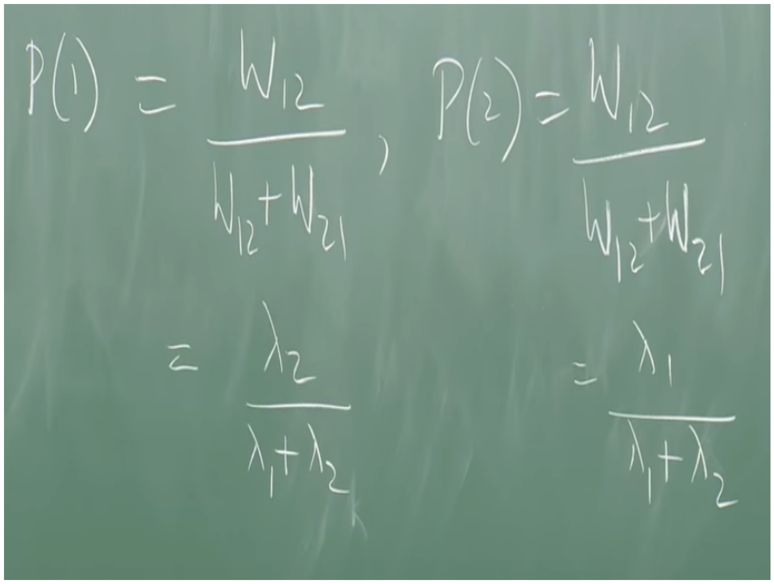

Then according to what we have here, it says P 1 must be equal to W 12 that is lambda 2 over lambda 1 + lambda 2 and P 2 must be equal to lambda 1 over lambda 1 + lambda 2 okay. Now what is the physical meaning of this lambda 1 and lambda 2?

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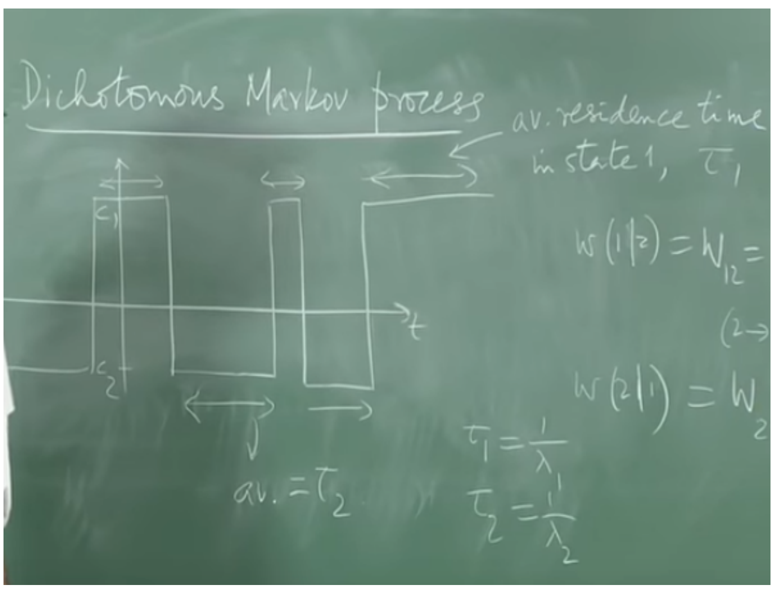

If you look at his picture you see that it stays, suppose it had started like this somewhere, it stays for a time interval in state 1 random time interval, another random time interval, another random time interval etc., and the average over all these things is the mean residence time in state 1. So if you say what is the average in between switches from state 2 to 1 in between what is the average time it spends in each, each time it reaches state 1.

Let us call it some tau 1. So average residence time in state 1, let us call it tau 1 and similarly the average duration of a stay in state 2, let us call this average equal to tau 2 okay. The clearly the switching rate is just the reciprocal of these things right.

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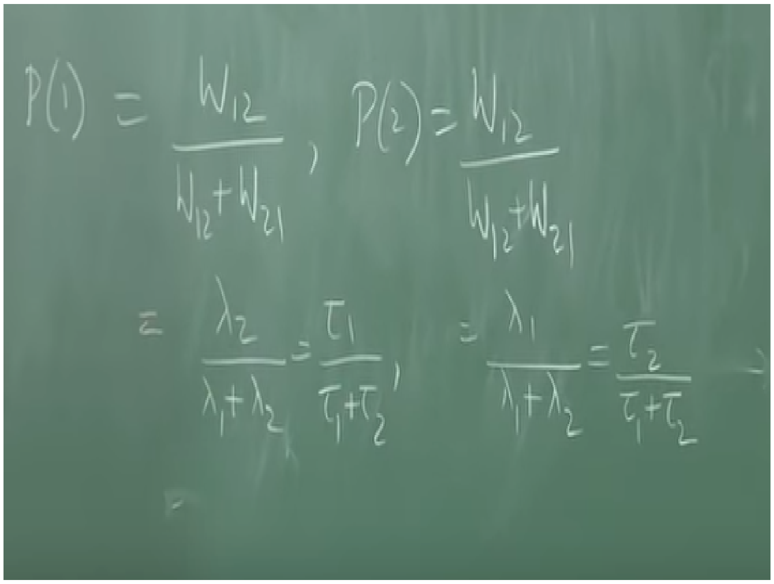

So it is clear that lambda 1 tau 1 equal to 1 over lambda 1 tau 2 equal to 1 over lambda 2. So it says this time is obviously equal this is equal to tau 1 over tau 1 + tau 2 and this is equal to tau 2 over tau 1 + tau 2 okay and that is exactly what you'd expect physically because now you would say ha if I look at it as a function of time and I ask I put my finger on one particular point in time I ask what is the stationary probability that it is in state 1 or state 2?

Well, the probability that it is in state 1 will be the fraction of the time that it spends in state 1 over a long period of time and similarly for state 2 and they are precisely proportional to the mean residence time in state 1 divided by the total residence time. These are the fractions okay. So we have a simple physical interpretation of what is meant by the equilibrium distribution for a Dichotomous Markov process. It is just the ratio of the fraction of the mean residence time in one of the states divided by the sum of the residence times in both the states okay.

So it is physically very clear this is what is happening out here okay. We still have to solve this problem of the time dependent problem. We have still not dealt with that but the stationary distribution is completely clear in this particular case. Now even if you have a three-state process the formulas that you write down in general without detailed balance are not so trivial. They are quite involved. They get more and more algebraically more complicated.

But when you have something like detailed balance then of course it simplifies enormously. But the most popular model is the Dichotomous Markov process in this case. Now what remains is to ask what happens as a function of time; what happens when you put in the time dependences everywhere and this is what happens.

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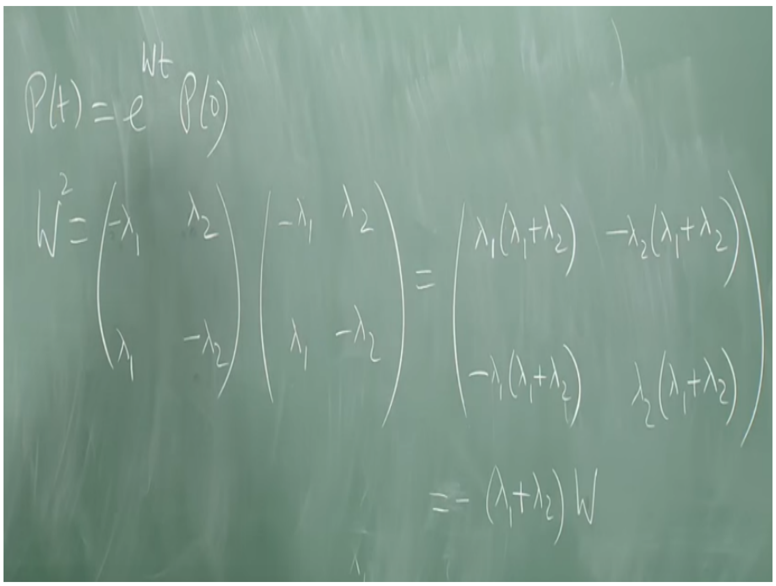

We need to solve this problem by going back and saying d over dt so we need, we will write the solution down P of t is e to the Wt P of 0 and we need to know what is W and W was equal to W 11, W 12 but W 12 was the rate to go from 2 to 1 on this side which was lambda 2 and similarly W 21 was the rate to go from 1 to 2 and the diagonal elements are minus those guys. So that is W the transition matrix and we need to find the exponential of this matrix.

So you square it, you cube it and things like that and find out what is the exponential but matters are made a little simpler by noticing that W square is - lambda 1, lambda 2, lambda 1 - lambda 2. This is W square and what is that equal to? It is lambda 1 square + lambda 1 lambda 2 so take out a lambda 1 and you get lambda 1 into lambda 1 + lambda 2 and then - lambda 1 square - lambda 1 lambda 2.

So again, you take out a lambda 1 and then on the other side you get - lambda 1 lambda 2 - lambda 2 square so it is - lambda 2 lambda 1 + lambda 2 and then finally lambda 1 lambda 2 + lambda 2 square so that is lambda 2 + lambda 2. So we can take out the lambda 1 + lambda 2 and write this as equal to lambda 1 + lambda 2 times lambda 1 - lambda 1 that is just - of this guy out here. So this is turning out to be equal to minus and then a W itself okay.

Now the rate to go from 1 to 2 is lambda 1 and 2 to 1 is lambda 2 and the average rate is half the sum right. So let us define the mean transition rate lambda is lambda 1 + lambda 2 over 2 okay. So it says that W square equal to - 2 lambda W. That of course immediately makes the problem of finding the exponential trivial because it says W cubed is proportional to W once again and so on so forth. So where does it get us?

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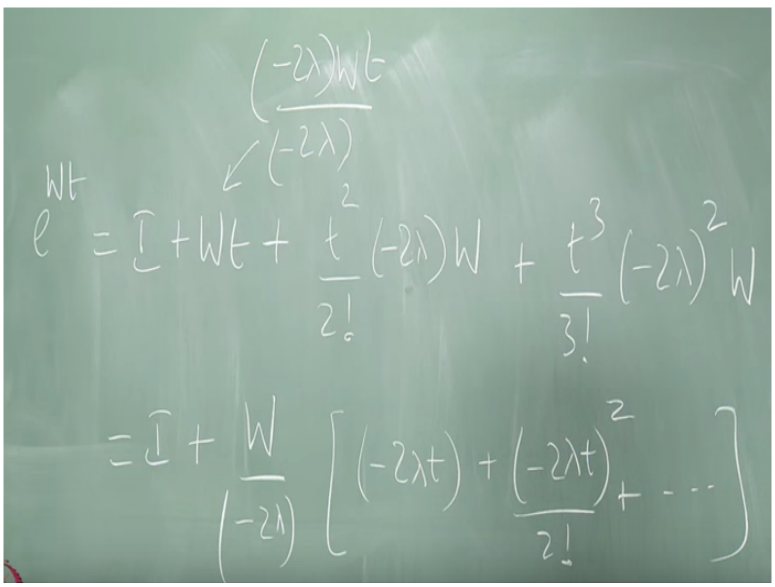

It says e to the Wt is the identity matrix + Wt + t square over 2! W square which is - 2 lambda W and then t cubed over 3! and then W cubed but that is W times W square and so on so it is - 2 lambda whole square W etc., all the way. So let us write this as I plus let us take out a W let us take out the W and then let us write this term as - 2 lambda Wt/2 lambda. Let us write it in that form.

So this becomes over - 2 lambda a - 2 lambda. Take out a - 2 lambda in this fashion and then inside I have - 2 lambda t + minus 2 lambda t whole square over 2! plus dot, dot, dot. I took out a

- 2 lambda so let me put that back in here so it matches this power out here. But what is this fellow? e to the -2 lambda t - 1 right?

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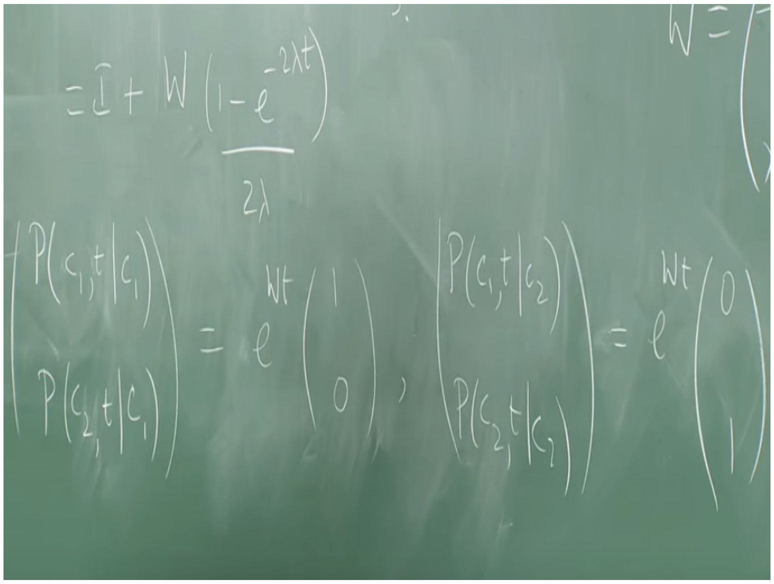

So let us erase this; 1 - e to - 2 lambda t and get rid of the minus sign and it is over right. Now we can write the full probability distribution down completely okay. I will leave that to u as an exercise to write this down so you should be able to write down P of now let us put in the values that we have for this process. So you should be able to find P of c 1, t given that you started in c 1; P of c 2, t given that you started in c 1. How would you find this?

What is this guy equal to? This is e to the Wt on the initial matrix and what is the initial matrix if I start with c 1 what is this equal to? It is equal to 1 and what is this? 0. So all you have to do is to apply that to this fellow. All you need to do is to apply this matrix which is a 2 by 2 matrix because we know both W and i, apply it to this fellow and read off these 2 numbers and similarly P of c 1, t c 2 and P of c 2 t c 2.

This column matrix is equal to e to the Wt, that is this matrix acting on 0 1 and you can write the full matrix down okay.

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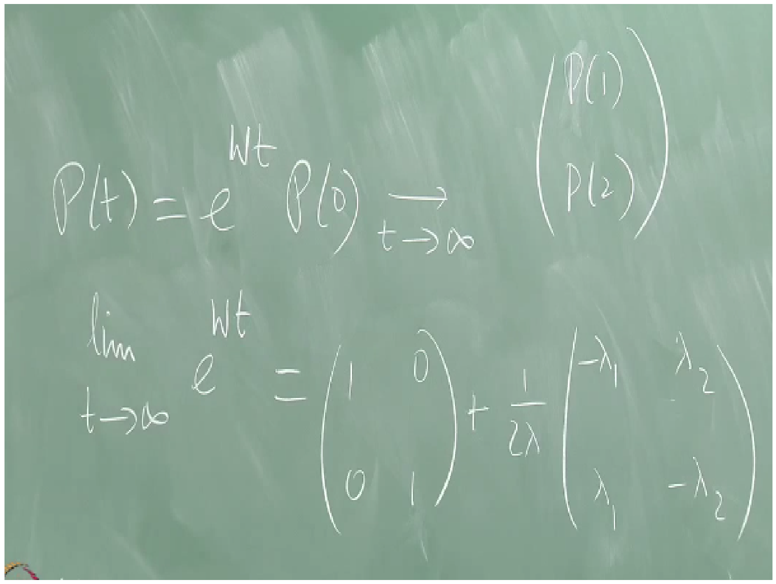

By the way just to check since we have this statement here if you let this, if you let t tends to infinity if you let t tend to infinity we better recover the values that we already had for the equilibrium distribution. What happens to this guy? It becomes a stationary distribution P 1 and P 2 right. So you have P 1 P 2 should be the limit of e to the Wt on the initial state whatever that state is whether it is in 1 or 2 we do not care.

Now what is the limit of e to the Wt? You can read it off from here. Ya, so we know that the limit t tends to infinity e to the Wt turns out to be the identity matrix plus W over 2 lambda because this goes away. So you have 1 0 0 1 + W over 2 lambda and now you have to tell me what is 1 over 2 lambda. What is W by the way it was lambda 1 here lambda 2 here - lambda 1 - lambda 2 in this fashion okay added to this guy out here. And this is lambda 1 + 2 okay. So what does this become equal to lambda 1 + lambda 2.

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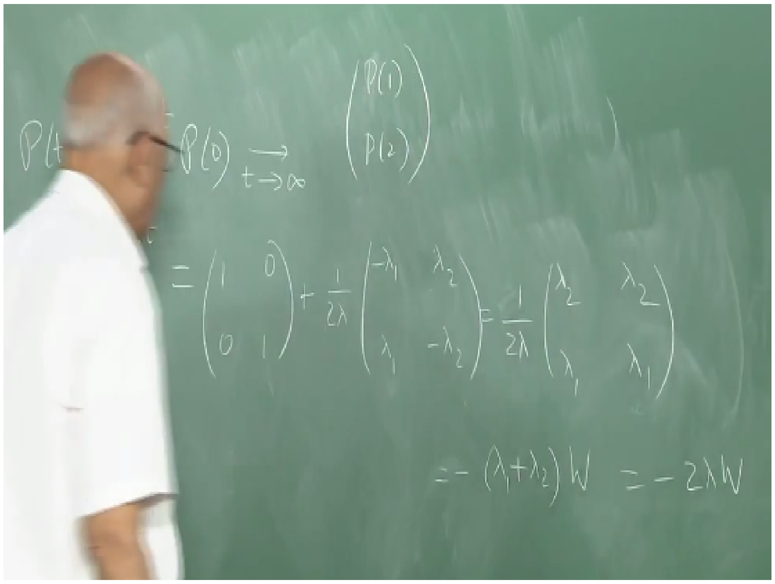

So it is lambda 2 again a lambda 2 and then a lambda 1 and a lambda 1, 1 over 2 lambda times that right on whatever initial state you want to put down. Now one of our articles of faith is that as time increases the initial value should not matter. So whatever be the initial distribution I should still get the equilibrium or stationary distribution to be the same thing. So whether I apply it on 0 1 or 1 0 it should not matter, it should not matter at all. Otherwise I am in trouble right? In fact I could have a of this and 1 - a of the other. So let us do that and see what happens.

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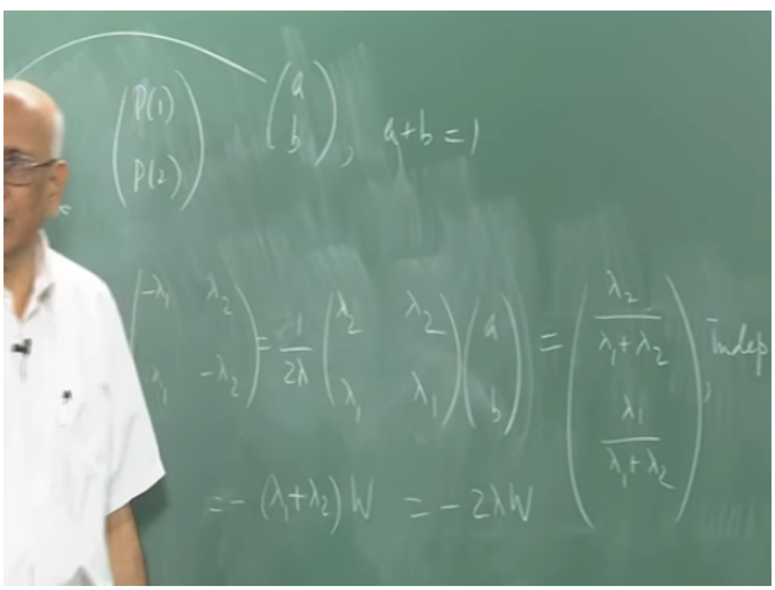

So suppose the initial state this fellow here had been some a b with a + b equal to 1 nonnegative numbers such that a + b is 1. So we apply it to a b and what do you get? You get lambda 2 a + lambda 2 b. So lambda 2 comes out and you have a + b which is 1. It goes away. So this indeed gives you lambda 2 over lambda 1 + lambda 2 and lambda 1 over lambda 1 + lambda 2 independent of a and b, well independent of a, b is 1 - a.

So that corroborates the requirement we had that this stationary distribution should not depend on the initial distribution at all okay so it does it does cancel out this thing does. That is why you had the same element in both places out here and that is exactly the this is exactly the equilibrium distribution we discovered earlier okay but you can write the time dependent one out here and write this out explicitly. I leave you to do this, a little bit of algebra but you get expressions.

What is the so what is the decay to equilibrium; what is it going like? There is a constant part in these probabilities which gives you the stationary part and then there is a part which decays. There is only one more eigenvalue and what is that other eigenvalue, - 2 lambda, - 2 lambda. So the correlation time you expect the correlation time in this process is going to be 2 lambda inverse and again that corroborates we are going to find the correlation time explicitly. I am going to define it in a minute.

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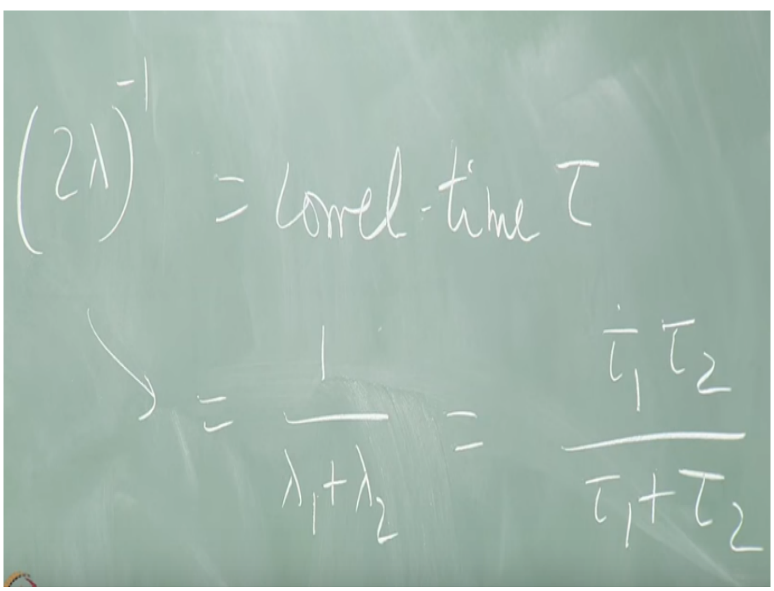

But you see it is this time scale that is the correlation time in a very specific way, we will see that. The autocorrelation will die down with this exponent out here. But what is this guy equal to? This is equal to 1 over lambda 1 + lambda 2 right which in turn remember this is it is equal to tau 1 tau 2 over tau 1 plus tau 2 because this is 1 over tau 1 that is 1 over tau 1 and what is this number? What sort of mean is it? It is the harmonic mean of the individual residence times right.

So we have this very simple relationship which says in a Dichotomous Markov process with mean residence times lambda tau 1 and tau 2 in the two states the correlation time of this process, the time on which it loses memory in some sense is the harmonic mean of these 2 individual times okay. Now let us define correlation. Well, some things can be written down immediately and then we will if time permits do this or I come back to this a little later.

We will write the answer down, at least in equilibrium, we will write the answer down. So I ask now, what is the mean value of this process.

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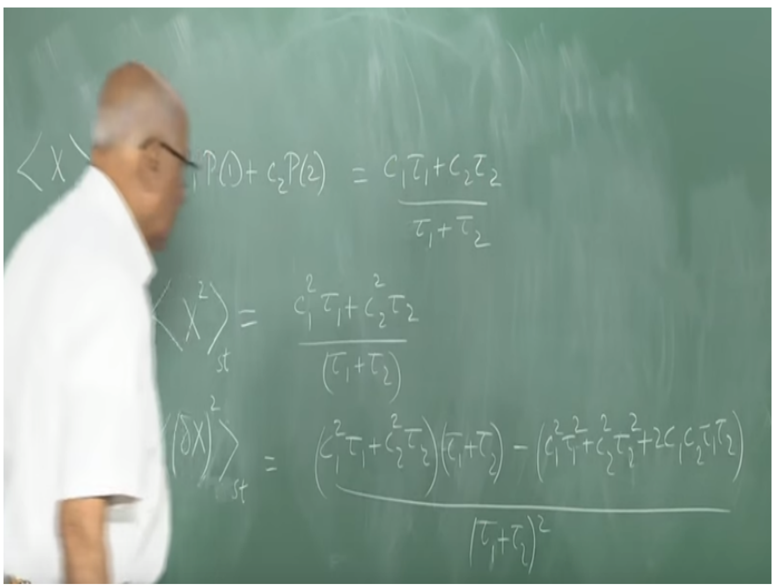

Let us call that process x. So what is the mean value of x? X has values c 1 and c 2. The sample space of x comprises the 2 values c 1 and c 2 and the system is switching randomly back and forth between these 2 values with mean residence time tau 1 and tau 2 respectively in the 2 states. So what do you expect is the average value of x? X stationary, x stationary is with respect to the stationary distribution. By definition this is equal to c 1 P 1 + c 2 P 2, by definition.

The stationary or t tending to infinity average has got to be this because P 1 P 2 are the stationary probabilities and c 1 c 2 are the corresponding values and therefore this is the weighted average and it is a normalized probability and what is this equal to. Not surprisingly this c 1 tau 1 + c 2 tau 2 over tau 1 + tau 2 or in terms of lambda it is c 1 lambda 2 + c 2 lambda 1 over lambda 1 + lambda 2. What is the mean square going to be?

So same guy but with squares right. So it is c 1 square tau 1 + c 2 square tau 2 over tau 1 + tau 2. Again, this is the stationary and what is the variance going to be. So what is delta x that is the difference between this and the mean square, stationary delta x whole square stationary. What is that going to be? That is equal to this minus the square of this fellow. Now the square of this first of all if I take this minus the square of that there is a tau 1 + tau 2 multiplying this fellow here.

So let us quickly do that where c 1 square tau 1 + c 2 square tau 2 tau 1 + tau 2 - c 1 square tau 1 square + c 2 square tau 2 square + 2 c 1 c 2 tau 1 tau 2 the whole thing divided by tau 1 + tau 2 square out here. So some terms definitely cancel out and we can write down a very simple formula.

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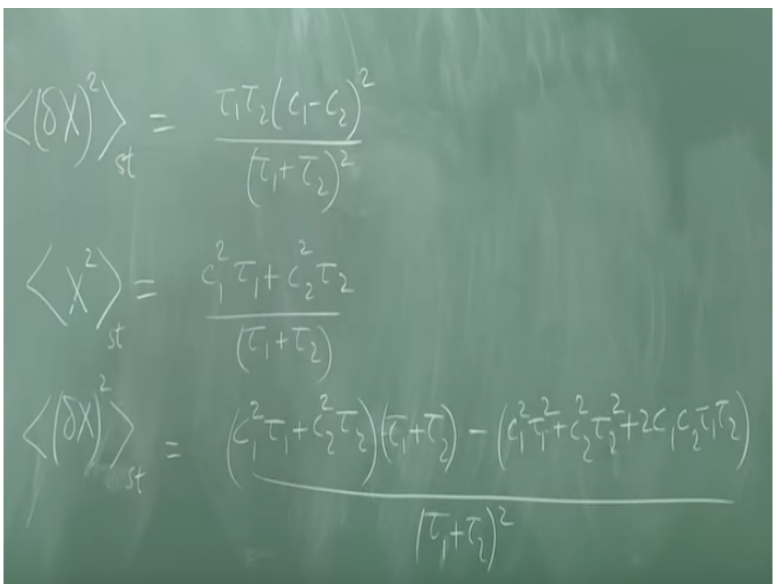

And that is delta x whole square in the stationary state is equal to first of all the tau 1 square cancels with this guy, the tau 2 square cancels with that and then you have a c 1 square tau 1 tau 2, a c 2 square tau 1 tau 2, and then you have - 2 c 1 c2 tau 1 tau 2. So this is equal to tau 1 tau 2 right and that is it okay. Now what is tau 1 tau 2 over tau 1 + tau 2? It is the correlation time right. So there is direct connection between this.

What happens if, what happens if c 2 = - c 1? This becomes a square of whatever value. So sometimes it switches between + 1 and - 1 or something like that then the formula is simplified. Let us call the symmetric dichotomous process. We will talk about that a little while later but what happens if there is time dependence. So the next question to ask is the autocorrelation the generalization of the variance and I will do that tomorrow.

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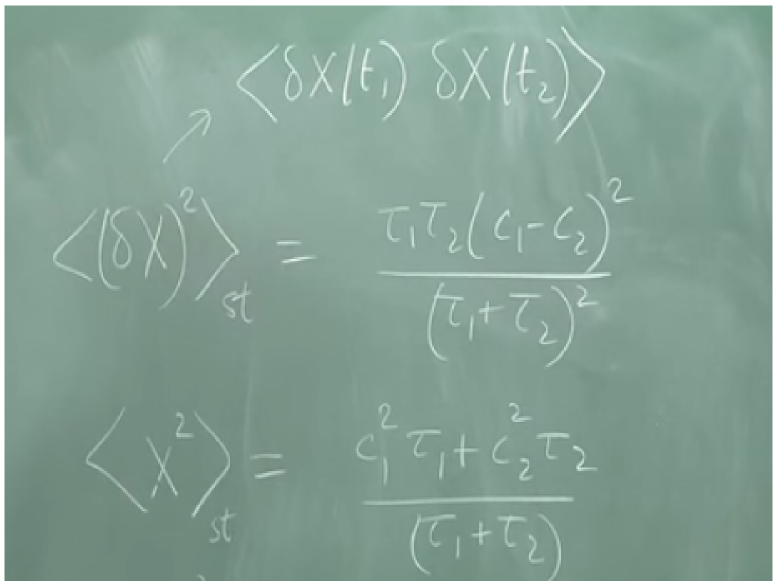

We need to generalize this to ask delta x at any instant t 1 delta x at any instant t 2, what is that equal to. That generalizes the variance but this is a stationary process. So this thing must be a function of t 2 - t 1 or something.

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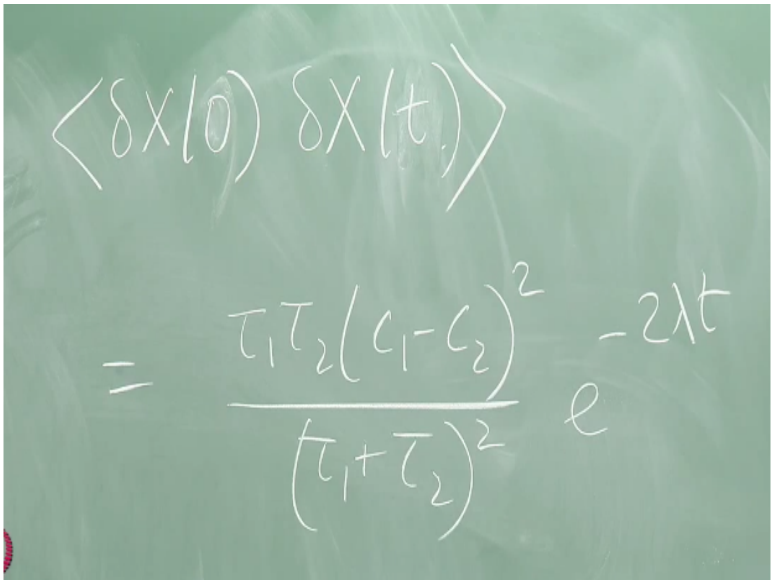

So I might as well look at delta x at any time 0 and then call this just t. What would this be? If t = 0 it should reduce to this right. So it is this times something or the other and you expect the memory is going to drop as a function of time exponentially with the correlation time which is 2 lambda inverse.

So what do you expect this formula to be? What do I expect this to be in general? That is equal to this multiplied by e power - 2 lambda. So the Dichotomous Markov process is exponentially correlated okay and we will prove this. We have to define the correlation time autocorrelation more precisely. We will do that and then we will take it from this point tomorrow.