Physical Applications of Stochastic Processes
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Lecture-29
Statistical aspects of deterministic dynamics (Part 2)

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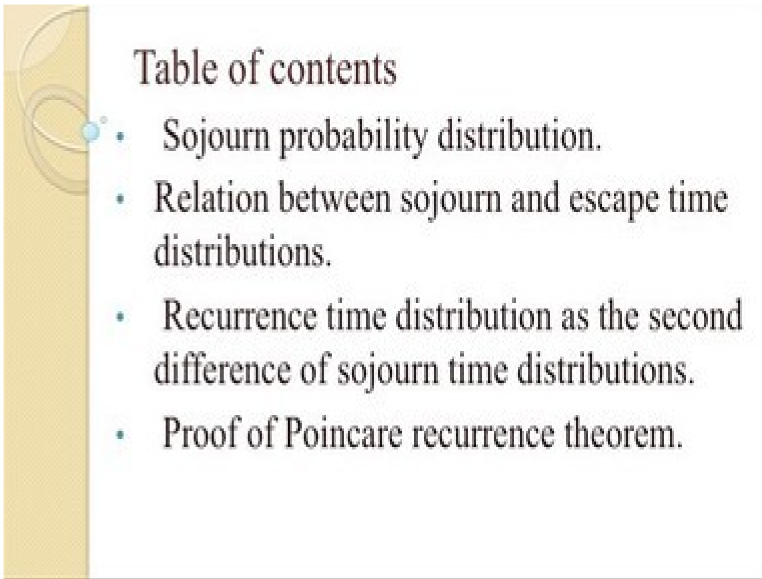

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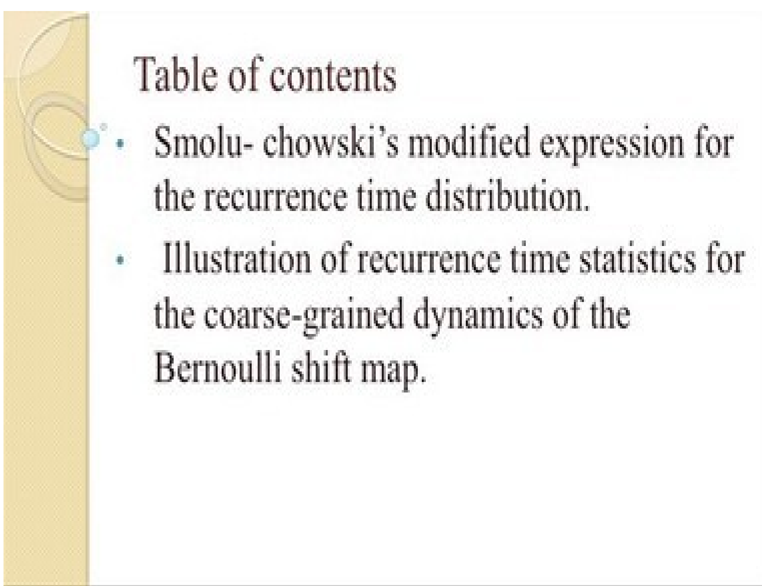

So, let us continue with our discussion of the probabilistic aspects of Coarse Grained dynamics in a dynamical system.

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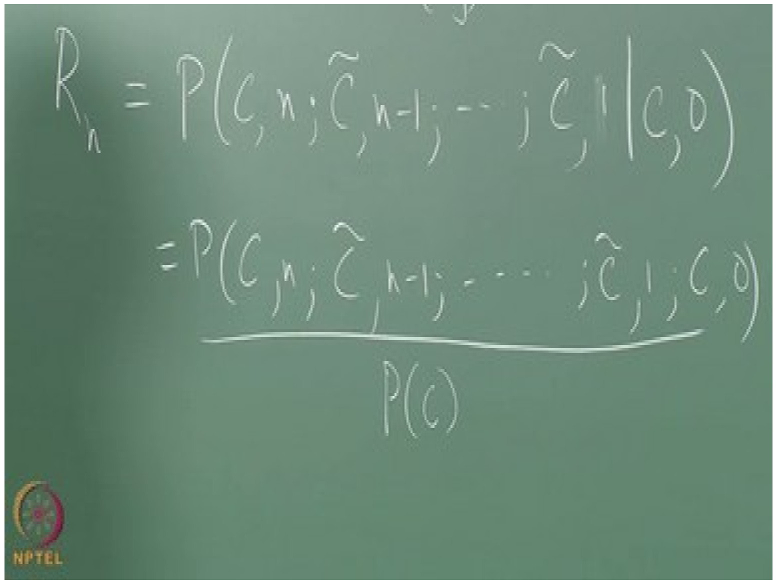

Just to recall to you what we are looking at we have a certain phase space gamma a region of phase space possibly into which the system is stuck and then we partition that into cells of some given resolution. We look at a typical cell C and the complement of it is C tilde the rest of it, so we mark the cell and then we are trying to ask for the statistics of recurrences to this cell. For this purpose we started with the following quantity.

We said there exist an invariant measure for each cell so Mu of Cj for Cj there exists such an invariant measure. The total invariant measure is = 1 this is the are priori probability that if I just put a pencil dot on any given on the phase space I would be in the cell Cj okay. Occasionally I will use the notation P of Cj for the same guy. The measure is a little more general because as I said there could be individual points in this cell which have a finite probability so on.

So, we are allowing for that then the quantity that we want to compute is the probability that you are going to come back to this cell after a certain time in and I call that R sub n for recurrence to this cell and this was nothing but the joint but the conditional probability that you are going to be back at cell C in time n and you are on the compliment in the previous instance C tilde 1 given that you started in this cell at time 0.

So, that is the recurrence time distribution as a function of n in running 1, 2, 3 etcetera. So, this is the quantity we want to compute but because it is a conditional probability we can also write this as P of C n C tilde n - 1 all the way up to C tilde 1 and C 0 and divide this whole thing by P of C because a product of this times this conditional density is = this probability here. So, we could always write it in this form the question is to compute this.

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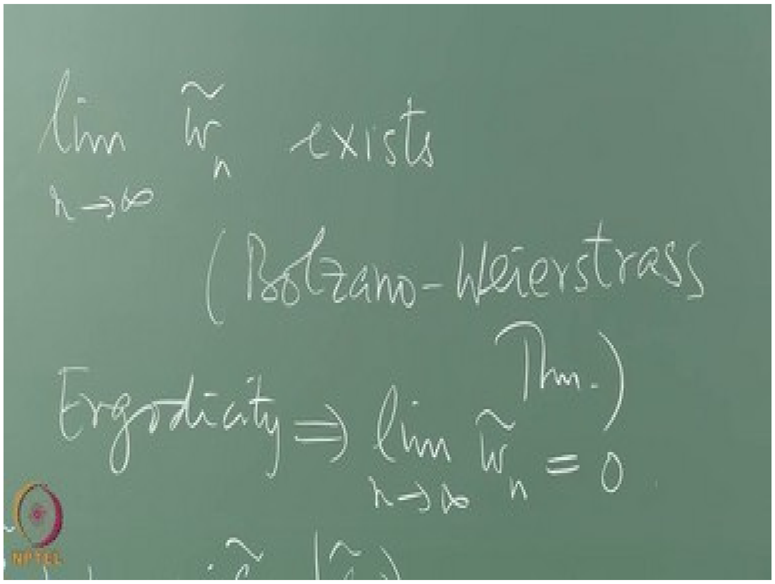

What we did was to say let us first define a probability W n tilde to show that it belongs to the compliment which is the probability that you are in the compliment at time n - 1 all the way up to the compliment at time 0. So, let us just define this joint probability that you started in the complement at 0 and you remain in this complement of C till time n – 1. So, that is the definition of W n and it is immediately clear that W1 tilde = P of 0 itself.

It is the invariant measure of the compliment because the whole thing is stationary so the time index for a single time probability does not exist. We also defined for convenience W naught tilde is 1 we define it in that fashion simply. So, that the formulas look very uniform, our target is this but we could start by asking what is the probability of a sojourn in the compliment for a given amount of time.

So, let us define sojourn probability H sub n tilde to be a sojourn in this in this in this compliment so it is the probability of C tilde at time n C tilde n -1 dot C tilde at time 1 given that you were in the compliment at time 0. So, all the normalizable probabilities will be those which are conditioned on something. So, I say I started in the compliment at t = 0 and what is the probability that I stay here till time in that is the sojourn probability it sub n okay.

H because halting I halted in this compliment here okay. But this is = the problem the joint probability with the C tilde 0 without the bar but a semicolon divided by this invariant measure we have C tilde but this is obvious that from here then this is = W n + 1 tilde over the probability of C tilde which is W1 and I emphasize that there is no question of normalizing this H sub n they are all probabilities.

There is no question of adding different ends because they are not mutually exclusive events what we do know is that this set of numbers is actually a decreasing sequence as n increases the probability that you are going to stay out of C is going to decrease because it is Ergodic sooner or later the point will move in to C. So, as n tends to infinity I expect W n tilde to actually go to 0 but we can show this quite rigorously without anything else because I know that Wn tilde is a non increasing is a decreasing sequence bounded from below by 0 because it is a probability.

So, it cannot be negative and there is a theorem and analysis called the Bolzano Weierstrass theorem. Which says that if you have a limit if you have a sequence a decreasing sequence which is bounded from below then the limit exists okay, I mean there is no reason why there should be a limit at all for Wn it could oscillate or something like that but the limit exists. So, we guaranteed limit n tends to infinity W n tilde exists and this is the so called Balzano. So, decreasing sequence but the limit need not exist at all.

But now we are guaranteed by this theorem that if it is a decreasing sequence bounded from below then the limit exists and it is bounded from below by 0 because it is a probability. Ergodicity then implies and this is a crucial point is actually 0 this probability has to go to 0 so that the probability of hitting C becomes non 0 going into C becomes non 0. So, this is all the input we need that this is a decreasing sequence and its limit point is 0 on this side.

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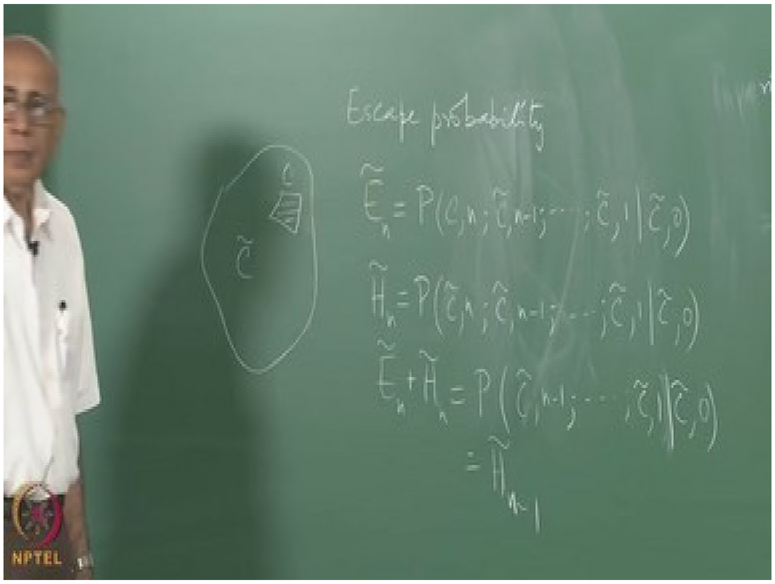

So, this gives us this sojourn probability and the next step is to ask what is the escape probability? So, the probability escape probability out of C tilde so let us put again En tilde this is the probability that you are out of C tilde at some time. So, Cn given that you are inside at all earlier times including C tilde 1 given that you were in C tilde at time 0. So, you start adjustable for reference let us keep this year so this is C and the rest of it is seated.

So, we start by saying I start out here but a time n I am out in I am inside here I have escaped from this compliment having stayed there till time n - 1 at time when I get in. So, the question is what is this probability = well again it is a joint probability divided by the invariant measure of this guy so we can certainly write it in that form. But we can do it in yet another way it is related and that is to consider H n tilde what is this = that is written down here.

So, just for reference let me write it down just below that P of C tilde n and the rest of the arguments are all exactly the same. Now what happens if you sum these 2 probabilities these 2 conditional probabilities well you started given that you are in C tilde at 0 and then you are asking for the probability that you are in C tilde to the n - 1 and what can happen in time in either it can go out and be in C or it must can stay in C tilde.

So, the sum of these 2 must necessarily be the probability that you have had a Sojourn until time n – 1, so it is immediately obvious but En tilde + Hn tilde must be the probability P of C tilde n -

1 C tilde 1 given that you are seated because it is a sum over possible final states next thing and that is the probability of whatever has happened till then. At that point but this quantity here is = H n – 1.

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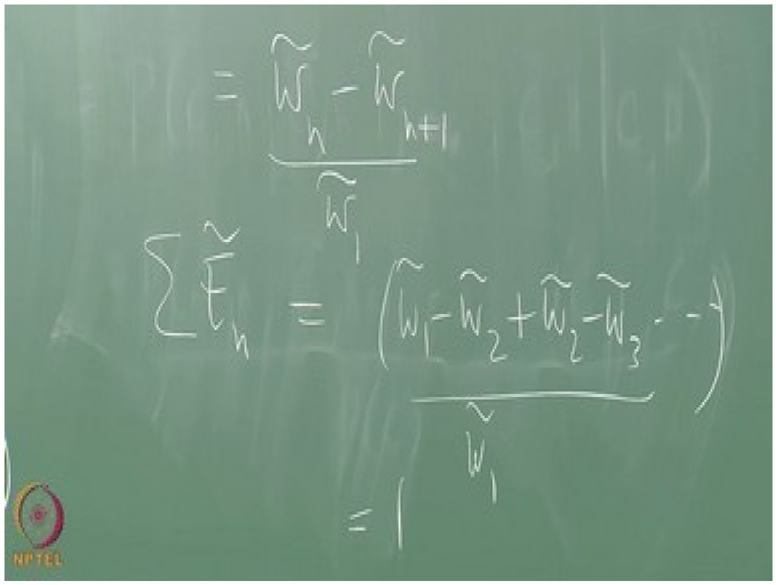

So, that gives us our first result which says that this escape probability En tilde is = Hn - 1 tilde - Hn tilde, so it is the first difference of the state probability this is like - d over dt is real in the continuous case and it is physically very clear but all you have done is to conserve probability. But we have got a formula for this and this therefore is = W H Hn tilde W n + 1 so Hn - 1 tilde Wn tilde - w n - n + 1 tilde divided by W1 tilde okay.

So, that gives us our first result it says if you compute these numbers Wn and we define those Wn's here, if you compute this number then you actually computed the escape probability in this case and now you can check that this fellow is normalized because it can either escape at time 1 or at time 2 or at time 3 whatever time. So, let us see what happens if you sum this En = well this is W1 - W2 + W2 - W 3 all tilda's etcetera divided by W1 tilde is = 1.

Because all the terms cancel out except the first so this is indeed a normalized probability. So, because that is the only thing that can happen we have assumed Ergodicity, so this immediately says that sooner or later things will escape but you allow for escape at any instant of time 1 onwards and you have to get 1 as the answer so this is normalized as it stands. You could ask for the mean escape time that is n times E sub n.

And you have to sum over and normalize it so you can that is a trivial thing to find out here what this n times is going to be okay. It may or may not be finite. So, let us find out what is the mean escape time meantime it is like asking what is the mean first passage time we already had experience with this we found that you were on a straight line with diffusion mean first passage time was infinite. Even though the probability of hitting that point in this case exiting see tilde is 1 you may still have an infinite mean. Well, let us see what that implies.

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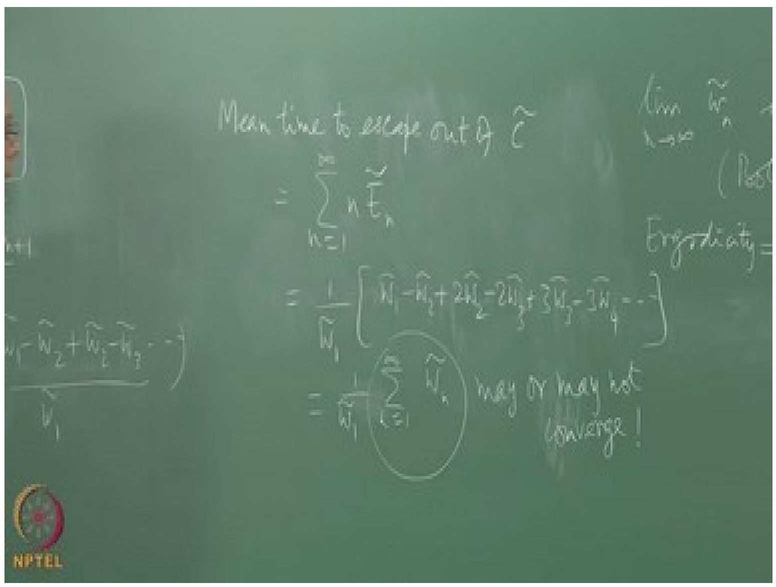

Mean escape time which is = we have a formula here, so it is = 1 over W 1 tilde times 1 times for this properly E1 is W1 - W2. So, it is + twice W2 - twice W3 + thrice W3 - Phi W for right which is = W1 tilde us everywhere W which is = summation n = 1 to infinity because you can see that each time it is going to be 1 shot of complete cancellation. So, it is W n tilde 1 over W is = this guy what can we say about this we know the Wn's all decrease it is a decreasing sequence and we know the limit n tends to infinity Wn is 0 that is a necessary condition for the series to converge then Wn must go to 0.

Otherwise the series will never converging but it has to converge, go fast enough for it to converge it has to go faster than 1 over n to converge but there is no such guarantee no such guarantee that this series converges. All we know is that Wn tilde goes to 0 as n tends to infinity so the sequence decreases but if it goes like for instance 1 over n then this is going to develop diverge logarithmically as you know the harmonic series is infinite.

So we have no guarantee at all and therefore the mean escape time might still be infinite. So, I want you to carefully note this because it is going to lead to a very interesting result if you start in this place by the way the C tilde could have been anyone cell you get similar results always. If you start here you are guaranteed to escape into this at some time but the mean time to do so may be infinite may be infinite.

But it may be finite we do not care but that does not tell us anything more unless you actually compute what Wn is and tell me what large and behavior is I cannot say right hand right away whether it converges or not may or may not be infinite. Now let us look at recurrence and see what happens. So, the lesson here is that the escape out of C tilde is the first difference of the sojourn in Ct by this simple formula.

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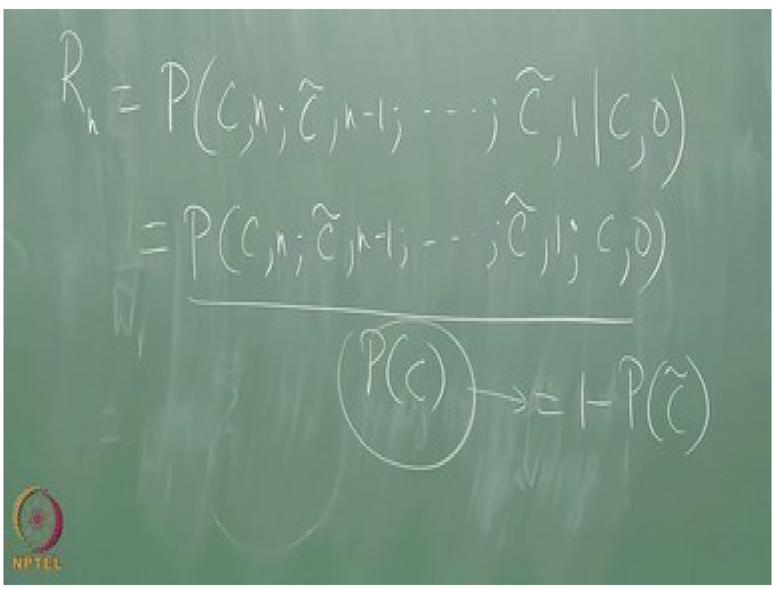

Now let us ask what about the recurrence itself and there I define what the recurrence was you defined it by saying the recurrence probability for to cell C our sub n is P of C and given C tilde n - 1 dot 1 C tilde 1C 0 notice now it is C, so the statement is I started I start inside here at t = 0 then I jump out and come back and a question is what is the probability of this happening in n time steps okay. So, I start with that probability here conditioned on this.

But I can also write this as P of C n 3 tilde n - 1 dot dot C tilde 1; C 0 divided by P of C the invariant measure of the cell itself. As always I write this conditional probability as a joint probability divided by whatever is the probability of the conditioning. But this thing here this here is = 1 - P of C tilde because the total probability measure is 1. For the full phase space right. But that is = W naught tilde - W1 because W naught tilde was defined to be 1 and W 1 tilde was P of C right.

So, in this case the denominator is not W1 tilde but it is this and this is a decreasing sequence so this number is smaller than that, it is positive that is guaranteed. Now what can we say about this P on top, we need to write it out in the idea is to write it out in terms of these W's but W is refer to C tilde so we have to be a little careful and let us see how we can write this out explicitly.

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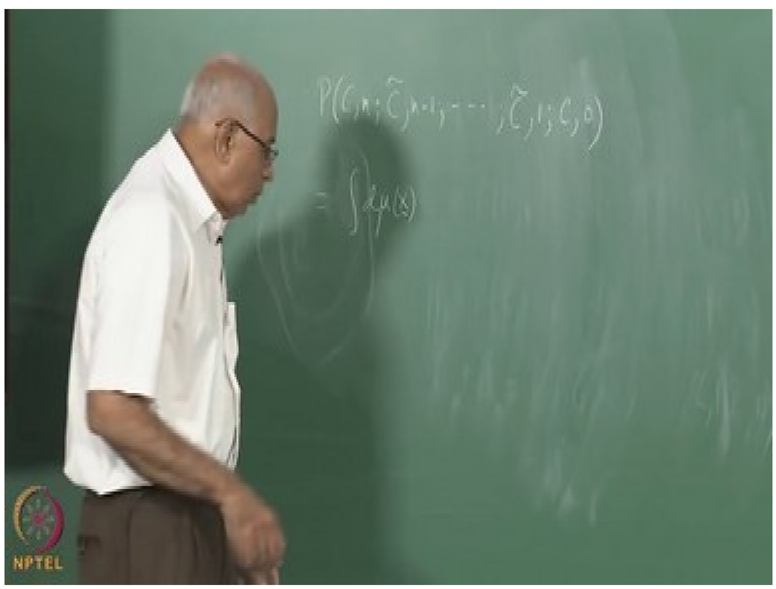

So, we want this joint probability C and C tilde n - 1 dot C tilde 1 and C 0 this is = an integral over the phase space over the invariant measure d Mu x you like this as Rho of x dx and x is the full over all the variables in the phase space multi dimensional integral formally times that set of points were at time 0 you are inside the cell and at time 1, 2, 3 up to n - 1 you are outside and a time when you are back in there.

So, the way to do this is to introduce theta functions which when the argument is positive is 1 and when the argument is negative is 0 and in general when you have a multi dimensional object like that it is called a characteristic function very bad name but that is the way it is. And it goes as follows.

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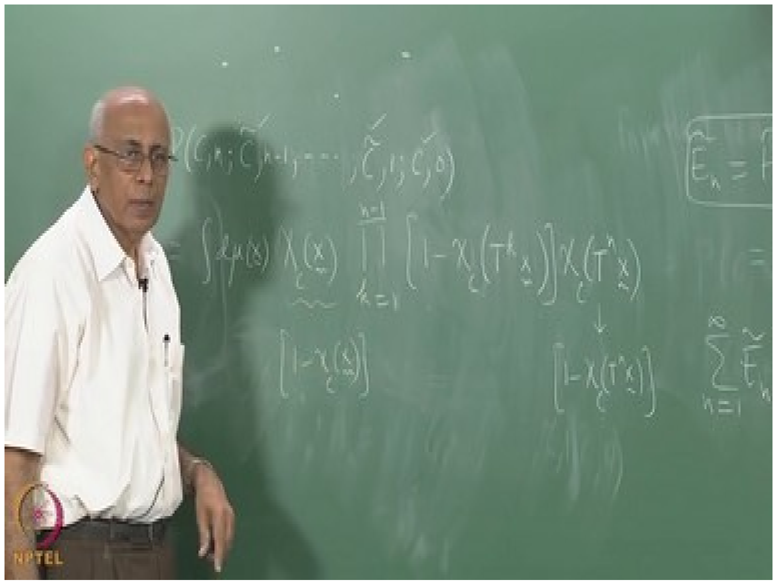

So, we define the so called indicator function or characteristic function but indicator function is better. It is just a big name called Chi subscript C of x and it is very simple it is = 1 if x is an element of C 0 if x is an element of the compliment just a theta function. But is a very useful device because it helps us to do these integrals with constraints of this kind then what is this fellow = this here you must integrate over all those points with this invariant measure which satisfy that so you have a Chi C of x.

That takes care of this portion because this integral will automatically be 0 of x is outside so the integral is now running over the full phase space but this function sees to it that x must lie inside here and then the remaining points. But remember that x evolves so what is happening is that x k is given by a time k is given by some operator t sometime development operator T acting on xk - 1 this is a time evolution operator.

We do not care what it is in a map it could be some very complicated nonlinear function this is what I call f of x last time. But it is formally some time operator and is completely general this whole thing is completely general. So, even in a continuous time dynamical system you can take time slices and what happens so the any phase space point at time k depends on where it was at time k - 1 and there is a map that carries you from k - 1 to k.

Let us call that map T and this is iterated each time the rule is not changing it is an autonomous system so this is = T squared x k - 2 = etc T to the n x0 the initial starting point. So, it is actually tracing the orbit if you have a point here at some time in the next instant it is their next instant rates here, next instant is here and that is given by this map here. So, this time evolution maps the phase space onto itself.

But it takes different points and maps them in different places and it is iterates give you how the time evolution occurs. So, we have ensured by this that the starting point is inside C but the first step it should be outside C therefore the next factor is 1 - Chi C of T on x this is the indicator function of the complement of C. So, if x is inside, if Tx is inside C is going to give you 0 because it is 1 - 1 gives you 0 if Tx is outside C and C tilde this will be 0 and you have got a 1.

So, you got the full measure and that should be true all the way up to n – 1, so this whole thing becomes a product k = 1 to n - 1 of 1 - Chi C of Tk, so just a compact way of writing something very complicated if you started writing it out in any arbitrary phase space. But this takes care of the fact that so this is taking care of this is taken care of right up to here is taken care. And then you got to be back inside here so the next 1 has to be Chi C of Tn and that is the formal expression for this joint probability okay.

There is no conditioning there is no conditioning on the initial state initial point except to say that it is we want the probability that it is in 0 it is in C not that given that it is in C. You want the probability so we are integrating over it. But notice that by this device of putting in this time operator here I have got rid of the need to introduce x1, x2, x3 these things as variables of integration and putting in delta functions all the time.

I buried all that by saying this map is iterated several times to give you this. So, if the first map x1 is some nonlinear function of x naught, x2 is the non-linear function of the nonlinear function of x naught, so it can be extremely complicated but we do not care notation wise this takes care of it completely. The next step is to say all right I want to relate this to the Wn's, so what we have to do is to replace this by 1 - Chi C of x and replace this by 1 – Chi C of T and x if you did that then you have added some terms and you have to subtract them out.

1 term you have already got but you have to subtract out the remaining terms that you put in I leave the details to you a couple of lines of algebra. What happens is simple if you took this here and this and this, this product then it says time 0 you are in the compliment time 1 you are in the compliment time n - 1 you are on the compliment time n also you are in the compliment.

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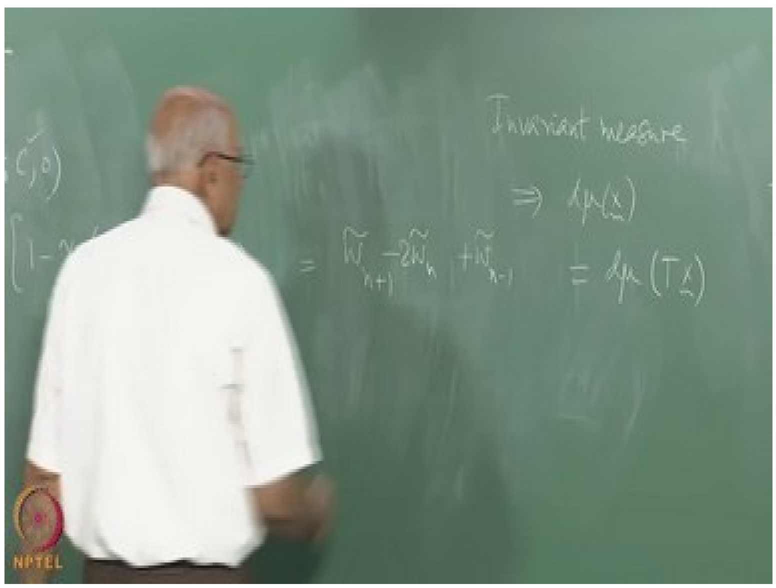

So, it is not hard to see that this guy here becomes = Wn + 1 tilde because from 0 to n you are in the compliment and that probability joint probability we call Wn tilde and then the term which involves just the Chi and Chi here on both these sides that is also sitting there and what would that mean this when you can you multiply the 1 with the 1, 2 of these guys are gone and you are left with 1 to n – 1.

But the integration variable is x on the other hand the argument is T times x, so you are a little bit of trouble till you realize that this is the invariant measure. So, it immediately says in this is very,

very crucial invariant measure implies remember what we said about the Frobenius Perron equation we said the density Rho of x the invariant density does not change under 1 iteration at all that is the whole reason why we called it the invariant density.

So, the formal terms this implies that d Mu x = T Mu of x so I change variables from x to y where y is this nonlinear function of x and the measure does not change the probability measure does not change at all. So, in the term which is 1 we change variables to Tx and then it will look exactly like another W and the W to look like is 2 less than this 0 to n which had n + 1 of these ones but now you got n - 1 of these points so there's going to be a + Wn - 1 tilde.

And then in between you have 2 terms where you have this 1 times this with a - sign and - this times 1 again with a - sign and I use the same trick of changing variables to Tx in each case and they will have the original thing had with the ones you had n - 1 time arguments with the full brackets who had n + 1 and with 1 of them firing and the other not filing you have n of them and the 2 are = each other by this invariance.

You have to check this out but I am just motivating it is very obvious here from this form. So, you have a - twice w until but that is the second difference of the W okay.

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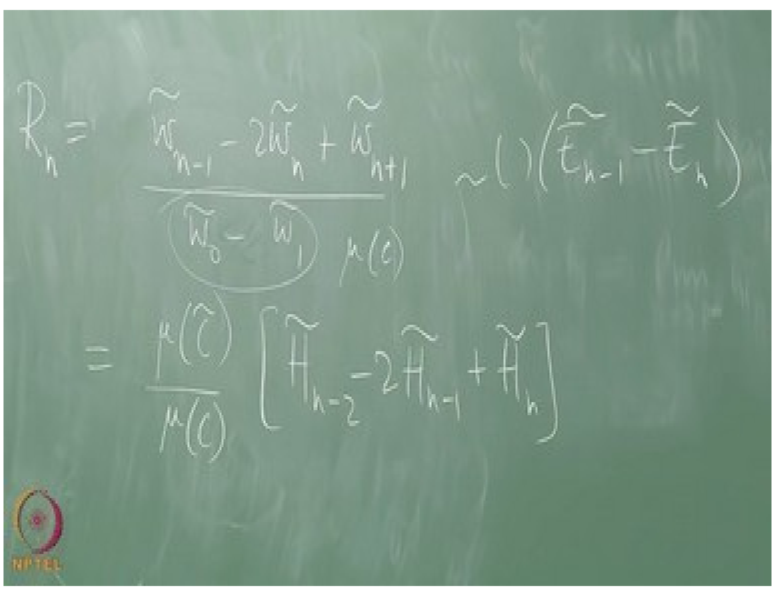

So, we are back here we now can write down what this famous recurrence probability is Rn therefore is Wn + 1 tilde - twice let us write it the other way because this is a decreasing sequence W n - 1 tilde - twice W n tilde + n + 1 divided by now W naught tilde – W1 you wanted that was Mu of C the invariant measure of the cell. So, we got a really interesting formula which says that the recurrence time distribution to the cell C is the second difference of sojourn probabilities in the complement of C.

The first difference give you the escape the second difference give you the return but to C okay. We can write this back in terms of I divide and multiply by Mu or C, so this is = Mu of C tilde divided by Mu of C this guy here is multiplied by Wn - 1 and then a Wn + 1 what was this fellow here in terms of sojourn probabilities W n + 1 was Hn, so this fellow is Hn - 2 - twice Hn - 1 n so it is the second difference of the sojourn probabilities.

And that is normalize we have to check normalization we still have to do this. So, that is a closed form expression this guy here and the only assumption we made is Ergodicity. We just said this sequence Wn is a decreasing sequence that is it with a limit point 0 and that immediately gives us this result for the actual distribution itself. You can it is obviously the first difference of escape that is also clear because you can also show that this is like apart from something else.

It is En - 1 - En times some constant because we already showed that En is Hn - 1 - Hn that is it so it is just En - 1 - En okay. So, that is an interesting physical way of looking at it they escaped but out of the compliment into the cell C occurs at time n with probability En tilde and En - 1 - En gives you the recurrence probability the second there is a non-trivial result it is not at all obvious to start with but all we did was to use the invariance of the measure.

The dimensionality did not matter the kind of dynamics did not matter how complicated it is did not matter we did not assume it was chaotic we will assume anything we just said it is a chaotic in some region of a space. So, even if you have a dissipative system in which the system falls into what is called a strange attractor this will apply on the strange attractor on that level. So, all that you need is that the system is in some region of phase space and it stays there forever.

Listen we do not care what kind of motion it performs inside there but it should be a Ergodic given enough time should visit the neighborhood of all points in this part of phase space then this follows at once. We need still to check normalization.

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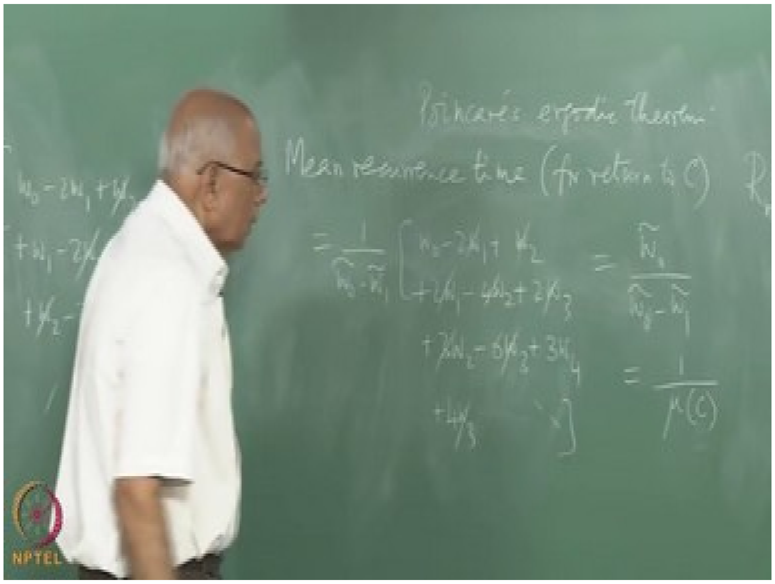

So, let us do that then we need to find the meantime so the normalization would say that Rn tilde summation n = 1 to infinity this is = 1 over the measure of C that is W naught - W 1 tilde times R1 is W naught let me just leave out the tilde - 2 W1 + W2 + W1- 2 W2 + W3 + W2 – 2Wt + W4 dot, dot, so it is clear that the W2's cancel the W3 is canceled and so on and you are left with W naught - W1 which is in the denominator also there is = 1. So, this series telescopes everything cancels out to except just the numerator and denominator and it is normalized to 1.

So, this shows that recurrence is a certain event sure event again at Ergodicity this if all parts of a space are guaranteed to be visited given enough time by a representative point then recurrence to any cell is a guaranteed event probability is 1. So, now we can ask what is the mean time to come back in the mean time mean recurrence time for recurrence to C for return to any C this is = 1 over W naught - W1 times I times this guy.

So, it is W naught - 2 W1 + W2 + 2 times the sky so it is 2 W1 - 4 W2 + 2 W3 + 3 W2 - 6 W3 + 3 W4 and so on so this cancels out 3 + 1 is 4, 2 + 4 is 6 because it is an external 4 W4 and so on W3 and so on so everything cancels out except W naught we talk so this is = W naught tilde them everyone W naught tilde - W1 but W naught tilde is 1 and this fellow is the invariant measure of the cell C.

So, we have a very elegant result this is Poincare Ergodic theorem which says that the mean recurrence time to a cell is the reciprocal of the invariant measure of that cell that is it. That is exactly what we have got here and that is the recurrence the Poincare theorem their body correctly there is a corresponding theorem for Markov chains and so on. Well we have already come across a trivial instance of it you know when you take a coin toss that is not a dynamical system because all the tosses are independent of each other.

But it is a Markov process because it is a Bernoulli process and then if you have n coin tosses this fellow at a geometric distribution for the first time you got a head or a tail or something like that. So, if you say the tail and the head are 2 cells in phase space and I ask what is the mean time it takes to get the head to get into the head the first time it goes like 1 over P the probability of a head okay, that is, but the measure of this that the probability measure of the success is P and 1 over P was the meantime.

So, that is actually an illustration of this theorem which this theorem at work in the case of a Markov process very, very simple Markov process. But here even if it is completely correlated we do not care what kind of dynamics it is you are still guaranteed to get this result. So, it is a remarkable result Poincare Ergodic theorem now it is got consequences it is got consequences for this problem with a problem of irreversibility in microscopic physics.

You may say well suppose you take an ideal case where you have classical dynamics operating and you have a statistical mechanical system say the gas in this room it is got a huge phase space right of the order of number in just the degrees of freedom and now I ask will it ever it is a chaotic system will it ever come back to any given cell in the initiative if you start with some initial conditions will it come back to it or not.

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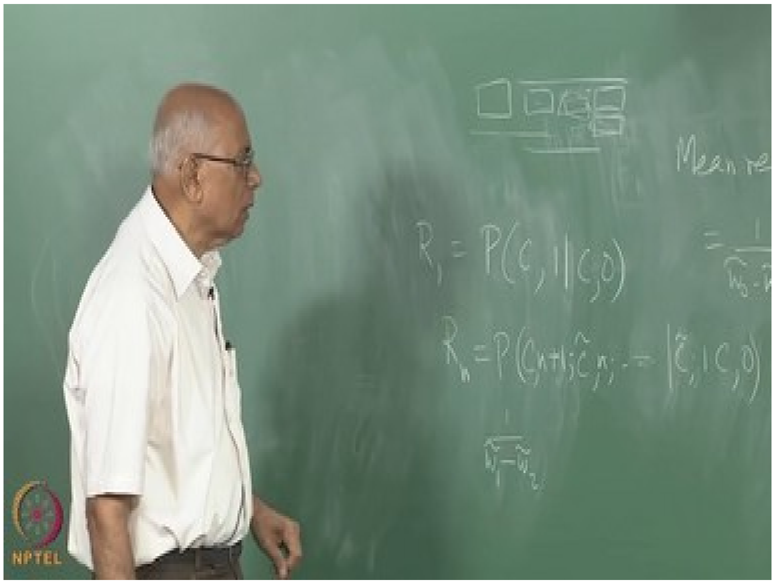

What is the recurrence time going to be like what do you think will happen this is the famous there is many, many models of this and the idea is that the famous model is the checkerboard model you have a checkerboard on which you start with some fleas and the fleas are jumping from 1 square to another and random. So, there is a discrete time say. Every time step each play jumps around somewhere yeah.

So, the question is when will the system return to its initial configuration it is an Ergodic system so it certainly will return but the question is what is the mean time it takes to do. So, it is a finite system completing your phase space is finite and so on. So, the question is now when is it going to happen will this happen at all as you can see if it is 1 flea and just 1 flea then all you are asking is how long does it take for it to come back to its initial point on they have it is 1 representative point.

But now you are asking where is the system going to come back to its initial state that means every flea must come back to the initial point that is started from here it is getting less and less probable as you can see. It is not hard to see in this case then what will happen now is that the meantime we will go like the exponential of the number of degrees of freedom okay. Be enormous because any given initial state has a very small measure compared to the rest of the states because a huge number of states possible.

And therefore this fellow will typically be of the order of e to the -n okay if you have 2 particles and each of them can be in 2 states then the total number of accessible states is 4, if you have n particle it is 2 to the end so it is exponential in this and the measure of any 1 of those states is 1 over 2 to the n, so if this is 1 over 2 to the n there is 2 to the n goes on top and the meantime becomes exponentially large in the number of degrees of freedom okay.

So, even in an ideal world with no other things happening the meantime of recurrence will go like the exponential of the number of degrees of freedom. Now even for the gas in this room that will be of the order of e to the power n Avogadro that is e to the 10 to the 23, it is so large a number that I do not bother to measure I do not bother to specify the time unit's right. Whether I call it picoseconds, femtoseconds or ages of the universe it still does not matter the age of the universe is only 10 to the 17 seconds.

And a picosecond is 10 to the femtoseconds is 10 to the – 15, so this is a factor of 32 orders of magnitude but we are here speaking of 10 to the 10 to the 23 okay. So, it is such a number large number that the system appears a reversible of course many other things happen in between. So, this explains why in large systems you do not in daily life you do not see reversibility it is highly improbable and even, even if it is a sure event.

It would not come exactly back to the same point come arbitrarily close to the initial state but it will take ages of the universe or even far more than the ages of the universe to come back. So, this kind of argument actually explains the reason for apparent irreversibility in observations it is simply that when you have so many possibilities a very specific thing that you want is not going to happen and irreversibility happens.

It can be said in many, many ways for instance when you park a car it is much easier to get out then to get in as you know and we have all seen this so assuming that people part normally in any normal country with human beings there are exceptions you have a spot like this you have cars everywhere you come along and then what you do is to go forward and reverse into this. In this fashion you pull out in the same way but it is much easier to pull out then to pull in much easier okay.

The reason is you may say why cannot, I just reverse my motion whatever I did to get out I just reverse it in exactly the same way. The fact is that when you are getting out when you get out and you are either like this or you are like this etcetera it does not matter there is enough phase space many access many possible microstates that you can get into. But when you are in here there is only 1 microstate you have to be like this if you are in any other direction you are going to hit somebody.

So, even though you have reversibility theoretically you can just reverse the motion exactly the aim has to be perfect while going in but it does not have to be perfect while going out. So, that is the reason for the ease and coming out but the difficulty in going in and that is an explanation of the fact that you have irreversibility in this problem. Even though the dynamics is completely reversible it is again has to do with how many microstates are accessible.

And that is the 1 manifestation of it is this kind of here now you might have 1 objection to this thing which is to say look all this is nice in discrete time but when i go to continuous time I am going to be in trouble because if I put a time step Tau then this is Tau over Mu C and now if I let the time step go to 0 this says the meantime of recurrence is 0 that is not true in continuous-time dynamics I still have recurrences but this gives you a wrong answer because it says you cannot go to 0 sampling time that you are in trouble.

The reason this flaw was already noticed by Small Kofsky way back in 1916 and he proposed modification to Poincare which goes like this the physical idea is the following. We included in recurrence R1 okay this correspond it took P of C1 C0 given this is not a recurrence you just stayed on, so there is no question of your going out and coming back we did not do that you just stayed on and it is a fake recurrence we counted this as a recurrence.

And we needed it because I showed you that summation Rn is 1 provided you sum from 1 to infinity which is some from 0 up to upwards then you do not get 1 right. So, I will quickly buried that and summed it up this thing here but this is not a recurrence should not be included as a recurrence. So, the way to do this is to say alright I have a Tau up there but in the denominator I must exclude this so I must call Rn = the probability that I start at C0 I get out at C tilde 1 and I stay out and get back at Cn + 1 Cn etc.

Now this is a true recurrence because you told you start here you get out at 1 and then you ask I come back at time n + 1 so the time difference is n again and that is R something okay. If you do that then you got to divide by this probability to find the joint probability go through similar manipulations but what is this fellow going to be this guy here says that you start inside the cell you are outside the cell at time 1 so you put the characteristic function etc and work it out.

And you discover this is W1 - W2 and not, W naught - W1 in the denominator and now you will sum over 1 to infinity and everything is going to be fine in this case. So, this modification because what happens is you have 1 over 2 in this case in the limit in which Tau goes to 0 this fellow will also vanish like Tau and so will the numerator and there is a finite limit and therefore you can go to a continuous. So, this modification was proposed by a Small Kofsky as I said way back in 1916.

And it is been looked at after that in several cases along with some collaborators I have a paper on this which actually tells you how to handle this thing in the continuous time limit and to extend it to stochastic systems we have not talked about them and talked about it here at all and then in the case of chaotic systems which have got phenomena like intermittency and so on then very strange things start happening.

There could be reason the phase space where you are stuck for a very long time and then you jump out etcetera excellent.

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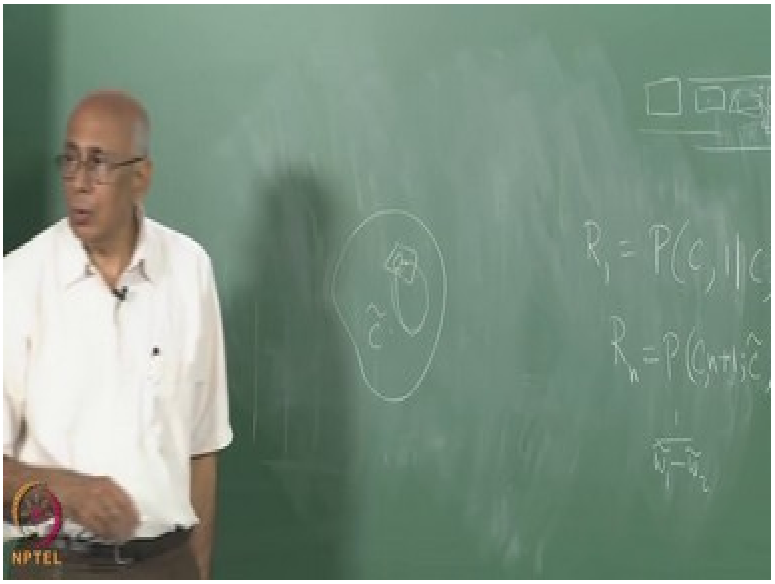

But now I want to post the following paradox to you we saw that it was possible that the mean escaped time out of C tilde so here is see here see Taylor and I said the meantime of escape out of C tilde could be infinite. But now I am saying the time mean time to start here and to come back here is finite that sounds weird because to start here and come back here I go out and then I escaped from out back in here.

So, how come the mean recurrence time is guaranteed to be finite for Ergodic systems by the Poincare Ergodic theorem but the mean escape time to go from here to there could be infinite depends on the convergence of this sequence Wn what do you think is a resolution to this paradox how do you think that is happening. You get the point right because it looks like recurrence involves an escape out of C into C tilde and a return escape out of C tilde in to C.

And this mean time could be infinite so could that be infinite but we are saying this round trip is finite mean time is finite. How is that happening, the mathematics is quite clear, so there is no mistake in that. The resolution is that when you say average time you are averaging over certain realizations of events the recurrence time probability is normalized. So, you are guaranteed that things come back and therefore you are averaging over all those realizations of this random process for which the return is guaranteed okay.

On the other hand in the escape case you are sampling over a different set of realizations you are sampling overall realizations when an escape from here to there is guaranteed that is it nothing more. This is a bigger set and therefore the average over it could well be infinite. So, it is a question of what you are averaging over once I normalize a probability in that fashion then I have said that I am averaging about all those realizations where the return is guaranteed.

And then you have a finite answer for the return distribution but this is over a different set of events just over the escape events and it is possible there are some for which the Wn is so slow decreases so slowly that it does not work. But another crude way of saying it here is another crude way of saying it, if Wn is proportional to 1 over n for instance then sigma over Wn is going to be fine infinity right okay.

Suppose it goes like 1 over n, 2 n squared what happens to sigma Wn that is finite right if it goes like this it is certainly to diverge in this case. But remember that when finding the recurrence time you are taking the second difference you are differentiating this guy twice more. So, it is going like 1 over n cubed and then things will be fine it is that so. Each time you take a difference you have a decreasing sequence the first difference will go to 0 faster than the sequence itself.

And the second difference will go to 0 even faster by a power of 1 each time. So, it is entirely likely that you may have a finite recurrence time but an infinite escape line this is entirely possible within this framework here. So, there is a lot more that 1 can say here about this but I thought I had given you a little flavor of what this looks like. One could ask that takes us into a different subject itself.

One could ask can be illustrated this general theorems using a simple model a map or something like that the answer is yes but it again involves partitioning phase space.

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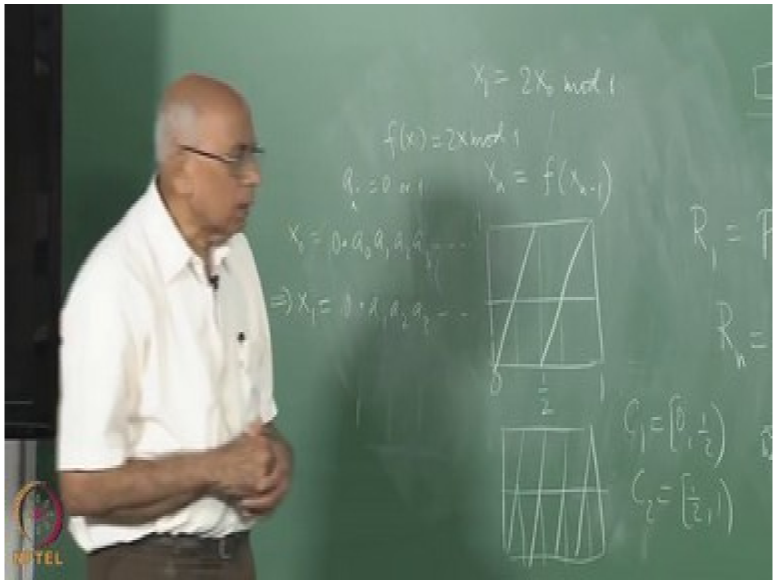

So, here is a simple model which will do the trick 1 dimensional phase space, so we have Xn = some function of Xn - 1 some nonlinear function and it is a 1 dimensional map of the unit interval for example 0 to 1, 0 to 1. And the map is simplicity itself but it is completely chaotic and the rule is the following start with any X naught between 0 and 1 and find X1 = twice X naught modulo 1.

So, this fellow says, just double the number and if it is less than 1 keep it, if it is greater than 1 throw away the integer that is it that is the rule it is called the Bernoulli shift this map here. And what is its graph look like what does the graph of this guy look like right f of X = 2X modulo 1 says 2X would be a graph like this slope to when X is 1 it is = 2 but you want modulo 1 so you cut that place and put it back here, so it would look like this and this is at a half that is the map.

And this is the rule therefore if you start with X naught X1 is 2X naught mod 1 times 2 is 2X 1 mod 1which is = 4 times X naught mod 1 and what does the iterated map look like it is got slope 4 and mod 1, so it goes like this and now the slope is 4 etcetera. No fixed points are stable in this case and all fixed points are unstable all periodic orbits are unstable etcetera.

And a typical point if you start with an X naught well as time goes along under this will fill the entire unit interval densely and uniform you can show that the invariant measure is constant in this case a typical point. But if you start with a rational point it will be part of a periodic orbit any rational point will be part of. For instance start with 1 3rd next time it becomes 2 3rds next time it becomes 4 3rds which is the same as 1 3rd is back to 1 3rd 2 3rd.

So, it is a period to you start with 1 5th goes to 2 5ths then it goes to 4 5ths and then it goes to 8 5th which is the same as 3 5ths and then it goes to 6 5ths which is the same as 1 5th and your back to for period for cycle and so on. So, any rational point will be a period to be a cycle of some finite period and those points will repeat themselves etcetera. But the rationales form a set of measures 0 among the in the unit interval.

And all irrational points are guaranteed to be part of non periodic orbits they are part of this chaotic orbit completely. So, you can find the invariant density you can now partition the cell the unit interval into say 10 parts 15 parts call each 1 of them a cell and keep track of only where the point is in which cell it is each time. And you can play this game you can find Wn in this problem you can actually compute what it is.

So, I suggest you to the following as a simplest case just do 2 cells so C1 = 0 to half C2 = half to 1 is the same as 0 by this mod business. So, call 1 of them C call 1 of them C tilde and find out what happens to Wn etc you can do this directly without too much difficulty. It is called the Bernoulli shift because you know what is happening is that if I start with an X naught and it is a number between 0 and 1.

So I can write a 0 point a naught, a1, a2, a3 etcetera you can write it irrational number will have an infinite decimal point expansion which is not recurring and I will do this in binary. So, if you do this in binary each ai is either 0 or a1 that is it. And what is this a not stand for this number actually stands for a naught divided by 2 to the 1 + a1 divided by 2 to the 2 + dot, dot, dot that is what this stands for in binary.

So each ai = 0 or 1 if this is X naught X1 doubles it and doubling it means multiplying by 2, so this becomes a naught divided by 2 to the 0 that means it comes out of the decimal point here so what the next step implies X1 = a naught point a1, a2, a3 dot, dot, dot but if a naught is 0 then the number is less than half to start with and when I did not doubled it, it was less than 1. But if they are not as 1 it means it exceeded 1 but I got to throw away the 1, so this is again 0.

That is what makes the map non-linear otherwise would have been just a linear map but now I brought it cut it down it is nonlinear. This means that if you give me an X1 there were 2 X naught's from which you could have emerged this got this X1. If you give me an X2 there were four knots from which you could have got the initial point. So, in the past they could have been 2 to the endpoints they are feeling into 1 point subsequent.

Or another way of saying it is in the forward direction a difference of epsilon will become twice epsilon next time to square epsilon the next time etcetera till it becomes 2 to the n epsilon where n increases without bound. And since you are modulo 1 it means that 2 points which are infinitesimally apart initially could become unit interval apart after given enough time that is what chaos is okay. So, this is called the Bernoulli shift because you lose information about this a naught the next time.

You lose information about a1, it is that is what this multiple value of mapping is saying telling you but you cannot retrace the back the past uniquely okay. And it is called the shift because the decimal point is just shifted each time 1 place to the right and whatever is on the left is thrown out. The map is fully chaotic at this place it is as random as a coin toss it has a lot of fantastic properties here.

But you can actually tell what is going to happen here and now interestingly what will happen you will discover is that if you break it up into these 2 cells and keep track of only the cell it is in call it left and right for example less than half its left right greater than half its right. So, you have a string sequence LR, LR, LR etcetera just a sequence of letters here then it will turn out that.

The mapping to go from at any given time to go from 1 C1 C2 to the next C1 C2 becomes a Markov process the 2-state Markov process. So, you can actually use the entire power of Markov process to now talk about recurrences here. Once does not always happen this is called a Markov partition but it is not necessarily true that will always happen it will happen if you break it up into 2 to the n pieces in this particular man. So, let me stop here and there are questions we will take them.