Physical Applications of Stochastic Processes

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Lecture-26
Recurrent and transient random walks

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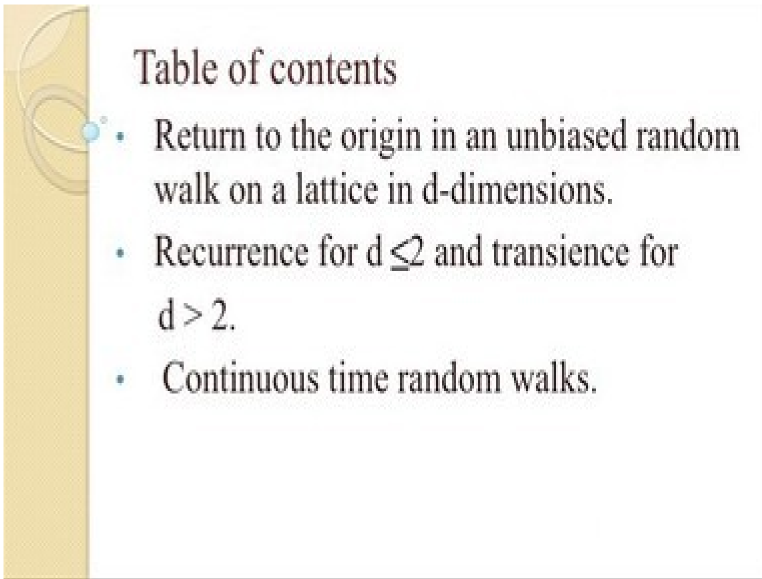

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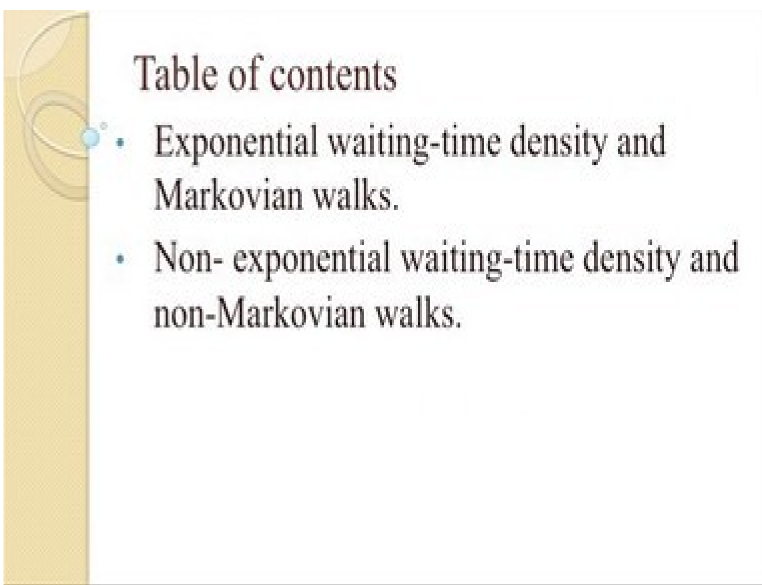

All right let us start today with trying to establish a little more carefully this problem in random walks about recurrence. I have been throughout mentioning that in 1 dimension on a line for instance the first passage from any point to any other point is certain for normal diffusion. It is a sure event and I have also been saying that the mean time for first passage is infinite however in the unbiased case.

We also saw what happens when you have bias this first passage to any point is not a sure event or a return to the origin in a linear lattice for example is not a sure event. I have also often not menti1d that in 1 and 2 dimensions walks are recurrent in the sense that if you do not have any bias then starting anywhere you are guaranteed to get to any other point return to this point any number of times an infinite number of times.

Possibly with a mean time between events which is infinite, we need to establish this. So, we will do that in 1 particular context and then this result can be actually generalized. After that I want to go back to what we started looking at namely non Markov walks and I want to put that in a slightly more general framework because there are lots and lots of physical processes which cannot be described by Markov processes or Markov chains.

And for which you need something for generalization 1 of these generalizations is called a renewal process and we will talk a little bit about renewal theory okay. So, first about the problem of recurrence and let me do this in the context of random box on lattices which are either linear or a square lattice or a cubic lattice or a hyper cubic lattice in d dimensions. The problem is the physical phenomenon of recurrence depends on the dimensionality and not on the actual lattice structure.

So, if a walk is recurrent for say simple cubic lattice it is also recommend for a face centered cubic and so on. There could be minor changes of details depending on the lattice structure for instance in 3 dimensions it will turn out that an unbiased walk is transient it is not recurrent. So, the total probability of return to the origin for example is less than 1 but what number that is between 0 and 1 will depend on the actual lattice structure okay.

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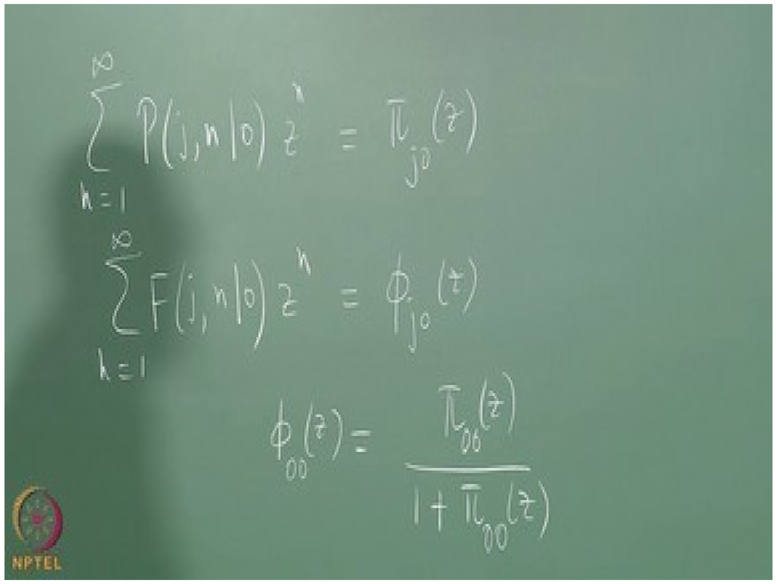

So, let us try to put this in a slightly general context I have in mind a d-dimensional, so this vector r is an element of it is a site in a d dimensional lattice and let us call its coordinates for example j1, j2, jd just to remind ourselves that these are integers. So, their lattice points they run over all the integers. And now I want to ask the question if I start at the origin I want to find for example P the probability that you are at any point r at time t or in discrete time it would be n given the fact that you started at say the origin at t = 0.

So, this is the quantity we need to compute which we can do fairly straightforwardly if it is a Markov process and then remember that the problem of recurrence came about in the following way in the 1-dimensional case we had explicitly a thing like P of j n 0 0 and then I said the generating function for this quantity for this probability is that to the power n summed over from n = 1 to infinity I had a symbol for this I do not remember what I called it Pi j n 0 or z recon that is the symbol I use for it.

And then I said that the first time if it hits the point j this would be F of j n 0 this is the probability that starting at 0 you the site j for the first time and the generating function for this was n = 1 to infinity I call this Phi j 0 or z, Phi j 0 of z right. It is summed over in and then the statement was that Phi 0 0 of z was = Pi 0 0 of z over 1 + that was the relation which we got by using the renewal equation between the generating functions.

And just to refresh your memory if Phi 0 0 1 is = 1then recurrence to the origin is a sure event because it says the sum over all these probabilities without this said is 1 okay. Now the way that can happen is for this to diverge that is the only way it can happen. So, this quantity tends to infinity then you are sure that recurrence to the origin is a sure event. And if you do this on an infinite lattice this is true for in general for Markov chains.

But if you do not do this on a lattice and talk about the random walk problem in that context then it is clear that for the random walk problem on a linear lattice.

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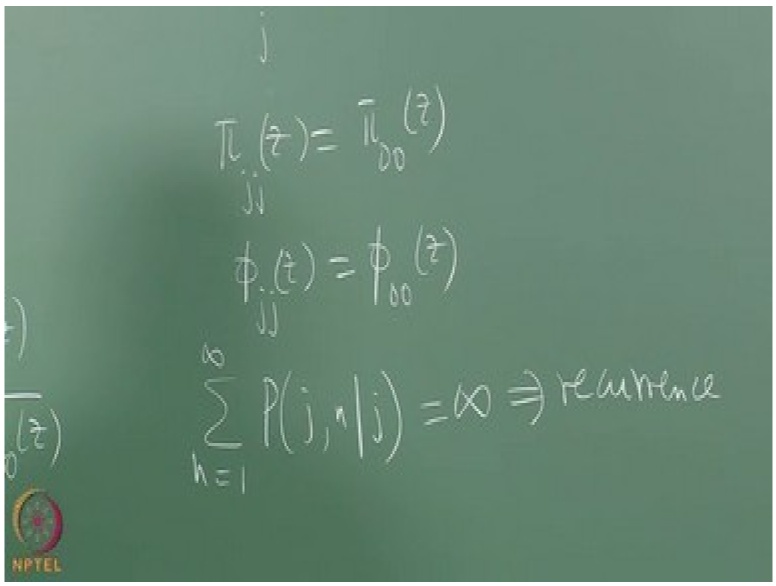

For example where the sites are label by j in this case so you have an random walk on a linear lattice in this case it is clear by translational invariance that Pi jj of z is = Pi 0 0 of z this probability depends only on you can choose any point as the origin essentially and similarly it also follows that Phi j j or z = Phi 0. So, although I spoke about return to the origin I could have chosen any point and said return to that point okay it is exactly the same thing.

So, we want this quantity to diverge in other words you want this quantity summation n = 1 to infinity P of j n j should be = infinity implies recurrence. So, we want to check if this quantity is divergent or not and that is what we want to do in higher dimensions in the general context of a Markovian random walk on a d dimensional lattice let us say it is a hyper cubic lattice and I want to find out if starting at any site say the origin.

The probability of coming back to that site at time in that probability if you sum over n does it give you infinity or not if it does then this quantity then this walk is record so this is what we want to check out. And I had like to specifically prove that for 1 in 2 dimensions this quantity will diverge but in 3 and higher dimensions it will converge and therefore the walk will not be recurrent this is what we would like to establish okay.

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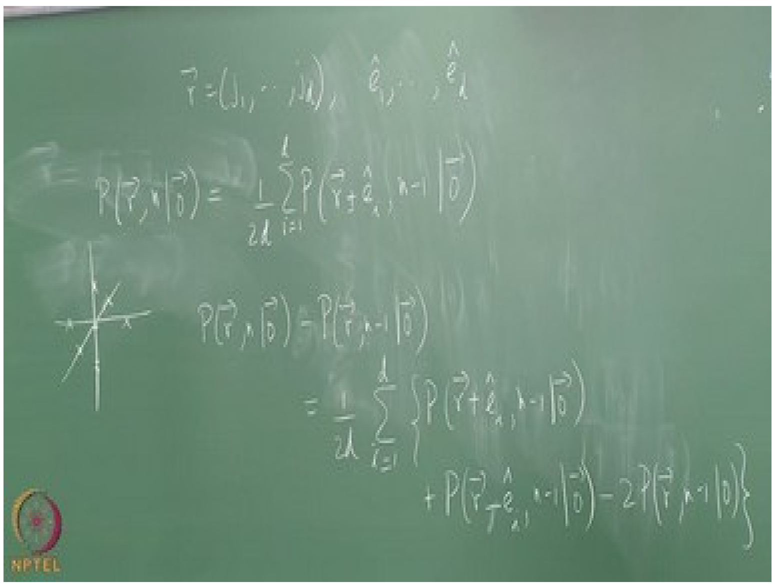

So, let us see how to do that and we will do this you would not spell out all the details but we will do this by writing out what this equation is actually now what is P of r n starting at the origin see what is this thing = well in any general lattice. Here is the point r the lattice point r and it is got a certain number of nearest neighbors in a hyper cubic lattice for instance there are neighbors on this side on this side and front and back so in all 3 dimensions.

There are neighbors and we need some unit vectors to denote these neighbors. So, these unit vectors would for example be x, y, z is that in a cubic lattice in 3 dimensions. So, let us suppose that you have r is therefore j1 to jd and you have unit vectors e1 up to ed. These are the unit vectors in the Cartesian along the Cartesian axis in d dimensional space. Now this gets fed in because it is a Markovian walk in the previous step you should have arrived at 1 of the nearest neighbors.

And then from that point from any of the nearest neighbors you are going to jump in to this point here with probability = 1 over 2d because the number of nearest neighbors in d dimensions is 2d in a cubic lattice right. So, this is = 1 over 2d P of r - ei summation from i = 1 to d right at time n - 1 and that is it that was the difference equation. We also subtracted from this - P of our n - 1 and then you can go to the continuum limit if you like or else you can call this quantity the discrete Laplacian or whatever.

So, writing it out explicitly this guy here was of the form so we had an equation of the form P of r n 0 was = 1 over - P of r n - 1 0 this quantity = 1 over 2d summation i = 1 to d P of r + or - you need both because you can have a jump from here into this or from this in to write both vectors you have got there are 2d nearest neighbors right. So, you can write this as P of r + e sub i at time n - 1 0 + P of r –e sub i n - 1 times 0 - twice P of n - 1 0 that was the discrete Laplacian right.

Now let us just check you are going to sum this d times so the 2 cancels against this and the d will cancel against the d here, so indeed I have subtracted 1 and now you can proceed to the continuum limit of this guy so, this is the usual Markovian walk right. But now how do you solve a thing like this the solution is obvious what we should do is to do a 4ier transform in space right.

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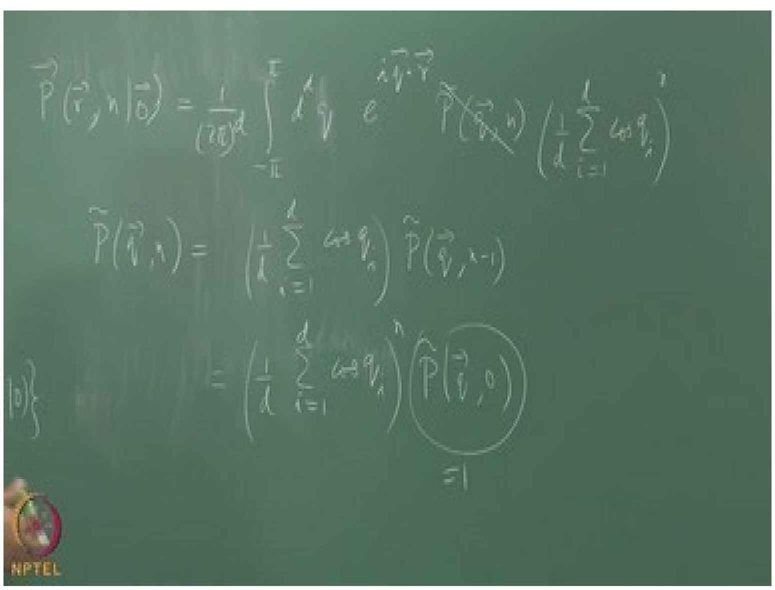

So, let us define a Fourier transform with respect to the space variables so let us write P of r n starting from the origin let us write this as = 1 over 2 Pi to the power d and integral dd k in d dimensions I use the symbol k very often for a lattice point. So, let us call it q or something like that this is the momentum the conjugate variable q d dimensional integral times e to the i q dot r times P tilde of q.

And so that is my Fourier transform but this is periodic this P is on a lattice this guy is on a lattice so you have to integrate over the fundamental period which is - Pi to Pi for each of the components of q because it is periodic. Now or another way of saying it is that this r is P of r is 0 except when r is a lattice point. So, it is just a bunch of supported only on the lattice points. So, formally 1 can write a thing like this.

Then what happens here if I plug this in into this equation if I do a Fourier transform here then it becomes clear that P tilde of q n = I am going to leave out factors and things like that 2 PI's etc we cannot just write this is = 1 over 2 Pi to the power d and integral etcetera. Let us plug that in and see what immediately what is going to happen you are going to have q dot r + ei this is what is going to appear inside.

So, write it as q dot you know the I q dot r + P sub I this is = e to the i q dot r + e to the i qi because that is a unit vector. So, the moment I put this in I am going to get an extra phase factor e to the i qi and from this guy I am going to get e to the - iq i and I somebody is I get 2 cos qi right and that is going to happen for each 1 of these fellows. So, this gives you finally 1 over d summation i = 1 to d cos qi.

The 2 goes away in the cosine in the definition of the cosine right. So, I got you to the iq i + e to the - iq i which is 2 cos whatever it is and these 2 cancels here and I am left with just 1 over d cross qi and this fellow multiplies P tilde of q n - 1 because that is the same thing out here. I have taken out that extra factor due to these 2 guys and you are left to adjust that from this equation. So, that is a very simple solution now.

It is just a recursion relation in which the Fourier transform at time in is some constant times the 4ier transform at time n - 1 right so this implies this is = 1 over d summation i = 1 to d cos qi to the power n P tilde of q n 0 with just this factor but what is P tilde of q , 0 it is the Fourier transform of the initial distribution the initial distribution we started with the assumption this fellow here satisfies P of r 0 0 vector = a delta function at d dimensional Delta function at 0.

It start from the origin it is just a d dimensional Delta function and we want the Fourier transform of that which is 1 with this normalization here I put this guy in here is = 1. So, that solves this guy here is = 1. So, you immediately see that all you have to do is to replace this by 1 over d the summation i = 1 to d cos qi to the power n. If you can do the inverse Fourier transform of that you got the probability distribution formally.

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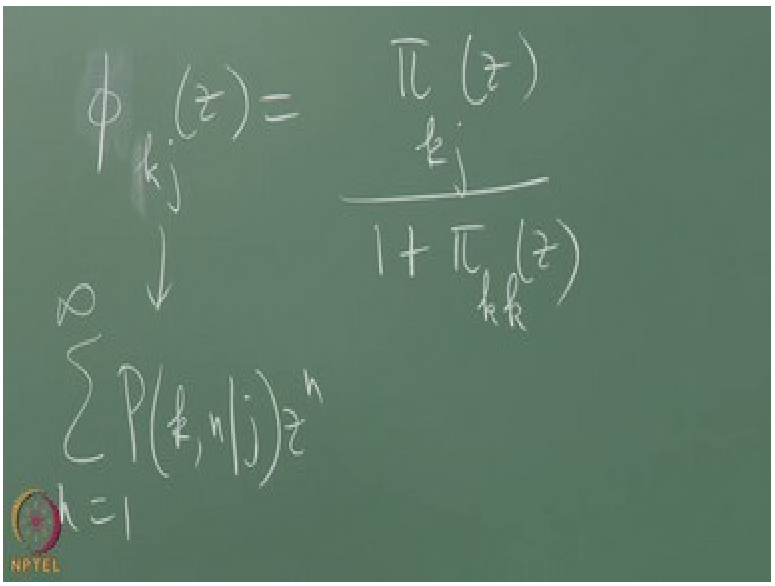

But now we cannot therefore ask what is Pi is 0 0 and these are vectors now 0 0 of z of 1 this is = a summation over n n = 0 to infinity n = 1 to infinity does not matter a 0 to infinity it adds a 1 but just we are going to show diverges so we are not interested in this 1 over 2 pi to the power d that well - Pi to dd q e to the iq dot r but I put r = 0 it is the origin right. So, that factor goes away and you are left with 1 over d summation cos qi i = 1 to d to the power n.

But that sum can be d1 it is just a geometric series we can do this sum and notice that these cosines are less than 1 in magnitude and as z sitting here. So, this number is a number less than 1 and therefore you can sum it the geometric series and it gives you 1 over 2 Pi to the power d integral - Pi to Pi d dq divided by 1 - 1 over d well let us write it out cos q1 and that is the answer for Pi 0 0.

Now this diverges if this integral diverges then the walk is recurrent return to the origin is sure. But if it converges we know that the probability of a return to the origin is less than 1. Now how would you do this integral these are magnitudes of q1 to qd. So, the way to do this is to try to do this in spherical polar coordinates this is called a Watson integral it appears in lattice dynamics very often.

When you study the normal modes of vibrations of lattices this sort of thing will appear this precise factor depends on the fact that we started with the hyper cubic lattice. So, you will get different factors here because the unit vectors would be different in different lattice systems but you would get something similar to this. Now, can it diverge at all, well if it diverges it is got to do so at the origin at q = 0.

You got to see what the behavior of this fellow is if q vector = 0 each component is 0. So, all these fellows become 1 and cancels against the d the 1 will cancel against this and give you a divergence in the denominator which should be could still be finite depends on what is happening on top here. So, we are looking at what is going on near the origin in the magnitude of this. So, if you did this in spherical polar coordinates for instance you wrote it out.

And I wrote out an expansion of this for small values of qi the leading term is q squared the 1 cancels against this here. So, the denominator goes like u squared magnitude. So, you certainly have q squared in the denominator but upstairs remember you have q to the d - 1 dq. So, near the origin this is going to behave like that if at all it diverges this factor is going to go like q squared where this q squared stands for q1 squared + etc 2q d squared square of the magnitude.

And upstairs you have that phase space factor volume element q to the d - 1 d q right so d = 1 you have integral 0 dq over q squared this tends to infinity no doubt about it d = 2 this is 0 q dq because in 2 dimensions the line element is rdr the volume element the area element is rd rd theta or whatever so there is a q dq divided by q squared which is like the dq over q this tends to infinity. So, in both 1 and 2 dimensions this is how you show that the walk is definitely recurrent.

In 3 you can see this integral becomes finite in 3 dimensions d = 3 you have integral q squared dq divided by q square near the origin this is less than infinity, this is finite. So, this is the reason why return to the origin is sure in 1 and 2 dimensions the walk is recurrent for unbiased walk and in 3 and higher dimensions it is transient. The actual probability total probability of return well you must compute the first by system that is it right.

Let us compute this integral and compute Pi's 0 0 and then extract the coefficient of said oh put z = 1 to get the total probability of return. Remember very carefully that this quantity itself this summation itself summation P of n 0 sum from 1 to infinity here this is not a sum over mutually exclusive events probabilities. This is not 1 I mean this quantity can be as large as you please that is why it is diverging out here.

On the other hand this quantity F of r n 0 this quantity cannot diverge it cannot be bigger than 1. If it is = 1 you know the event of first return is a sure event otherwise it must be less than 1 right. Because it is very structure as you can see in the general case we saw a relation like Pi j kj of z was = Phi Pi kj of z was Pi k Pi k this thing here was the generating function for P of k n j to the power n sum from n = 1 to infinity.

This quantity here we said z = 1 here then this quantity here on this side this cannot diverge this 1 there is some over probabilities that we put set = 1, it cannot diverge. On the other hand Pi can this quantity may well be divergent and that is what we have shown here this object here is divergent. If it is convergent then we need to find that ratio to find out what is the actual first by such time or return time distribution like or what is generating functions like okay.

I hope this is clear this is not this is not a probability this guy is a probability there are some over probabilities here this is some our first time return probabilities and the total probability cannot exceed 1. So, we have been very careful to discussed the issue computing this requires you to compute this and the behavior of this dictates whether it will recur or not all right.

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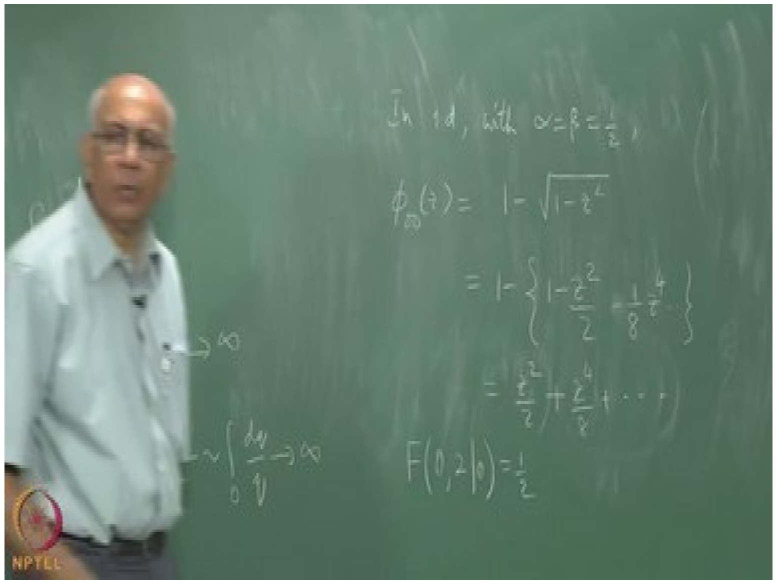

So, much for this return probability I computed in the 1 dimensional case we actually saw what the return distribution was. For instance return to the origin I think we got something like in 1d with alpha = beta = half we found that Phi is 0 0 of that was 1 - square root of 1 - z square that is the result we got right. For the generating function of return to the origin, now of course if you did an expansion of this in power series it is clear from this that it is singular at z = 1 straight away you can see that.

You put z = 1 you get 1 which means the walk is recurrent in the unbiased case. But the coefficient of z to whatever power is that to the power k will give you the first return probability that the first return occurs at time k. Now the fact that this is a function of z squared tells you that only even powers of z can appear. That is a reflection of the fact that if you start at the origin on a linear lattice you can get back to the origin only in an even number of steps.

So, P F will be 0 for odd in right and we can expand what this fellow is this is = the first term there is a 1 – 1 - z squared over 2 and then + half into half - 1 so that is - 1 and then 2 factorial so 1 8th z4 dot, dot. So, this gives you z squared over 2 + z4 over 8 + all terms are positive yes they have to be positive for z = 1 okay. Now this series as it stands will converge this binomial series converges for mod z less than = 4 mod z less than = 1.

It is it is got a singularity at set = 1 the square root singularity. But it is quite finite at that point and now you can see that the probability that you are going to start you are going to come back to the origin at the second step = half that is the coefficient of z squared and that is trivial to establish.

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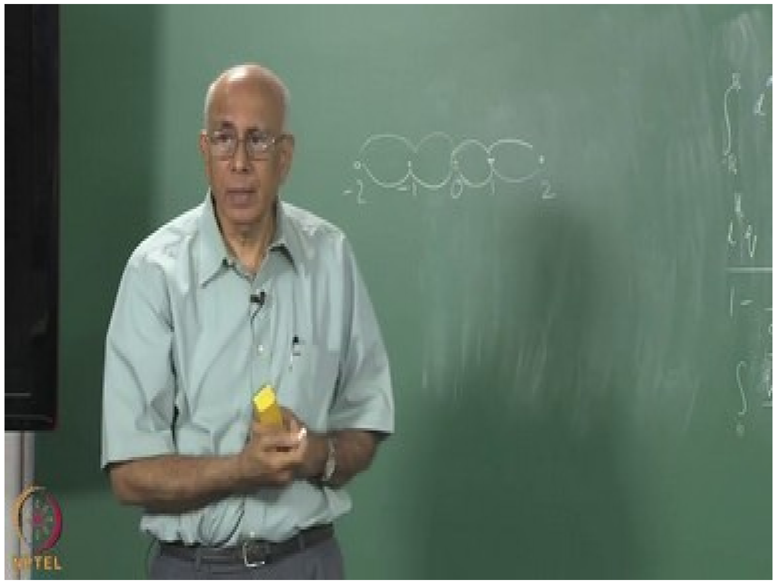

This is completely trivial to establish because you are on this lattice here is a site 0 this is 1 this is - 1 you want the probability that you come back to the origin and the second step okay. There are only 2 ways of doing this 1 of them is to go here and come back here and the probability of that is 1 4th because you have to go there have to come back here instead of going off and the other side have to come here you have to come there.

So, that is 1 4th and 1 4th + 1 4th is half so that is an immediate enumeration similarly F 0 4 0 = 1 8th now what are the possible ways in which you can do this. So, here 0, 1 – 1, 2, - 2 and so on we want the probability that you come back on the to the origin on the 4th step. Well clearly you cannot do this because you are back on the second step right. So, you have got to go here and you have got to go there and then back here back here right.

What is the probability of that happening there are 4 of these fellows each of them you have a probability half factor so that is 1 16th and the only other way you can do it is to come here and go back here in here that is 1 16th you add the 2 you get 1 in okay. So, that is okay and next 1 is going to have non-trivial factors you are going to be there is going to be 3 factorial in the denominator. And things like that; so there more than 1 way of doing that right.

So, clearly what is going to happen is that you could go up to this point and then come back in 6 steps but you could do this and then come here. So, you have to allow for all those factors in other words you have to enumerate all these blocks but this is doing it automatically. This fellow is automatically doing this so once you write the binomial expansion of this the matters over. It is actually automatically telling you what is the coefficient?

So, what has happened is actually an enumeration of walks you counted you have counted all distinct paths such that on the end to end step you are back into 0 yeah but we want to come back in 4 steps so here is 1 here is 2 so I come here I cannot come back there I have to go here and then I got to come back into the 4th step to are over already so the only way I can do it is to come back in this fashion there is no other possibility right.

So, there are only 2 walks but the moment you have go to the next step 6 etc you have more possibilities and of course you go to 2n in general you have a large number of possibilities the number of these walks would increase.

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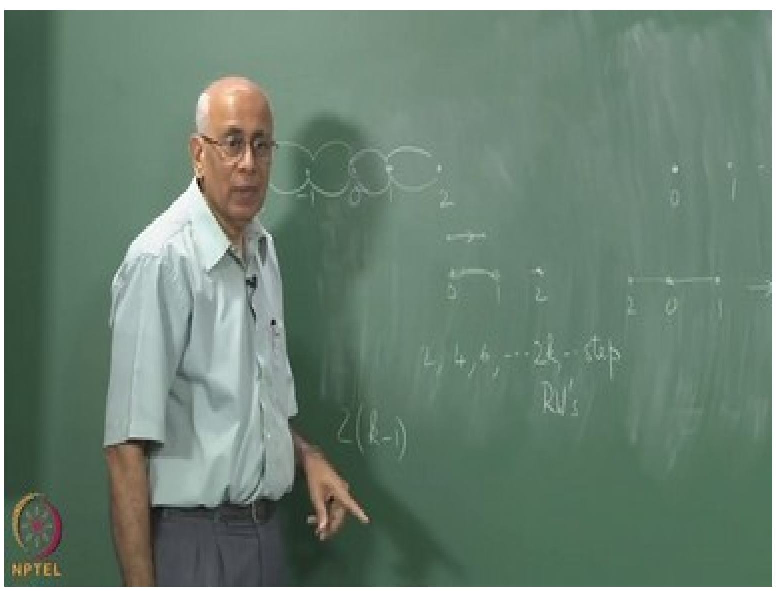

Incidentally that also brings me to the following and this is a good place to do that but I will come back to this because it also leads to the concept of what is called renormalization of random box. We saw already in the case of a linear lattice that it takes on the average it takes 4 times as long to go twice as far. So, we put barriers at 2 ends at + - j and I said the first passage time to go from 0 to j is just j squared on this lattice.

We solved a set of difference equations and I said it is just j squared. Incidentally we did that problem by saying here is 0 here is 1 here is j and on this side I said - 1 and this is - j and I said there is a barrier here barrier here and starting from 0 what is the mean time to get to either of these fellows at distance j right. But and then we got the answer j squared so this means if I increase it j from j I go to 2j then the answer is going to be 4 times as long it is going to go to 4j right. So, that is the reason the walk dimension was 2 in this problem okay.

You need to have d1 it that way you could have just said although I have a lattice starting at 0 on 1 side. I asked what is the time to go here and ask what is the time to go twice as far and this thing is in some sense self-similar I can make this structure a self-similar 1 as follows. Just like in the Sierpinski gasket instead of considering lattices with 1 side and then 2 sides and 3 sides, 4 sides and so on. Instead of doing that I can rescale it in the following sense, I start at 0, 1, 2 pretend there are bonds here and I put I decorate it by putting a site extra site in the middle and then blowing it up.

So, from this the next step would be to double the size so what I have d1 is to put a site here and a site here and double the same that is and it gives me that okay. I do the same thing the next step how many sites will there be well there is 1, 2, 3, 4 more I am going to add to the existing pipe right so it is going to be 9 next time and the next time it is going to be 16, 17. So, it is 2 to the n + 1 the fellow in the middle and then there is a toe to the end because I am doubling it each time right.

So, this structure is now self similar although the original lattice is a regular Euclidean lattice. If I consider lattices with 3 sides, 5 sides and then after that 9 sides and so on 2 to the n + 1 sites that is a hierarchical structure right and you can play the same game on it and discover that the time doubles each time. In fact once I know that I can do it even more cheaply I can say I start at 0 and ask what is the time to go to 1 meantime it is 1 unit.

Now I ask what is the mean time to go there - what would that be what is the answer I should get I should get 4 for the average right. How does that happen how does that happen in the first case to go from 0 to 1 there is just 1 there is just 1 walk that is it I flipped that with probability 1 I m out I finished. So, the mean time is just the deterministic time it is 1 right. The next time around I can start here go here come back go do any number of times and then finally go there.

So, I should be able to prove by counting over all the time steps here and all the walks that I end up with an answer which is 4. Now to go to this place I can only do so in an even number of side steps. So, you only have to sum over you cannot do it in 0 steps you cannot do it in 2 steps right. So, you have to sum over 2, 4, 6, 2k step random box. You have to sum over all these walks and for each 1 of them you have to find out what is the time taken multiplied by the time taken and calculate the average.

And the answer should be 4, for instance how many 4 steps works would there be to go here. Well I go here and I must do it in 4 steps so I cannot go here, I must come back that is the second step 3rd step 4th step right and there is only 1 set walk there is only 1 set walk and it is weighted by 4 number of steps the = time so it is weighted by 4. How many 6 steps walks are there I got go 1, 2, 3, 4, 5, 6 and how many such walks are there, just 1.So, each and this is a trivial problem it is a trivial problem these 2k steps these fellows of that the last 2 steps must be 1 2 finished.

So, that leaves 2k - 2 steps right. So, you have 2k - 2 which means twice k - 1 pairs of going back and forth and that is all and each time when you are here remember the probability of jumping here is half. You have to be careful about that so I leave you to do this as an exercise to show that the weighted average the mean time to go here is going to be 4. You must count all the walks carefully weighted with appropriate probability factors and so.

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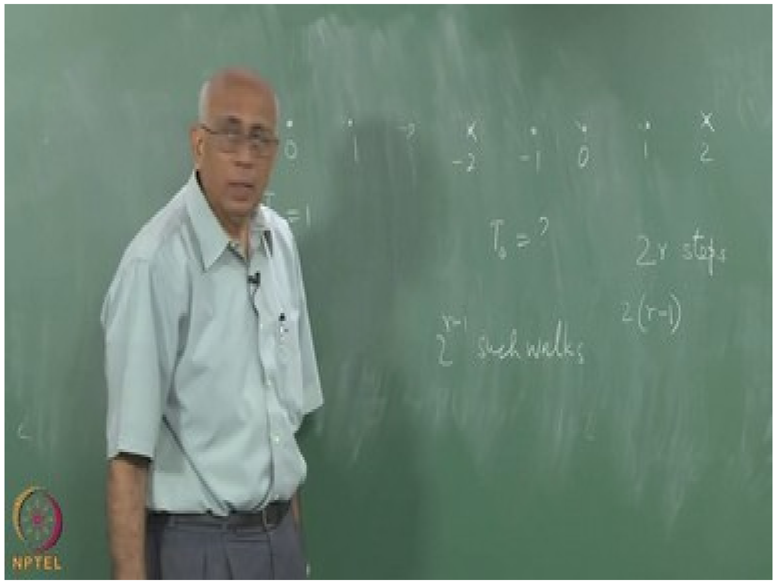

If you did the same thing here on this and this is going to take us to a non-trivial problem so I have 0 I have 1 and I have - 1 and now I say alright I start at the origin I have barriers at + 1 and - 1 and I ask what is the mean time it takes me to hit a barrier and the answer is 1. Because whether I go right or whether I go left I hit a barrier and that is it I had unit distance right. So, in this case T naught = 1 let me call the mean time to hit the trap T naught this 1.

Now from this I go to this, so here 0 –1, 1, -2, 2 and that traps are here what is T not equal T they should be 4 this is what we have been saying all along it should be 4 right but you can do this now either by writing the difference equation which we wrote down for a Markov walk or better still by actually counting all the walks with appropriate probabilities. And I want to do that for a specific reason because I want to generalize to non Markov box.

So I do not want to use the Markovian backward Kolmogorov equation which had those nice difference equations or something I am going to do is somewhat more general kind of random walk so now I say all right how do I figure out when I am going to hit this what are the all the possible walks again remember to hit this point or this point you can only do so an even number of steps so let us suppose that you do so in 2r steps the last 2 steps must be straight like that that is the only possibility there is only way you can go from here to there from the origin right.

So, clearly what is happening is that you have twice r - 1 steps which consists of this or this entertaining between these 2 and then at the end you jump from here to there in this fashion right. So, if you want to compute the number of walks from the origin to 2 or - 2 in an even number of steps. You have to enumerate all possible walks go to ask how many of these walks there are and clearly.

The last 2 steps are quite deterministic depending on where you want to go here or here but then this many steps this many walks are either 2 + 1 and back to 0 or - 1 and back to 0 because you shouldn't hit + or - 2 you want to hit it for the first time in the 2r-1 step and you have to know add all such walks. And there are clearly 2 to the r + 1 such walks 2 to the r - 1 such walk okay. Because there are r -1 pairs and each pair is determined by do you go to the right or go to the left.

You have a choice and therefore you have 2 to the r - 1 choices such works. So, we will use this fact I will come back and we use this factor to show how a walk on the linear lattice can be renormalized even in the more general case of non Markov walks okay. Now let us switch gears and go to what a non Markovian walk is.

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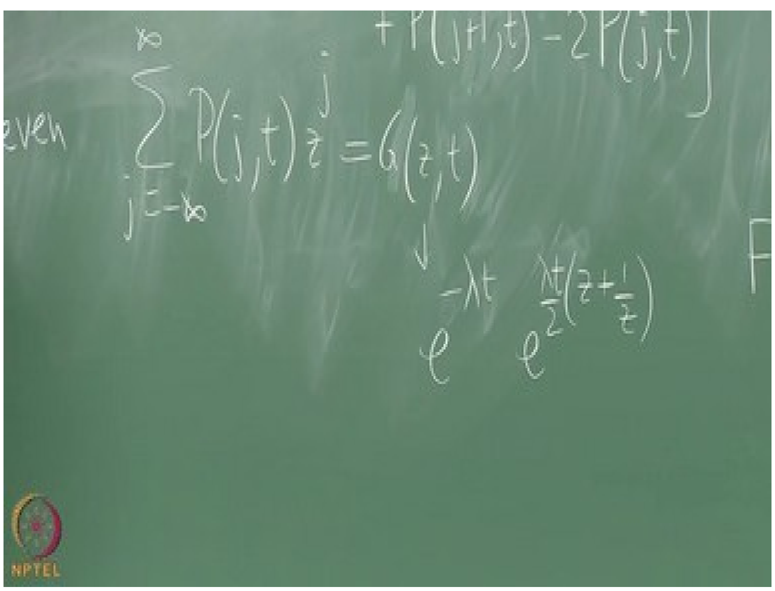

And we will do this in specific framework called continuous time random walks. This is a technical term which is used in the physics literature it does not just refer to the fact that time is continuous because we looked at Markovian box by the time was a continuous variable. So, that is not what is in meant by this what is denoted by continuous time random walk is a very specific non Markovian or generalization of a Markov in random walk in continuous time okay and it is the problem is like this.

If you go back to the original 1 dimensional random walk problem we did this in the Markov case in several ways first we said let us look at a lattice 1 dimensional lattice discrete space and then I said discrete time as well. So, we ask what is the probability that if you start at the origin you are at a site j at time step n okay we found the solution to that, in fact the solution to that was if you started with the origin here and some site j here.

Then the P of being a j at time n starting from the origin this was = n in n steps the random variable here was j because you are given n time step. So, now as where and where do you end up you end up at some j and the distribution of that j is given by this binomial distribution n - j over 2 1 over 2n. Well let us look at the unbiased case for simplicity and what were the conditions here you are given n the number of time steps.

So, what are the conditions here for this probability what should j be, are there any restrictions on j, it is an integer of course but what are the restrictions on this j, it should be less than = n, so this is mod j less than = n and n - j even you cannot end up at a site for an even number of time steps you can only end up with an even site not on add site. So, P was = 0 otherwise, otherwise it was = this binomial distribution so that was 1 way of doing it.

Then I said okay let us look at this problem not in discrete space we leave space discrete but let us look at it in continuous time and then I wrote different equations with the differential on the left hand side with respect to time so I said dP over dt so we said dP j t over dt was = and I said huh you had to reach you had to jump from the site j - 1 or the site j + 1 and with some mean rate lambda. So, I put some lambda here and said this is P j - 1 t - sorry j +1.

These were the gain terms but you could also jump out from this fellow with the total probability 1 so this guy was - so I had a set of different differential equations of this kind t was continuous here and the statement was that in the time axis these jumps occur completely at random dictated by a Poisson process with some mean rate lambda. So, that lambda over 2 is a mean rate of jumps to the right and lambda over 2 is a mean rate of jumps to the left because it was an unbiased walk.

So, what is the solution to this we have got an explicit solution here it is a binomial distribution what is the solution there? Now there is no time step it is g1 so there is no question of even number of steps or number of steps time is continuous out here and this I did by solving I solve this by saying let us find a generating function for this fellow sum over z to the person so I put P of j, t is e to the power j sum from j = - infinity to infinity I call this sum generating function.

Let us call this G I do not know what symbol I use then G of z , t, it is not a power series and positive powers of z because this fellow here is running - infinity doing it is a Laurent series you do not care what and what did this give you finally we got a simple equation for this guy for this G and then I read out the coefficient. In fact the answer I got for this was e to the - lambda t e to the power lambda t is that + 1 over z, that is the answer I got for that.

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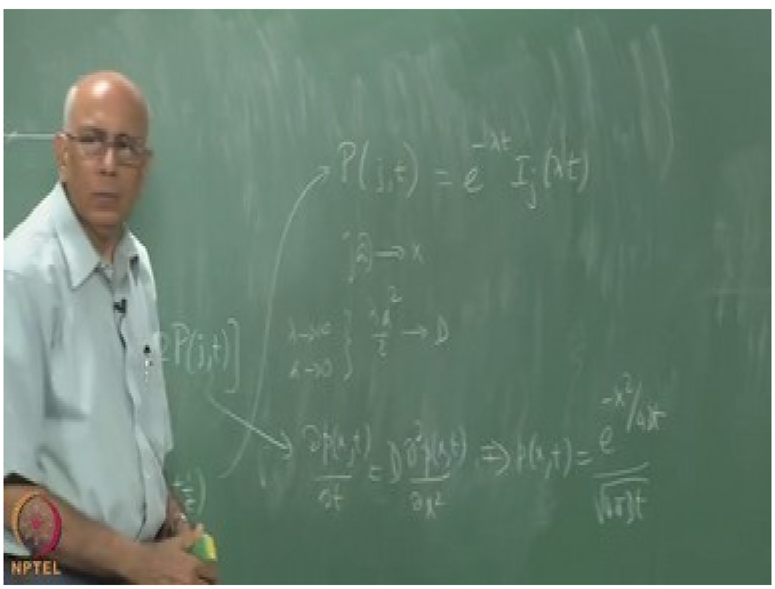

And from there I said okay I now look at the coefficient of z in it and what was the solution this got salt and P of the j , t starting from the origin of course was = e to the - lambda t the modified Bessel function of indexing and I - j is = I + j because this is completely symmetric with j 2 - j unbiased walk if I had a bias then I had an alpha over beta to the power j over 2 sitting there and then this was lambda t's twice lambda t square root of alpha beta but in the unbiased case you had this.

So, discrete space discrete time discrete space continuous time again a Markov process that was a Markov chain is a Markov process and then I went to the continuum limit in the space variable by putting a lattice constant and putting in limit. So, I said j times a tends to x this is the lattice constant and I said lambda a squared over 2 tends to a finite number d. So, lambda tends to infinity a tends to 0 such that this fellow tends to be and I got the diffusion equation.

So, the solution for the probability density function in x was the famous Gaussian e to the - x squared over 4 Dt and so this equation this thing here went over into Delta P of xt over delta t and I found the Gaussian solution to this right. So, what really happened was that from this if you know went over to continuous time by saying that these jumps are happening at random instants of time guided by some Poisson process, uncorrelated jumps.

Then you end up with a Markov process whose solution is this Bessel function here and if you now went to a continuous space limit as well with this particular limit you are the diffusion equation and its solution was the Gaussian solution so this would imply with the same initial conditions it told you that P of xt was e to the - x squared over 4Dt square root of xt.

So, that is a continuous Markov process it is a Brownian motion essentially x of t was a stochastic process the Wiener process which is a non stationary Markov process and there is a fundamental Gaussian solution for the probability density okay. Now if I want to say ok at what stage can I make this process non Markov? 1 way to do this 1 way to do this from this is the following is to say alright this is ok.

This is the probability that you are at the site j at step in nth time step and let us suppose I always start from the origin so for the ease of notation let me write it like this.

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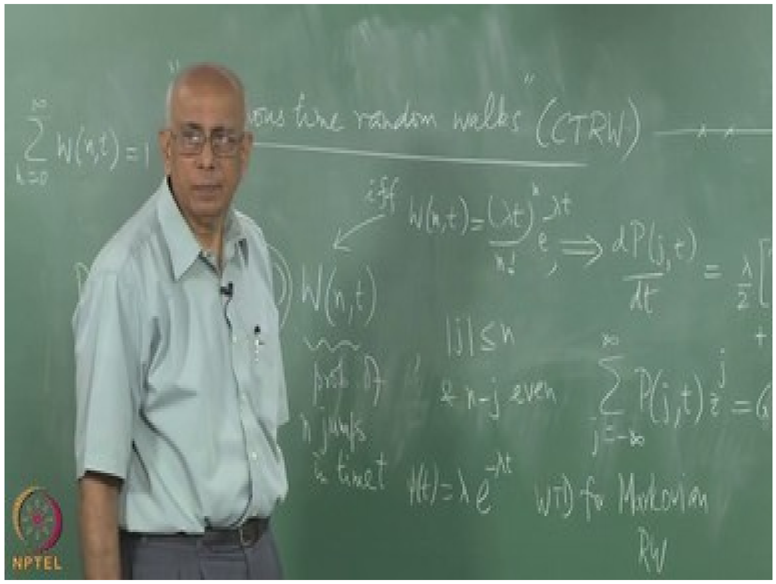

But now I say look in a given interval of time I might have made any number of jumps. I do not have to do this at uniform 1 second intervals or a times fixed time step I might have made it randomly with some random process underlying I might have made any number of jumps. Then this quantity here P of jt this will become a summation out here over n the number of jumps times the probability that let me use a subscript here so that with the idea is clear P n of j let me call this the binomial coefficient that we had.

I want to distinguish it from this function here times the probability that you make n jumps in time t so this is the probability of n jumps or n times in time t or in large steps in time t summed over n from 0 to infinity in principle. So, what is it that we are doing we are saying that the same random walk that I have in discrete space by saying at the end of every second or every time stepped out I flip a coin I moved to the right or left gave me this guy.

So in n steps the probability of being at some point j and this was a binomial coefficient this is the guy which was in n - j over 2 but now I say all right I could have made any number of steps I could have taken in a given time t continuous time because these steps are not being taken randomly. All I have to then do is to say all right the probability of reaching the geometrical point j in n steps is this that is the combinatorial factor with this probability factor that I put in because I can go right or left.

But now that is multiplied by the probability but you take exactly n steps in any given time t and then you sum over all n over all possibilities and you are guaranteed to get the probability that you are going to be at the side j at time t because you could have reached that in any number of steps going back and forth okay. And all the constraints about j less than n etcetera are included here in this guy. Whatever this is this is quite independent.

Now, under what circumstances what kind of W is going to give you that what a Poisson distribution of course because that is the whole point this is a these jumps are happening in an uncorrelated way completely. So, the only way that can happen it can become a Markov process is if the number of the probability that you have n jumps in time t is a Poisson process its Poisson distributed with the same lambda.

So, if this guy if and only if W of n t = lambda t to the power n over n factorial e to the - lambda t then it implies this the Markovian walk any other normalizable distribution of course what you are trying to say is that at any given time t at any given time t you must certainly have this summation n = 0 to infinity W of n = 1 for any positive t. Because any given time t you must make it a 0 jumps or 1 jump or 2 jumps normalized okay.

So, this is a probability distribution in the number n for a given t in the random variable n which takes on non-negative integer values and when that becomes a Poisson with this mean rate lambda it is exactly equivalent to saying that the random walk in continuous time is a Markov process which satisfies this difference differential equation. Therefore any other choice of this probability distribution is going to give you jumps which are correlated to each other.

So, this is not un correlated sequence crosses on the time axis there is some memory. So, any functional form other than the personal form for this W of n and t is going to give you a non Markovian random walk okay and this is called a continuous time random walk. Now what kind of process do we want here and put anything you like but we would like to have something which is a generalization of this Markovian walk.

What characterized this walk actually what characterized it was the fact that in any time interval Dt you had a probability lambda Dt that there was a cross in it and a probability 1 - lambda Dt that there was no cross in it the probability in an infinitesimal time interval of having 2 crosses was of order Dt squared and higher order that was the whole point about this Poisson process right.

Now the basic point is what is the probability waiting time distribution for a jump to happen for an event to happen that is that was a crucial point because everything got generated from that right. So, the question is starting the clock at 0 what is the waiting time distribution such that between t and t + Dt you had a cross. What is that distribution? Well for the Poisson we know the answer what is the probability that till time t nothing happens e to the - lambda t right.

The rate of change of that probability with a - sign is going to be the probability that you have a transition. So, e to the - lambda t is the probability that if you start the clock at 0 till time in t nothing has happened no jumps right. Now you want the probability of a jump that is the holding time waiting time or holding time distribution in Reenroll theory. And this holding time is got to be some function.

So, this thing is also called holding time it is some function Psi of t which must satisfy the following properties first of all it is a distribution probability function so it cannot be negative. It must satisfy 0 to infinity dt Psi of t = 1 means we wait long enough that has to be a jump so it is normalized and it must be the time derivative of this guy e to the - lambda t with a - sign because at the rate at which like the survival probability the - the time derivative of it gave you the first passage time distribution.

Exactly the same way what is - the derivative of this guy lambda e to the - lambda t, so, this means that Psi of t = lambda e to the - lambda t this guy you can see that it immediately satisfies that normalization condition is the waiting time distribution for a Markovian random walk or more generally for a Markovian process.

Now of course once you say that 1 event occurs we saw how to generate the Poisson sequence from the 0 event probability you can find the problem probability that 1 event will occur by multiplying this by lambda Dt integrating and so on and you generate the rest of the Poisson sequence. So, a general statement is that if you give me an arbitrary Psi of t which satisfies this condition non-negative Psi of t which satisfies this integral this normalization condition.

I have a non Markovian walk in general but a very special kind of walk in the sense that it is the same waiting time density for all these events even that need not be true it could be that the waiting time for the first step is different from the waiting time for the second step or the third step and so on. Then I lose translation invariance in time right but if I say look there is a common Psi of t and the waiting time is independent of the step number is drawn from the same common distribution this is a much simpler problem.

It is called a renewal process it is a generalization of a Markov process. So, a non exponential waiting time density this is a density probability density because integrated I get total probability this non exponential waiting time density implies a non Markovian walk. But what is the great advantage of choosing this Psi of t a common Psi of t well what will be the waiting time density W of nt for n such guys what will W of n and t be.

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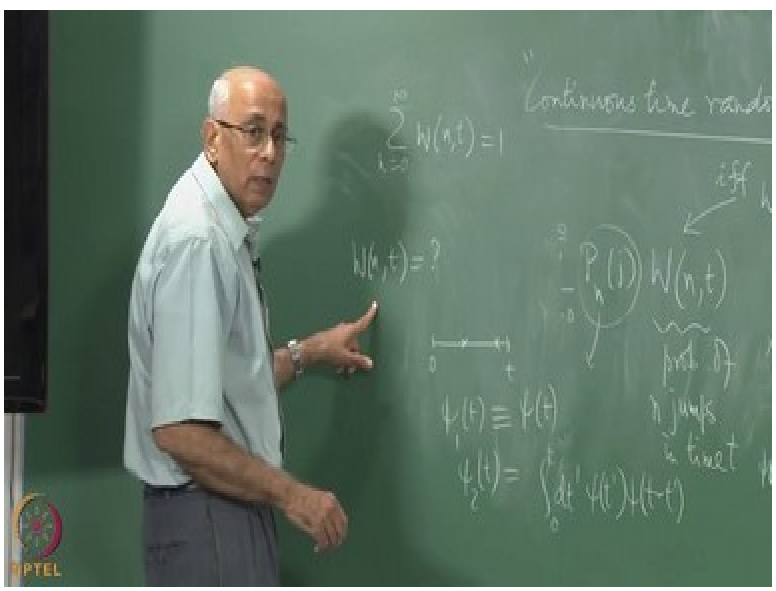

So, now I tell you the waiting time density for a jump is Psi of t some for probability density functions Psi of t what is W of n and t and I also tell you further that it is the same waiting time density for all these guys. So, what will Psi of t n, t? There is translation invariance here. So, what is the what is w of 2, t w of 1, t is Phi of t by definition what is w of 2, t why should it be so it is clear that you want 0 here and in this interval P, I want 2 of these guys.

There could happen anywhere 1 of them happens with some Psi of t that leaves or Psi of t Prime that leaves time t - t prime for the other guy to happen right. So, what will the waiting time density for 2 steps we Psi 1 of t is identically Psi of t what is Psi 2 of t, it is got to be = 0 to t st prime Psi of t prime Psi of t - t prime because that is the time left for the second guy to happen and we already said that the waiting time density for 1 fellow to happen is precisely Psi of t.

So, this is the time available for it and what is this in the form of a convolution right. So, you immediately see that we do not have a simple expression for this but we do have 1 for w tilde of n , s it is Psi tilde of s to the power n that is great because this now tells you we work in a + space all the time we work in terms of Laplace transforms and finally we will take a inverse transform. So, it is really telling you that we can solve this problem.

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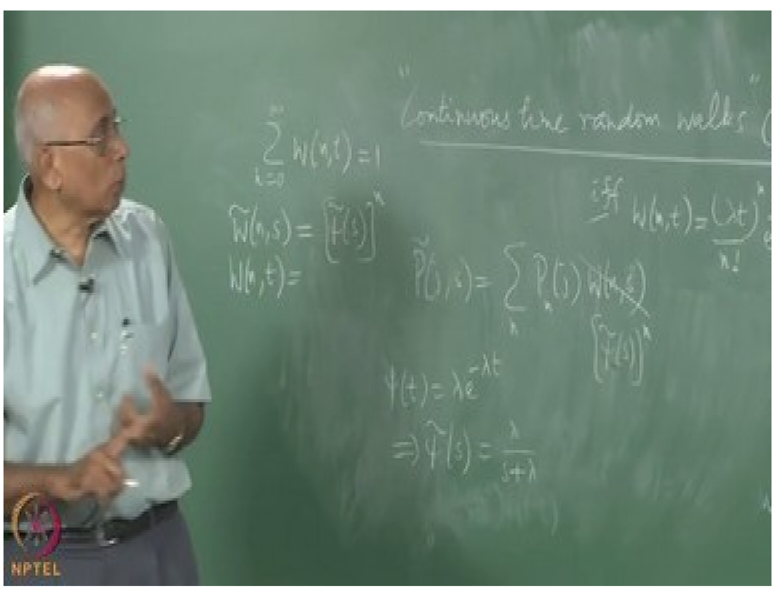

Because now look at what is going to happen to my P of j , t there is a P of j , t and this was = P n j and then there was a W of n , t summed over n but I take the Laplace transform of this I get this out here and that gives me this and I cancel this and write this as Psi tilde of s to the power n that is like the generating function for this fellow with z replaced by this function. So, if I can write the generating function and substitute Psi tilde of s then in principle I can invert the transform and find this P of j , t without going through any master equations or anything like that at all.

So, we will do this tomorrow okay so you see the strategy and the idea that how you generalize from a Markov process of course you check all the time by putting Psi of t is lambda e to the - lambda t, you check whether it is going to work or not in the other case what is what Psi tilde of s for the Markov case when it is lambda e to the - lambda t, of course remember Psi tilde of 0 must be = 1 because of this.

You can ask what is the mean waiting time that is of course integral t Psi of t Dt which is the first derivative of cited office with - at s = 0 so once you actually give me this I do not even go back to time if you give me this I work entirely in terms of Laplace transforms I can find any moment or whatever I like so the thing to do now is to generalize this to higher dimensions I will show you how to solve this in general lattice in arbitrary dimensions by exploiting the fact that for a renewal process.

All that happens is that this probability in the Laplace space the probability of n jumps becomes a power of some transform of some waiting time density okay and then we can determine from that that is enough to determine what is going to be the long time behavior of this walk long time in t space is = s going to 0 in Laplace transform space. So, we are going to use these asymptotic theorems to discover what is going to happen at long times. If it is diffusive or not sub diffusive or whatever so this is the general strategy we really do this.