Physical Applications of Stochastic Processes

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Lecture-28
Statistical aspects of deterministic dynamics (Part 1)

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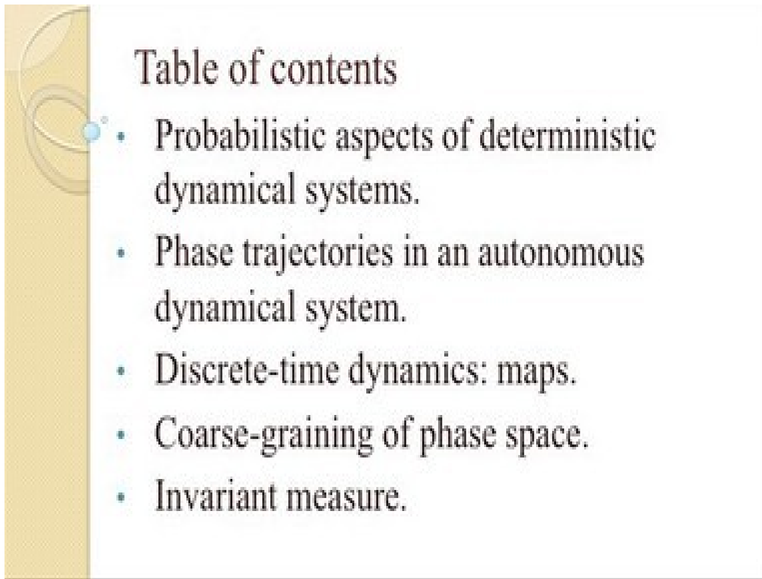

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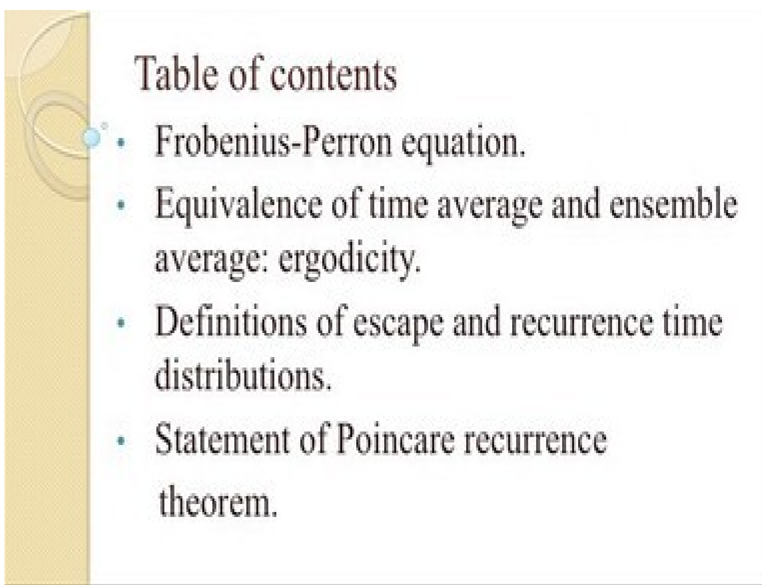

Today let us turn to a topic of some importance in the study of dynamical systems and specifically I want to show you how probabilistic methods help you to discover things about complicated dynamical systems in particular systems where the dynamical behavior is perhaps chaotic or something like that.

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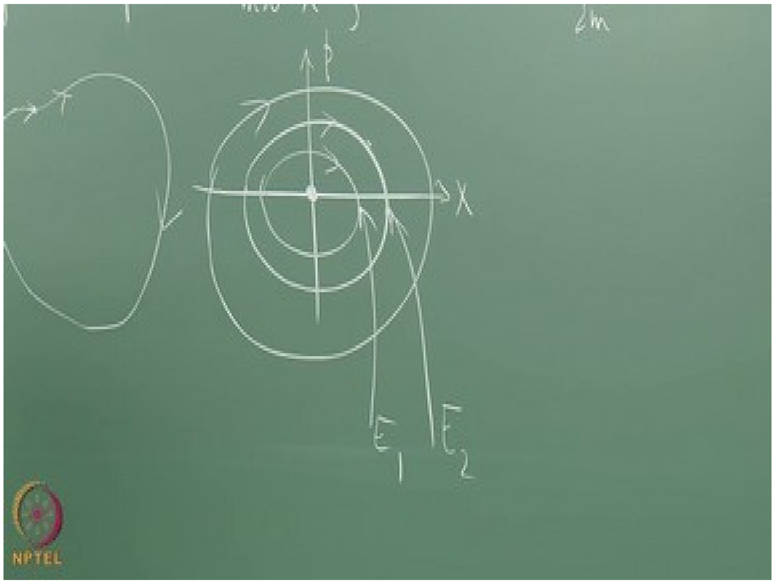

So, the specific things I want to discuss the following I want to discuss Sojourn escape and recurrence in deterministic dynamical systems well explain what I mean by this gradually. So, we are going to talk about classical dynamical systems for which the rule of evolution is specified there is no randomness, no external imposed randomness, no thermal fluctuations quantum fluctuations or no sources of fluctuations.

But completely deterministic dynamical systems and the kind of systems we have in mind are either in discrete time or in continuous time it does not matter. And specifically just to be very concrete about it if you had dynamical variable x which perhaps stands for x1 up to xn say in dynamical variables in the context of mechanics these could be positions and momenta of various particles or whatever angular momentum.

We do not care some dynamical variables of this kind and their evolution in time is given by a set of differential equations. And the differential equations look like x dot first order differential equations. So, you have a sufficient number of variables x1, x2 up to xn to describe the system completely and then the idea is that these up change with time according to a rule of the form f of x possibly dependent on time as well in this dot stands for that time derivative.

This could be a very complicated nonlinear function of x and t and x is the set of dynamical way we perceive. In particular if it is a Hamiltonian system or a Lagrangian system then I have generalized coordinates in generalized velocities or momenta and they would all come under the umbrella of these dynamical variables. If I have an explicit t dependence here the system is said to be non autonomous that means the rule of evolution itself is changing with time and it is a little more complicated than if you did not have that we just had f of x.

So we will remove this and will consider systems which are autonomous which means that this t dependence is removed so the time evolution rule is translation invariant in time it is exactly the same rule now of course this is a very, very general problem. And in general we do not have closed form solutions for it for arbitrary nonlinearities and the whole subject of dynamical systems is concerned with solving this problem in seeing to what extent you can extract the information.

Now to cut a very long story short the way to do it is to define a phase trajectory in phase space namely the space of all these x's some phase space and for simplicity. Let us just take it to be n-dimensional Euclidean space. Then the state of the system at any point is specified by giving x at any instant of time let us say x at time 0 so a point in this n-dimensional space tells you all the values of the variables know the way of values of all the variables at initial instant of time and then when once you solve this set of equations in principle you know the values at later instants of time ok.

Now the explicit solution in closed form may not be possible in general if the function is nonlinear it has a large number of variables this is impossible in most cases what is possible however is to do numerical integration of this equation. So, that you can instant by instant if you have a sufficiently small time step you can actually numerically integrate and find out what happens at any later instant of time you can join all those points together and you have a Phase trajectory.

This word phase space is borrowed from mechanics borrowed from Hamiltonian Lagrangian and Hamiltonian dynamics. So, we call it a phase trajectory here. Now if the system is autonomous in this fashion then under suitable conditions on f of X the solution is unique namely if you specify the initial value X of 0 it is a well-posed initial value problem and in principle the solution is unique which means that as time evolves this trajectory which is meandering around in this n dimensional phase space cannot ever do a thing like that it cannot intersect itself.

Because if it did so and you had an intersection of this kind then you see once you reach this point the future is unique there is only one outward trajectory with arrow pointing outwards that you can get from here because I could start with that point as the initial condition and the future has to be unique but if this situation is permitted you have this and you have that so this is not possible not possible.

A trajectory in an autonomous dynamical system a phase trajectory cannot intersect itself nor can it intersect any other phase trajectory nor can it intersects something where the initial condition was that and this is some other trajectory this too cannot intersect that because at the point of intersection the future becomes non unique. But this is not much of a constraint because when you have an n-dimensional space and n is large a ball of thread winding around the space will hardly ever intersect itself you never do so in general okay.

There is only one case in which this is possible and that is if the trajectory goes out like this in this fashion comes and joins itself again in this way that is perfectly all right. But this means that if you start here after certain finite amount of time it comes back here and then again because of the uniqueness theorem it is going to go in the same direction and come back here repeatedly which means that a closed phase trajectory a simple closed curve is the only thing possible means the motion is periodic.

Because all the dynamical variables have returned to their initial values right, so, the two things we know right away are that for an autonomous dynamical system phase trajectories cannot intersect themselves or each other. And the second thing is every periodic motion is necessarily a phase trajectory a closed curve is a periodic motion and vice versa a simple closed phase trajectory is periodic motion.

And all periodic motion every periodic motion is described by a simple closed phase trajectory. Now look at the fact look at the simplest example you take a harmonic oscillator for instance then the phase space in this case the equations are to be right down you have x dot is P over m and P dot the rate of change of momentum is the force on this particle which is = - m Omega squared x where Omega is the natural frequency and m is the mass of the oscillator P is the canonical momentum conjugate to the position x and these are the two equations of motion.

So, in this case this vector x this the phase space is 2 dimensional this comprises x and P and it is part of this kind of set of equations but this is even more special than that what is so special about this right hand side here their linear it is linear in x and P. So, therefore the problem is solvable in closed form no complications occur here at all. And of course we know immediately that in the phase space of x and P, so here is P here is x the phase trajectories are all simple closed curves they are ellipses.

They are ellipses corresponding to the fact the ellipses are also written down explicitly these two will imply that half m Omega squared x squared + p squared over 2m = a constant e which in this case has the physical meaning of being the total energy of this oscillator and that is an ellipse in general. So, in this case all motion is periodic for a simple harmonic oscillator. No matter what positive energy you specify the motion is periodic motion and the energy determines the amplitude and different energies would correspond to different amp ellipses.

So, you see in this case the phase space is a phase plane and the whole plane is laminated by these ellipses each ellipse corresponds to a fixed energy. And there is one very special value of the energy which is 0 which corresponds to a critical point in equilibrium point because at that point the right hand sides of this vanish. So, both x dot and p dot are 0 and if you start with 0 values for both of them then you are going to remain at 0 values because these are first order equations.

So, this point here is a phase trajectory all by itself it corresponds to the equilibrium point. So, this problem is utterly trivial here. Now you can ask the question and when does the system come back to its original state does it wrecker. And here because the motion is periodic every initial condition wreckers is very clear that once you are here you are going to come back to this after finite amount of time the time period and ditto for the other thing and so on.

So, the motion is periodic in this case all motion is periodic so in that trivial sense every initial condition will recur after a fixed amount of time. There is something even more special in this problem about this periodicity and what is that? So let us suppose this energy is some energy E1 this is some E2 and so on. All of them are positive numbers different numbers and the larger the energy the bigger the ellipse because the bigger the amplitude.

What is special about the recurrence time in this case the recurrence time is just the period of every orbit what is special about it in this particular pardon me it is independent of the amplitude that is not true in general it is independent of the amplitude only for simple harmonic motion for any motion other than any periodic motion other than simple harmonic the amplitude depends on the time period depends on the amplitude.

If I drop a ball from this height and it is a perfectly elastic ball perfect elastic collision with the bottom of the floor it is going to come right back to this and the periodic motion but if I drop it from a greater height the time period changes. So, every time you have a potential which is not the harmonic oscillator potential you have an amplitude or a time period which depends on the amplitude or the total energy whichever way.

So, this is a very, very special problem in this case not the most general case is much more complicated than this is a very special case. Our focus is not on dynamical systems here but we want to look at recurrence properties in particular. So, the first kind of recurrence you have is periodic motion as a very trivial kind of recurrence. Once you solve the dynamical equations you compute the time period that is the end of the matter here.

But already if you complicate this a little bit and take two simple harmonic oscillators, let us say a particle moves in a plane and it is connected by a spring to the origin and it moves in the plane that is the combination of two simple harmonic oscillators at right angles to each other then already a complication starts arising in that case they have got an x and a y-coordinate so you have a 4 dimensional phase space.

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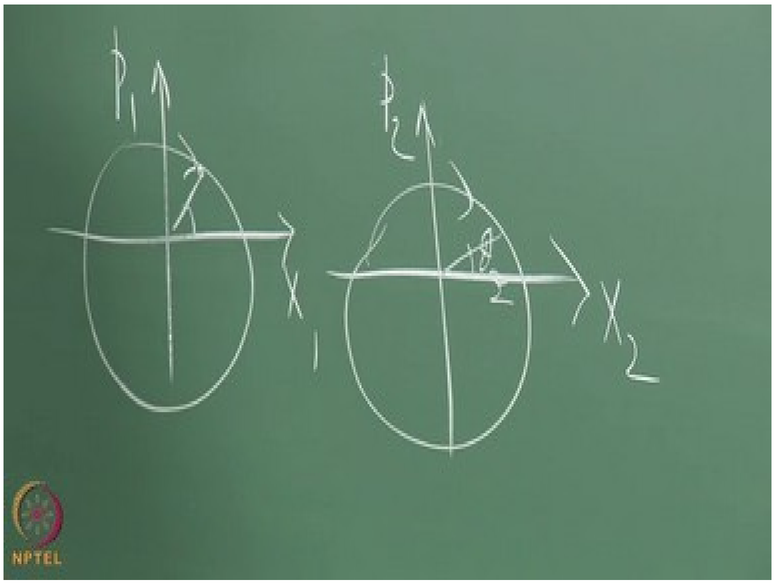

And if I plot I cannot plot this phase space because it is 4 dimensional what I can do is to plot x1 p1 p2 x2 and I write down the usual total energy or Hamiltonian of this oscillator 2 different oscillators at right angles to each other the total Hamiltonian is p1 squared + p 2 squared over 2m + the potential energy in this case and this potential energies of the form half m Omega 1 squared x squared + Omega 2 squared y squared or so take the scale of 1 of them to be 1.

So, it is x1 squared + Omega square x2 square in general right. So, if I took this oscillator I have set the frequency of 1 of them to be = 1 and the other one to be Omega some number Omega there is no reason why the frequency should be the same in the 2 orthogonal directions. Is this motion periodic? Is this motion going to be periodic? It depends on Omega it depends on Omega if Omega is = 1 then of course it is periodic immediately.

What happens if Omega is too it is still periodic because the two frequencies are commensurate with each other. So, as long as Omega is some rational number the motion is periodic but the moment is irrational the motion is no longer periodic it will never come back to its initial point. What will happen that will come arbitrarily close to its initial value every initial condition will come arbitrarily close but never come back to its exact initial value the motion is said to be quasi periodic.

In this case and you can sort of tell what is the nature of this quasi periodicity by saying that basically what I want to know is moving around the trajectories look like this the sections of the trajectories look like this, so what is relevant is once you specify some initial conditions such as the initial value of the total energy of this oscillator and that you specified these ellipses then the question is if you start at this point and you go around you start here and you go around you do not come back to the starting point there when you finish a revolution here.

And if the frequencies are in commensurate there is no time at which you will come back to exactly the same values for all the 4 dynamical variables right. What is relevant in this case is the angle so it is sort of making by changing units you can make these circles like then all you want to know is the angle here is theta 1 and the angle here is theta 2, the state of the particle the values of the variables at any point at any time can be written in terms of two angles theta 1 and theta 2 where are you on these circles for given values of the energies of the two oscillators.

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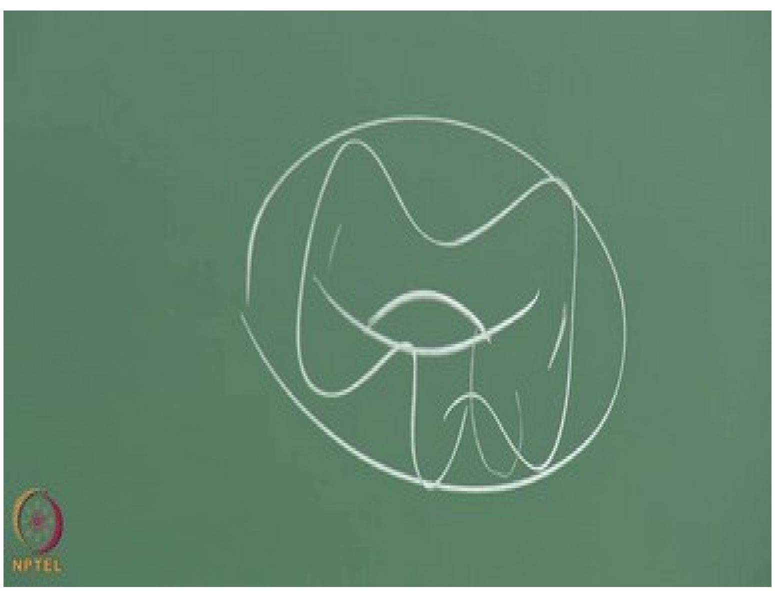

So, you can write that as a point on a torus in which this angle here with some reference direction with respect to some reference direction gives u theta 1 and in the cross-section here of the storis you get data to basically you want to know your phase space is now reduced to the space of two angles each running from 0 to 2Pi. So, at every value of theta 1 you have another variable which runs 0 to 2Pi so it is like a donut surface of a donut right.

And this torus on this tour is the phase trajectory is doing this kind of thing it cannot intersect itself so not all on the system and if the frequencies are in commensurate it is not hard to show that any individual initial condition will whine tightly around this torus an infinite number of times never intersecting itself but coming arbitrarily close to its initial value. And there are theorems which will tell you what will be the density with which this typical point will cover this torus and the answer in this case is in uniform we uniformly covered.

So, now you could ask what about recurrence what about if I start with a small patch here set of initial conditions and I let the system evolved. Since it is quite regular it is not diverging or anything like that this patch will move around neighboring initial conditions will remain neighboring initial condition neighboring points as time goes along but this patch this little patch will move around this torus and so on.

And you could ask when does a typical trajectory come back to this region? So, if I specify a small cell in this phase space I could ask a trajectory which starts at this point how long does it take to come back that is it how long on the average does it take to come back. So, we already have our first example of a recurrence problem in which you have quasi periodic motion and you are asking what is the mean time of return to this patch sometimes.

It will of course depend on the size of this patch possibly it will depend also on what is the resolution with which you made this patch etc all these details exist can be worked out. But the question is what is the mean time over which it comes back this problem has a closed answer it has a straightforward and simple answer based on number theory which we will not talk about right.

Now but you see the first of these recurrence problems appearing but we want to go to a more general case I want to go to the most general case possible and say you have a phase space some complicated dynamics is going on there is a complicated phase trajectory going on and now in this n dimensional space I identify some small cell and say suppose I start in this cell the system starts in this cell initial conditions specified to some resolution.

How long will it be before it comes back will it come back at all is a question and if so how long will it be on the average before it comes back? So, it is a very, very general question of which these are very simple examples this cause I periodic recurrence and so on. But we are asking an extremely general question.

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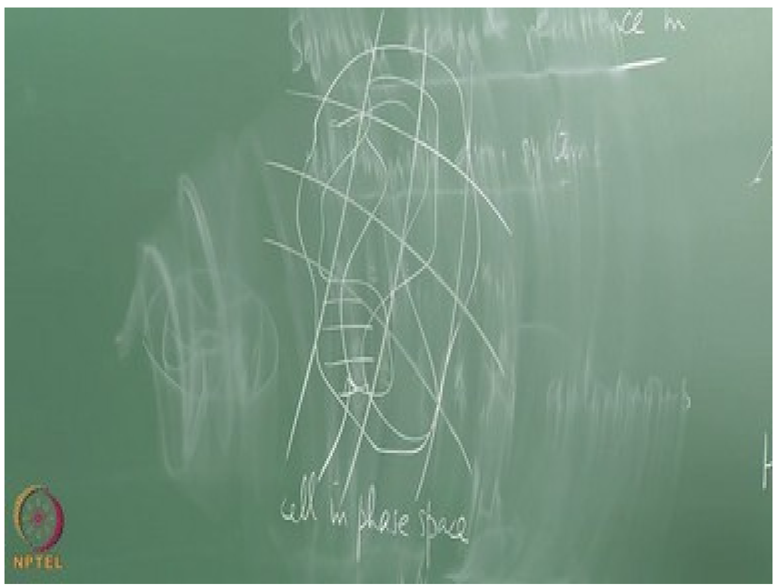

And here is how one can make some headway and answering it. So, schematically let us suppose that we have a big phase space since I cannot draw an n-dimensional space stations draw this blob and say that is the phase space in which the system is. And typically a trajectory will do some very complicated and meandering in this directly I have shown it to be intersecting itself it is not supposed to intersect itself which is just a projection on the plane of this trajectory.

So, it is like a ball of wool is never going to intersect itself but it will come arbitrarily close to itself and I would like to know something about recurrence. So, what I do is to simplify the problem by saying look I have some resolution with which I can identify points. So, I have some coarse graining by which I identify volumes in phase space. So, I partition this phase space into cells and this is a typical cell.

This whole thing is a typical cell, cell in phase space and I do not keep track of individual trajectories at all. I simply ask is my representative point in the cell or is it not in the cell where is it with respect to the cell. So, I give a label to each cell c1, c2, c3 etcetera and ask at a given time is my typical trajectory is any factory that I am following in which cell is it. And so, so I work to that resolution and I also do this with some sampling time.

I generally do not look at the trajectory at all times I have no information I do not solve this problem exactly I simply look at it as a discrete in discrete time with some time step then the dynamical problem that I have becomes slightly different.

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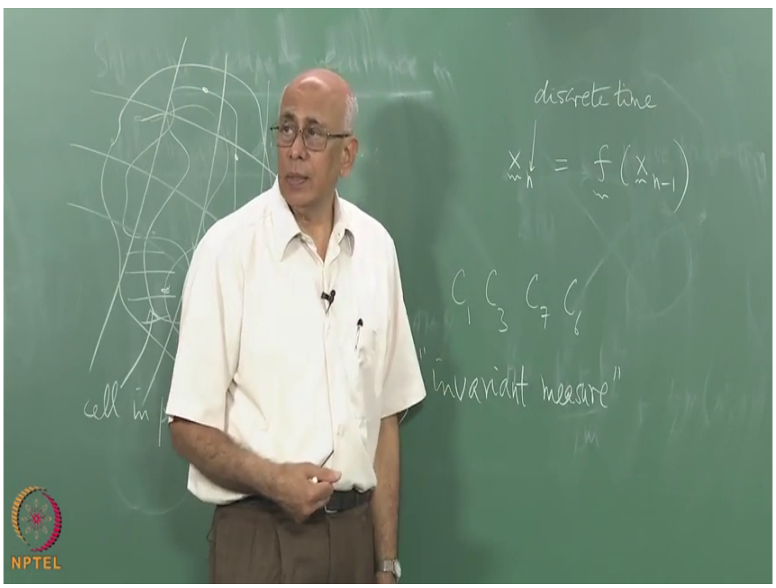

It becomes not a flow but a map so now what I have is a situation where with typical point x at a given time n is = some functions are nonlinear function in general of this point at time n – 1, so this is discrete time. In other words instead of following the trajectory I am following this these are the points where it is at discrete time steps. And then I ask where is this, which cell is this point in itself?

So, now I have essentially I replaced a continuous trajectory by a symbolic dynamics I simply say that my point to start with is in some cell c let us call it c1 it jumps to cell c3 when it comes to c7 then it jumps to c6 etc so I have a string of letters following a typical trajectory. Saying it is in this cell or that cell of this cell or that cell and the question asked is what is the mean time it takes to come back to the original cell, but it started from.

In the most complicated situation possible neighboring trajectories will actually diverge and have very different histories because the system could be chaotic. In that case you have exponentially sensitive in system sensitive to initial conditions exponentially. So, initial trajectories which are very close to each other initial phase space point will diverge typically exponentially till they become as for big as a space itself.

Separated as much as the phase space itself so we will look at cases where the full phase space is bounded some bounding volume and we will now look at a case where we assume that this process has been going on for a long time there are no reasons of a space where the system gets stuck and the system is moving about in a portion of a space or the whole of a space with some invariant measure or probability with some steady state probability.

So, the idea is that the trajectory has gone on for a long, long time and now it is visiting different regions of phase space here with probability measures which do not depend on time. This is the equivalent of saying the system is in thermal equilibrium in all the equilibrium problems we looked at right. So, there is an invader measure just as the Maxwellian distribution of velocities is the invariant measure for velocities for an ideal classical gas. Similarly we assume there is some invariant distribution.

Now what is meant by this distribution, so this is equivalent to saying that which each cell Cji associates a probability? That the system is a priori in that in other words I should close my eyes I put pen on the paper and asked p of Cj is the probability that the point it is Cj that the typical trajectory hits Cj right. What would this be actually well think of it this way if I toss a coin and I say the probability of a head is half what I really mean is if I toss the coin in n times the total number of times that I get heads to the total number of tosses approaches the limit half that is what is meant by the a priori probability.

In exactly the same way P of Cj would be the actual fraction of time that a typical trajectory spends in the cell Cj over an infinite amount of time you take the fraction of the time that it spends in Cj and that of course is going to be the apriori probability that the system is going to be that the representative point is going to be in cell Cj. So, we have an invariant measure it is like the stationary probability.

Now in this case in contrast to noisy problems where we had thermal fluctuations or some source of fluctuations I have to specify for you the statistics of the noise whenever we did Brownian motion or random box headset I had to specify for you the distributions each time it was put into the system but here the dynamics generates this P of Cj because in principle it is deterministic dynamics and there is nothing outside that is coming in.

So, from that from this equation I should be able to determine what this invariant measure actually is explicitly we should be able to do that right from the equation itself. Now how is that done okay and that is a little trick it is called the Frobenius Perron equation. And let me tell you what it is because idea is extremely simple.

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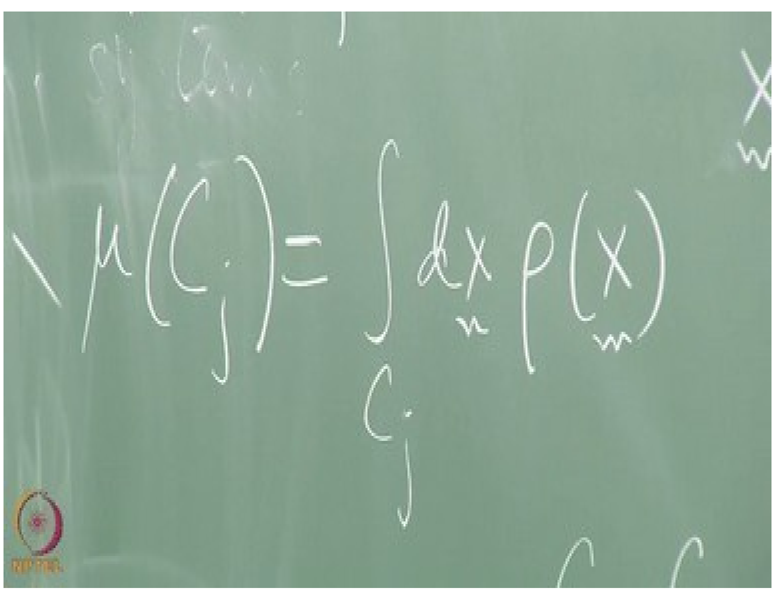

Suppose you say that at any given time in n the density of points in this space at any given time let us suppose off when you have this invariant measure is some Rho to start with I specify an initial distribution of initial conditions. And let us suppose that is Rho 0 at time 0 of x and I ask what happens after one time step what happens to this density. Well if you start with a point here and it is on a trajectory going there after one time step it is there.

If I start here and is the trajectory going here it is here etc, so this density spreads out into some other density after 1, 1 unit of time let us call that Rho 1. So, I want to find out what is Rho 1 of x given Rho naught of x how one approach this. Well the point the point y goes to the point x in one time step where x is = f of y by this rule. So, if you give me a point y it goes to f of y and I call that the point new point x at time one right.

Therefore it is clear that Rho 1 of x must be = an integral over dy Rho naught of y you are integrating over the initial distribution but each y goes to an x which is given by f of y. So, this has got to be multiplied by a delta function of x - f of y there is a delta function in all the variables because that is all it can be we know where each y goes deterministically completely. Therefore the distribution has to be a delta function.

This way and now you tell me the initial value itself was uncertain there was many, many possibilities there was a distribution you prescribe and therefore that distribution goes to a new distribution which is = this given by this. Now this is a tall order to compute what this is because remember that you are integrating over y but the Delta function is in x, so you have to invert this function and in general it is nonlinear so it is multiple valuable.

So, this is where the complex it is but it in this doing it even in the simplest one dimensional case is very, very non-trivial for nonlinear functions okay. But in principle that is what it is but now you can extrapolate this in time and ask what is Rho n of x given Rho n - 1 of y again the same rule so at time n - 1 if you had this then at time n you have to have this density. Now we want the invariant density the equilibrium density the t go into infinity limit.

So, what would that correspond to what limit should I take here. Well time has been replaced by discrete time n so I need to let n go to infinity but when I let n go to infinity I call this whole thing this is identically = Rho of x, Rho invariant I do not put a subscript this is the invariant density and that is got to be = integral dy Rho of why the same Rho of y Delta x - f of y the same Rho because it is not supposed to change.

It is exactly like the Maxwellian distribution of velocities in this rule. If I take an instantaneous snapshot the particles are all distributed by the Maxwellian distribution of velocities an instant later the individual velocities have all changed but the distribution has not changed it is exactly the same distribution. Although different particles will occupy different parts of this distribution now and that is exactly what has happened.

So, it is the same Rho of y which has to fold back under this time evolution to give you this guy here ok now you can see that this is now an equation for Rho of Y determined by the dynamics and the dynamics is here so this specifies the dynamics completely the nonlinear dynamics. And this says under that dynamics find that Rho which is invariant finds, that function which is ingredient this is an integral equation for Rho.

This is the kernel of this integral equation in n dimensions of course it is a singular kernel because there is a delta function sitting here. And it is a homogeneous equation because there is a Rho here and a Rho here same Rho. So, it is a homogeneous integral equation but with a singular kernel and as you know for a homogeneous equation you can multiply it by any constant and it will still remain a solution.

If you have a solution already so in that sense Rho is only determined up to a multiplicative constant how will you find what the constant is by normalizing it, as always. So, you would also have say if this whole thing if this whole phase space is some gamma you would say integral dx Rho of x = 1, so that will fix the overall constant and this equation is called the Frobenius Perron equation not guaranteed that it always has a solution not guaranteed that always has a normalizable solution.

That it should be non-negative Rho of x cannot be negative being a density probability density but this thing here is called the invariant density. When I multiply Rho of x at any point x by a volume element dx in this phase space I get the invariant measure of this set so I get d Mu of x = Rho of x a measure is more general than a density because there could be points Delta functions there could be points where there is a finite probability at one particular point and so on but we will use the use language of the density itself.

You and then what is the invariant measure of a given cell of the whole cell. So, what is Mu of Cj what is this = it is = an integral over C j dx Rho of x of course and the total Mu of Cj summed over all the j's is going to be 1. So, now we have in place the machinery we are supposed to be given a deterministic evolution equation f of x we are then supposed to discover the solution to this equation by solving the Frobenius Perron equation it is possible.

Getting an invariant density and then computing the invariant measure of each cell by integrating over the invariant density okay. Now we have assumed their Ergodicity in the sense that we have said given enough time a typical trajectory will spend time in all the possible cells does not get stuck anywhere then the property of Ergodicity is equivalent to saying that the longtime time averaged over a single trajectory is equal entirely equivalent to an ensemble average with this invariant density. So, this is the whole point of this discussion.

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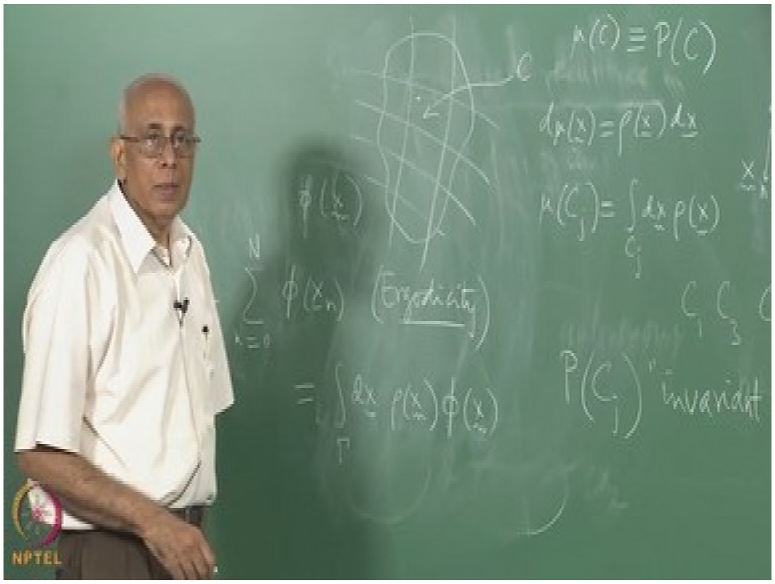

In other words if you have if you have some function let us call it something else Phi of the dynamical variables x and I want to find out what is the mean value of this Phi of x over a long time okay. Then what I do is to take Phi of x n at time n the any representative point is at x sub n I take this Phi of x sub n sum from n = 0 to capital N 1 over N and take the limit as n tends to infinity that is the long time average of this function of the dynamical variables some given any given for any function of the dynamical variables right.

But if the system is Ergodic then this time average can be replaced by an ensemble average. So, this would be entirely = integral on the right hand side and what should I write here integral dx over the phase space Rho of x Phi x this is a Ergodicity, this is the property of Ergodicity that the time average is = the ensemble average. This is the whole point of statistical methods since you cannot do long time, time averages if the system is chaotic.

You cannot compute what xn is given x naught with any precision then you resort to the fact that you do not need it you use if you can discover them invariant measure that is it. This integral is guaranteed to be that. So, that is the way one way of handling chaotic systems where there is so much sensitivity to initial conditions that any initial error gets exponentially amplified, multiplied in time and after a while it is nonsense okay.

But if you find the invariant measure you can circumvent that by using their Ergodicity property to rewrite it in this form. Exactly as in equilibrium statistical mechanics if you cannot find, you want to find out what is the pressure of the gas in this rule on a wall you cannot find or track individual particles and find out what the force they are exerting on the wall is that mean instant of time. Instead you say the system is in thermal equilibrium I discover the probability measure with which particles moving with various velocities.

And then I compute the average ensemble average with some equilibrium on some density matrix okay. In exactly the same philosophy you replace this time average by an ensemble average here. So, everything is contained in this fellow. Now comes the question which we want to answer which is given this phase space and I partition it into cells here is a typical cell see the questions I want to ask out the following.

I am assuming that there is an invariant measure and I know this measure, so I know it for each of the cells so I know Mu of C this by the way is the same as saying the probability that a typical particle is in the cell C its independent of time because it is invariant. And the question asked is suppose I tell you that I start at this point at t = 0, n = 0 what is the statistics of the time for which I stay in this cell without jumping out.

I stay for a random amount of time because if I start at 1 point I may jump out after 1 time step if I start somewhere else I am a jump out after 10 time steps etcetera. So, I want to know what is the statistics of the state probability the next question I want to ask is what is the statistics the rate at which I escaped from the cell and the recurrence problem corresponds to saying what is the statistics or probability distributions of the time at which I come back to this cell you.

So, I have Sojourn which is stay in some given region of a space then I have escaped which is like a first passage out of the system and the boundaries of the system and then a recurrence back to this system. So, we need to discover the statistics of these quantities.

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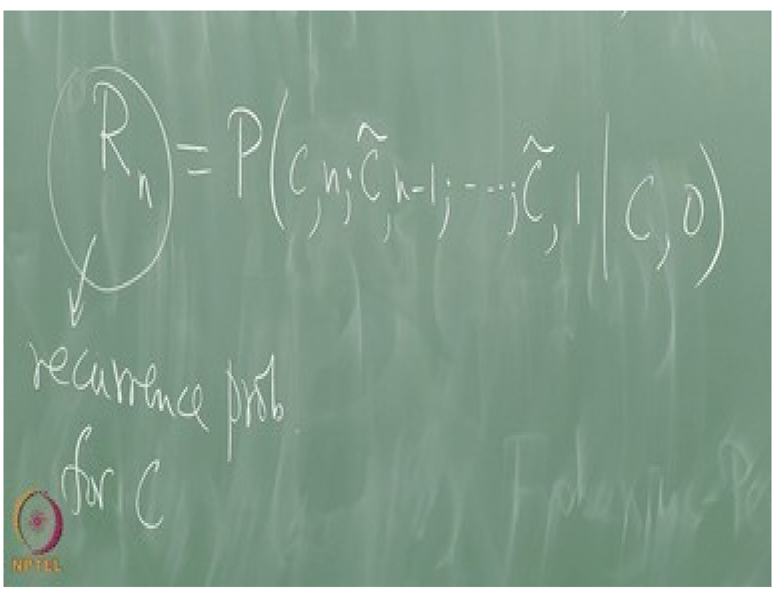

In particular we want to find out the following I want to find the probability that in the case of escape for instance I want to find the probability that if you start from the cell at time 0 in our notation now. I would like to find the probability that at time n I escaped out of this cell. So, I want see an escape so escape is the complement of this the rest of it let me call see Taylor equal to compliment of C that is the rest of phase space.

So, here is my designated self and the rest of phase space I call C tilde, so I want to escape at time n so at time n I am out in this fashion but before that I was still inside the cell, so Cn - 1 dot C1 given that I was in this cell at time 0, I want to find this joint probability for general values of n. What would the recurrence probability look like? So, I want the probability let us call it R sub n this is the probability that I start with C in 0 and at time n I come back, so Cn but till then I was out so this is the probability a master recurrence probably for the cell C.

And that is the quantity we are aiming at to compute on the way we will compute the escape the Sojourn etcetera probabilities but this is the one we want to compute. What is the mean time of recurrence then? Remember we are in discrete time so n is time first I got to make sure this is normalized so I got sum over all possible n I should get 1 because recurrence in mutually exclusive events.

If I say I come back up for the first time after three steps that is a distinct event from saying I come back up to 4 steps these are all distinct guys. So, this has to be a normalized probability by sum over n and then i multiplied by n and wait it with this P, I get the meantime of recurrence to the cell C. Similarly if I say I stay in the cell C at t = 0 at n = 0 and I escaped at n = 1 or a escape at n = 2 or 3 etcetera these are all distinct mutually exclusive events.

If you escape a 3 you could not have escaped at 2 or 1 at time 1 therefore this too is set of mutually exclusive event. So, which you can assign probabilities? Sojourn is not like that because if I say I am staying in this cell till time 10 it implies you have already stayed till 9 right. So, different Sojourn time probabilities are not mutually exclusive events. So, we will not be able to normalize Sojourn time probabilities.

But we be able to normalize escape time and recurrence time probabilities. So, that is worth bearing in mind so I will do this next time tomorrow will write down all these probabilities and write it but let me start the initial step for this.

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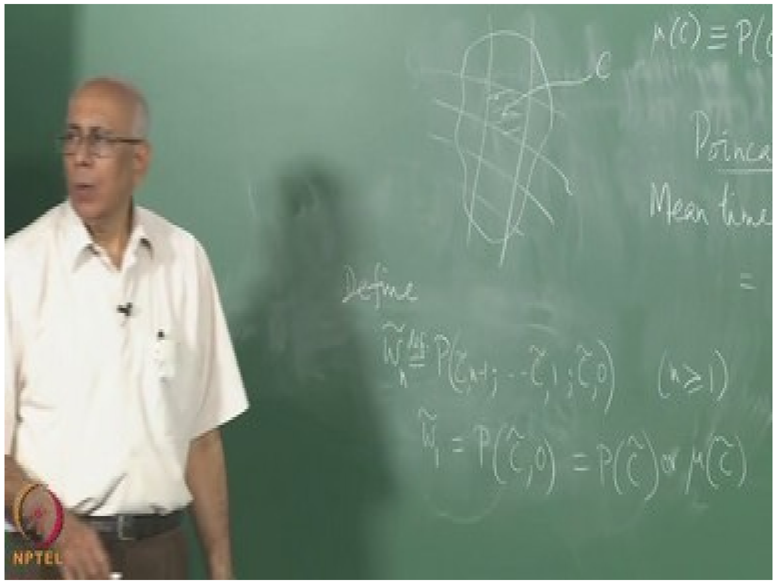

And that is the following and the entire trick lies there so is the target clear we assume we have a dynamical system with some n dimensional phase space some complicated dynamics going on there in general not regular not periodic not cause a periodic possibly chaotic. We have a region of a space or the whole of a space in which this system remains forever once it is got in, and there is an invariant measure.

It is a pre-assigned probability distribution perhaps discovered from solving the Frobenius Perron equation or numerically all you have to do is to run the trajectory for a long time and find out what is the fraction of the time that it stays in different cells on the average and that gives you the R priority probabilities are invariant measure. Then having got that you want to answer this question what is the mean time of recurrence to a given cell okay.

And here there is a theorem called the Poincare Recurrence theorem and I will state this theorem will prove it the theorem says the following mean time of recurrence to cell C is = 1 over the measure of a cell in time steps of one group taking time and this is called the Poincare recurrence theorem. It is remarkable how general this theorem is the only thing you need to prove it is a Ergodicity.

If the system given enough time comes arbitrarily close if every representative point comes arbitrarily close to every point of this space accessible part of a space then you are guaranteed that the mean time of recurrence to cell C is 1 over the measure of this cell. By the way if you have a single point the measure of that point is 0 so it will never come back and that is equivalent to saying the motion is not periodic in general.

But it will come arbitrarily close to it and as soon as the cell has got a finite extent this is it the smaller the cell the longer it takes to come. Of course if the cell is the whole of phase space the meantime is 1 because it is there already so in that limit to its okay. But this is a non-trivial theorem here and it is remarkable in its generality and you will see that the proof is really very elegant and does not require anything more than a Ergodicity.

But you exploit this Ergodicity property carefully the first step in the proof is the following. We want to focus we want to look at recurrences and so on to this LC so let us start with the compliments of it. And let us define W so any compliments quantity I put with a tilde on top let us define this quantity to be = the probability that you have a Sojourn outside, so you start outside the cell at 0 you remain outside the cell at 1 dot, dot, dot.

And you remain outside the cell at time n – 1 n greater than = 1 define, these are joint probabilities not conditional joint probabilities what is W1 it is P of C tilde at time 0 but remember that we are talking about the invariant probabilities. So, this time argument is irrelevant it is = P of C tilde or Mu of C, so that is true we define a sequence of numbers W1, W2, W3 etcetera where W1 is just the measure of the complement of the cell see that you are interested in.

Let us also define W0 tilde by definition = 1 this just makes the notation easier so let us put W 0 tilde = 1 what is the Sojourn probability now that you stay in a given cell C tilde in this case till time n what is the Sojourn probability?

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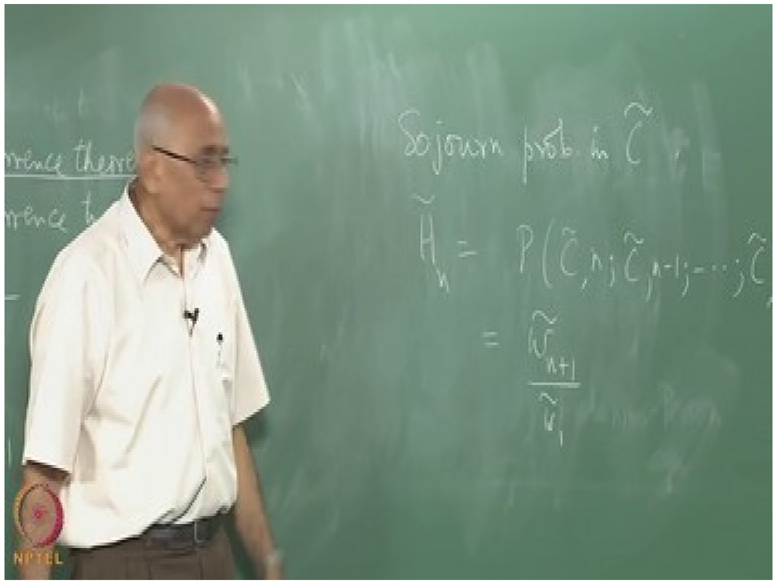

Let us call that H sub n tilde because it refers to the compliment it is the probability that your C tilde at time n C tilde at time - 1 dot, dot C tilde at time 1 given that you are in C tilde at time 0 given that now we are talking about actual physical probability so they have to be conditional probabilities if I tell you I start in the cell C tilde in the in the compliment C tilde at 0 and I now ask what is the probability that I am still there at time n and I never went in to C at all.

Well can I write that in terms of this remember this is a up to n - 1 and that is up to n and it is a conditional probability. So, if you multiply this probability by P of C tilde then you get the joint probability right so since if not multiplied it you have to divide this guy here so if you multiply you are going to get P of C tilde n from C tilde of 0 but you call up to n - 1 Wn. So, what is that quantity of going to be Wn + 1?

So this is = Wn + 1 tilde but you must divide W of C tilde which is = W in other words if you compute this set of numbers you have computed that. What should be the normalization condition on H sub n, is there a normalization condition, no because sojourn is not mutually exclusive different n's are not mutually exclusive events. So, there is no normalization condition on this.

But can we say something about the w ends if the system is Ergodic it is clear that Wn must be a decreasing function of n this is the probability that it just stays in C tilde without entering C and if it is regarding sooner or later it is got to enter C. So, this is a decreasing sequence starts with the value W1 tilde or even with the value 1 if you like this is a number less than 1 and decreases till at n = infinity it should go to 0 right.

If it is Ergodic it should go to 0 we know that so we are going to use that fact will use that fact with some with rigorous theorem and analysis to try and discover what the escape probabilities and then what the recurrence probabilities. So, the arguments are very, very general they do not really involve any detailed details of the dynamics at all but they are powerful precisely because they are so general and tomorrow we will do the rest of this okay.