Physical Applications of Stochastic Processes
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Lecture-25
First Passage and Recurrence in Markov Chains

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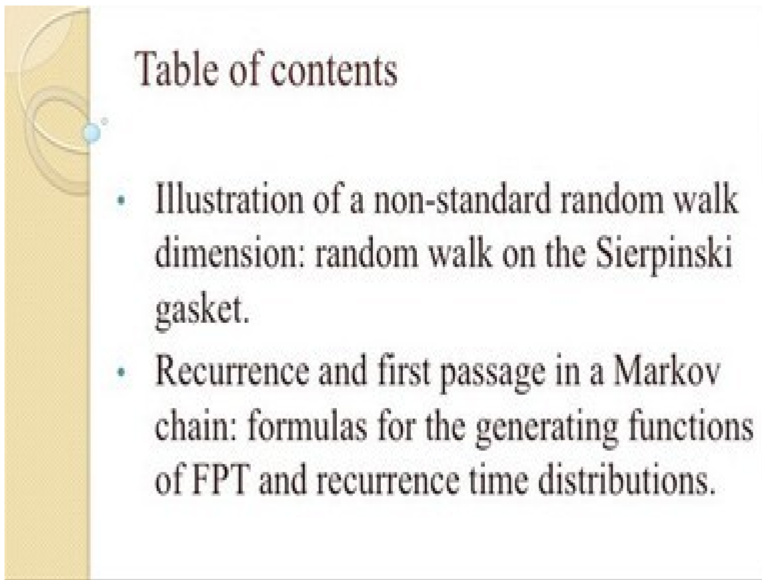

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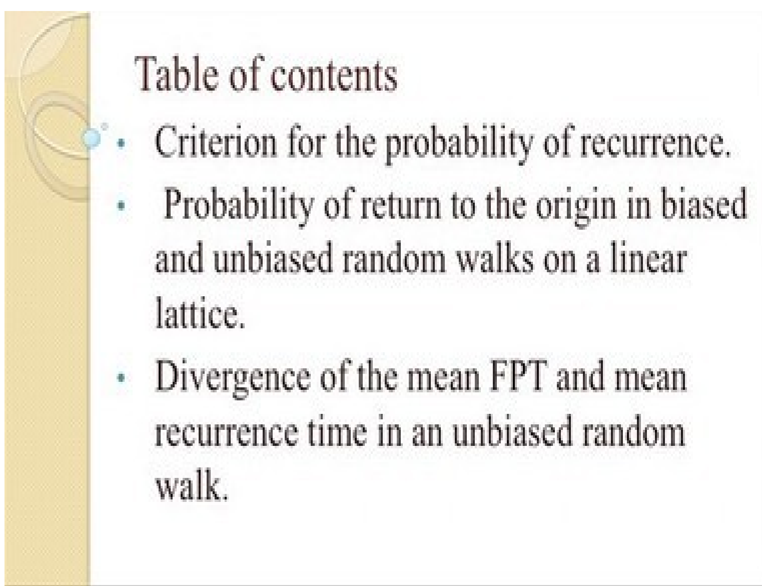

We were talking about the idea of a random walk dimension and I had started mentioning that on structures like fractals the walk dimension could be other than two and let me show you with a simple example how this follows and after that we will go back a little bit and talk about the problem of recurrence in Markov chains in general with application to random box.

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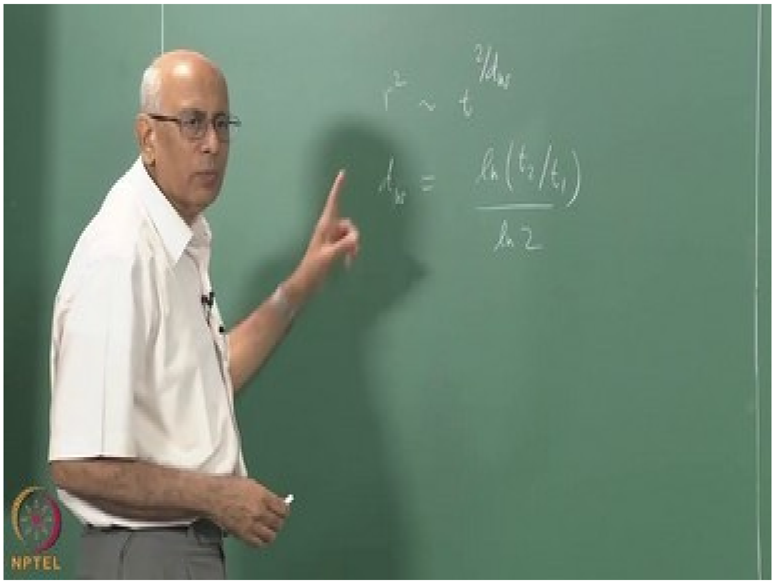

Recall that I mentioned that if you say the square of the distance goes like the time 2 over dw where either this or that could be a random variable and we mean a suitable average could say for a given time the mean square distance that you cover or you could look at for a given distance the time it takes the mean time it takes to cover this distance either way. When you have a relation of this kind a scaling relation then one says that the random walk dimension of that particular process is dw.

And for normal diffusion it turns out dw is 2 because we know r squared goes like t it scales like t but the formal expression for this can be found in the following way. Suppose you have two distances say one of unit distance and one which is double that then if the corresponding times are t1 and t2 it is clear that dw then is = log t2 over t1 divided by log 2. So, how long does it take to cover twice the distance? So, t1 is a time taken to cover some distance and t2 is the distance time taken to cover twice the distance.

You take the ratio of these two take the log and divide by log 2 and that gives you dw, so this is what we would like to do. Now look at the following problem on a hierarchical structure and the way we do this is defining a procedure for constructing a deterministic fractal.

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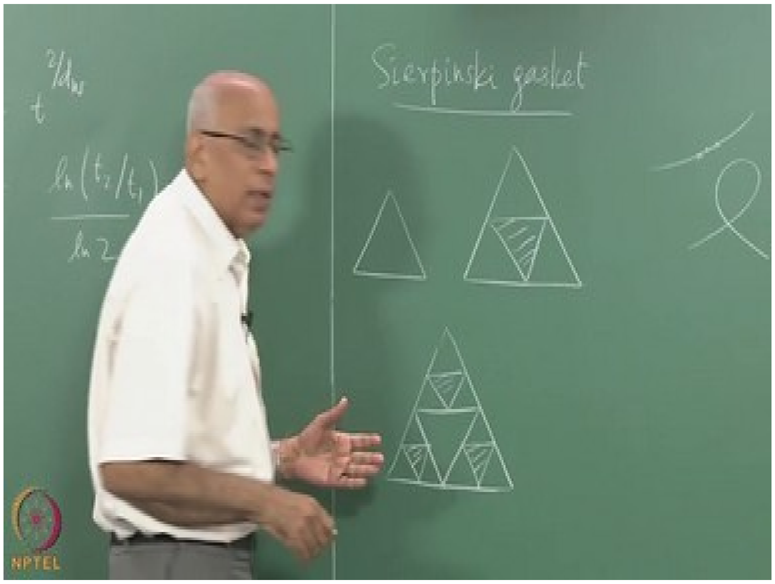

And this fractal is called the Sierpinski gasket in two dimensions and there are generalizations in higher dimensions. And so on and the way it is constructed is as follows and I will tell you what the reason for constructing this in the first place was and has to do with the following fact. If you have a uniform curve it is got at any point 2 nearest neighbors at a given distance from this point but if the curve has a self intersection then at this point there are 4 such nearest neighbors.

And the question was whether you could find a curve or some construction where every point has 4 nearest neighbors in this specific sense. And the idea of Sierpinski was the following you start with an equilateral triangle does not have to be equilateral but it is for simplicity equilateral triangle and then double it in size and at the same time remove the middle third. So, then the figure at the next stage I am not able to double it and draw it to scale.

But the next stage you remove this open set then at the next stage you do that for each of the three triangles that remains. So, in the next generation you have a figure which looks like this in which that is missing, this is missing and this is missing. So, you have a structure with more and more holes in it of all sizes but what we are interested in is the following. We are interested in a graph essentially a graph of some kind.

So, I will focus only on the nodes of whatever remains and say that is the Sierpinski graph in two dimensions and there are generalizations to higher dimensions also. So, basically what is happening is you start with this simplex to this triangle and you break it up into three simplexes remove the 4th one here and that gets broken up into 9 of them and so on. And each time there is also a scaling up so that the figure looks bigger and bigger or scaling down we do not care whichever way.

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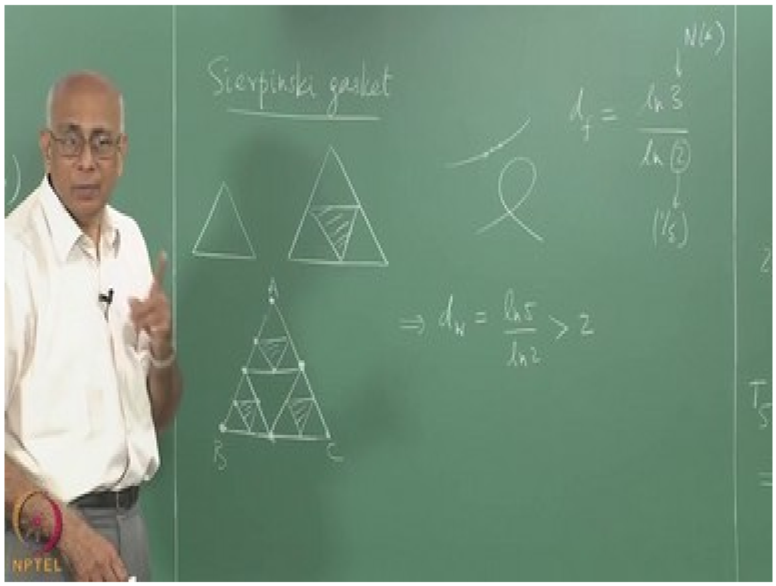

Now one way of constructing the graph is as follows it is to say I start with a triangle of some size and I decorate each bond each edge of this triangle with a new site so in the next stage so in the first stage you have three sites call them 1, 2, 3 and in the next stage you decorate the midpoints of these sides with new sites, so now we have 1, 2, 3, 4, 5, 6 and then in the next stage you have more sites and 15 sites and so on and so forth.

It is easy to see that if you call this generation 0 generation 1, 2, 3 etcetera that the number of sites goes on increasing exponentially with N go like 3 to the N essentially. So, the question is what happens if you do a random walk on this you can blow up the size of the triangle appropriately to make sure that the size of each triangle the smallest angle is always unity or something like that okay.

So, the question asked is to go twice as far on this structure assuming that this is 1 that is 1 etc etcetera how long does it take on the average. Now it does not matter where you put this is a completely deterministic graph because given the N - 1 generation you can find the Nth generation by decorating the midpoint of each of these sites by trying by with a point and then adding all those nodes to the graph okay.

So, the question is if you start here a random Walker and there is always jumps only 2 nearest neighbors at any stage how long does it take to go unit distance. Now in the original graph the sites 2 and 3 are unit distance away. So, it is clear that this walker if he jumps at the end of every time step with equal probability to all the nearest neighbors then these two points if they are traps if you reach us this or that you've traveled unit distance that is the end of it.

So, the mean time taken t1 is = 1 and that is the end of it, to go to this trap or this trap. Now ask what happens in the next generation. So, again you start at site 1 but you would like to go twice the distance so you would like to go to either 4 or 5 that is where the traps are. So, the question is it scaled up and now to go double that instance how long does it take. So, the assumption the idea is to ask you have a random walker on this graph, who jumps from one site to nearest neighbor site with equal probability.

If the site has two nearest neighbors with probability half but the generic site apart from these sites apart from this and this you notice that every one of these sites has 4 nearest neighbors always so the coordination number is 4 in general except for those corner sites. So, now I ask I start here and with equal probability the walker jumps to nearest neighbor sites what is the mean time for the walker to hit either 4 or 6 and if you hit either 4 or 6 the walk is over okay.

Now we can do the same thing exactly as we did for the linear lattice except that you have to now allow for the fact that it is possible that the walker will get stuck in this loop here and keep going for a long time or go back to 1 etcetera. And then eventually hit 6. So, the question is what is the mean time? We saw that on a linear lattice it was to go twice as far it was just 4 times as long.

Because on a lattice with j sites you start at 0 to go to j for the first time it took time j squared and therefore to go to a site 2j it will take for j squared and that was it but what is it here well. Let us write these equations down and see what happens in this case now T1 is = with probability half the first jump will take you either to 2 or to 3. So, it is 1 half T2 + 1 because you used up a step at a time step in doing that + 1 half T3 + 1 and that is it that is what the mean time to hit the traps from T1 starting from 1 is okay.

But what about T2 and T3 it is clear by symmetry since the traps are situated symmetrically you can use a little bit of symmetry here I do not write the equations for everything T2 must be = T3 because the two traps you are completely symmetrical. So, let us call this T2 and this will imply T1 = T2 + 1, what is T2 itself, what would be the recursion relation well where are the possible places to which t from two you can jump to 1, 3, 5 and 4 if you jump back to 1 you wasted a time step.

But you are back to T1 right so it is clear this is 1 4th T1 + the next near the other nearest neighbors are T3 + T5 + T4 and each time you used up a step. So, this is + 1 quarter times 4 which is 1 once again but T3 is the same as T2 so we can kill this and write this as T2, T5 is a distinct point we cannot do anything about that and what is T4 0 by definition because you are at the trap and that is it so this is gone.

And what is T5 we do not need to write a separate equation for T3 because it is the same as T2 T5 is a distinct sight and what is T5 with probability quarter you go T2 or T3 but these two are = each other so that gives you a half times T2 I have included it T3 contribution in there and then what about these T4 and T6 are 0 and you go there with probability half quarter + quarter right + 1 that is it. Why yeah so it is half of T2 + 1 + half and the half's add up and give you a 1.

So, the 1 is conservation of probability simply saying that one time step has been used up over all the nearest neighbors wherever you jump right so we have three equations for three unknowns T1 T2 and T5 then you can solve the set of equations fairly trivially and without any difficulty you establish that T1 is in fact 5. So, on this graph on this graph if I start at the apex A and I want to go to either the point B or C how long will it take well we saw that on this graph.

If you start with the apex and you want to go to this or this it takes you time 5 relative to 1 here in this graph how long will it take to go from a to either B or C the mean time to go twice as far it takes 5 times as long right this is 4 steps away 1, 2, 3 and 4 steps away 25. So, the time unit will be 25 you have to work out the full random walk problem on this over all possible excursions of this random walker.

So, there are lots of loops where the walker can get stuck for a very long time and eventually you hit either this point or that point okay and these are all connected. So, there is no doubt that this walk will eventually hit either B or C because it is Ergodic but the mean time to do so there is not going to be 25, each time it is multiplied by 5. So, what is d walk on this structure it says T2 is 5 times T1 right so this is log 5 over log 2 which is greater than 2.

So, it immediately says that r squared on this structure is like 2 over d walk which is log 5 by log 2 and this is greater than 1 greater than 2 it immediately says that r square goes like a power of T which is less than 1 not 1 so the process is sub diffusive. You can ask her is this an artifact of where we started we are not started at such an isometric point what would happen well you can start here.

You can start on this graph at this point for instance and say what are the nearest neighbor points that is this, this, this or this so you can ask how long does it take on the average to go from here to this, this, this or this and the answer of course is 1 the first step you are off. Then you want to go 2 points twice as far from this point in other words you want to go either here, here, here or here it is just scaled up twice and the answer will turn out to be 5 once again.

So, I urge you to work it out on this graph because now you have interesting points you have this point or that point and there could be lots of endings here without hitting these boundary points. But when you work it out it will be exactly the same as that you will again discover that the walk dimension here is log 5 over log 4. By the way what is the fractal dimension of this structure? What is the fractal going to be?

Remember when you scale things up or scale it down break a unit interval into pieces of size epsilon by multiplied by epsilon then it is clear that exactly as we saw in the triangular triadic Koch curve what is happening here is that one triangle is being replaced by 1, 2, 3 triangles and what is the size of each of these, I am putting a point here point here pointing in decorating so the size is a half therefore de fractal = log 3 there are that is N of epsilon over log 2 and this is 1 over epsilon, epsilon is half.

So, the fractal dimension once again as you would expect is between 1 and 2 it is greater than 1 less than 2 because you are no Euclidean plane you could play this game not with triangles but with simplex is for example Tetrahedra in 3 dimensional space. So, in 3dimensional Euclidean space you embed Tetrahedra. So, you take a tetrahedron and at each apex reduce the size by half and at each apex you place a tetrahedron this I do not want to draw it here.

And then you made this and so on and so forth okay and now what would be the fractal dimension of that pardon me, it would be well the fractal dimension there if you are in d dimensions if you want d dimensions its log d + 1 over log2 - okay whereas in 2 dimensions it was log 3 over log 2. So, if your d Euclidean dimensions where d is greater than = 2 the fractal dimension turns out to be log d + 1 over log 2.

And what would be the walk dimension that requires a little harder work you have to find out how many nearest neighbor the neighbors there are here there are 2 nearest neighbors in this case but in that graph the more complicated graph in a tetrahedron you would end up having a larger number of the nearest neighbors and you have to include those things and it turns out that the walk dimensionality is log d + 3 over log 2.

That is why it became d log 5 here, so for the Sierpinski fractal in d Euclidean dimensions you can write down what is the fractal dimension you and I do not want the walk dimension is and so on. But the essential point is that in these structures with lots of nooks and crannies it takes longer to go a given distance than it does on a regular graph and that is why the motion is sub diffusive.

You cannot describe even though this is a Markov process you cannot write down the usual kind of Gaussian solutions for the diffusion equation in the continuum approximation nor is this going to be nor is this process going to lead to the conventional kind of diffusion in the continuum limit it is going to lead to a very peculiar sub diffusive behavior. We will talk a little bit about it when I talk about non-Markovian diffusion in terms of what are called continuous type random box.

We will come back to this we will come back this graph we will see what is needed in order to understand graph of this kind completely how to work an arbitrary random walk on this kind of graph. Let me go now to a topic which I should have covered when I talked about Markov chains but which I mentioned off and on and that had to do with recurrence in Markov chains. When does the system come back to its starting point?

In particular we have been worried about the linear lattice and we have been worried about random box on a lattice and I have said things like the probability of a return to the origin is not 1 if you have a bias in the random walk it is = 1 when you have no bias at all we need to establish this. I already have set the stage for it by writing this renewal equation for the first by system density. But let us do it in the language of discrete time discrete Markov chains. So, that you have clear understanding of where this comes from okay.

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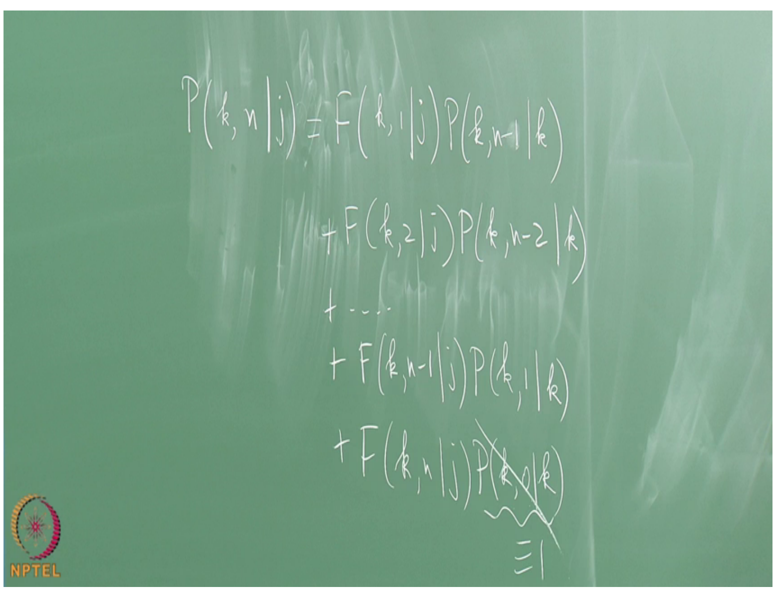

So, let us call it recurrence and first passage in Markov chains, so I have not mind a Markov chain in which the states are labeled by some integer j and the time is discrete labeled by n for instance and we have been talking about the conditional probability P that you hit for instance a state k at time n given that you started at state j at time 0 will consider stationary Markov chains. So, the origin can be shifted as you please and time is measured in discrete units n is the label for time k and j are state labels.

So, this is the probability that if you started at t equal at 0 time in the state j you are at the state k at time in by a Markov of sequence of transitions. So, for compactness let me write it in this form. Now it is clear that if I start with j I hit the state k to go to the state at time n I must have hit the state k at some time in between 0 and n either 1 or 2 or 3 for the first time okay.

And if I hit the thing for the first time at a particular time that is a unique event in the sense that hitting it at time 3 is excludes hitting it at time 2 for the first time or time 1 not so for P itself but for the time I hit it the first time that is a unique event always. So, I can decompose this probability this I can decompose it by asking when do, I hit this state for the first time and let us call the probability F that I hit k for the first time starting from j.

So, this is k 1 j and I hit it in the first time at time 1 let us say and after that this is multiplied by P k n - 1 okay after that I started 1 because it is a Markov process it renews itself. And then in the remaining n - 1 time units I wander around I come back to k this is a contribution to this probability from the event in which I started j and hit k for the first time. But to this I must add F of k2 P of k n - 2 because these two are not mutually exclusive.

So what I am doing is to decompose this in terms of mutually exclusive events based on the time of first hitting that final state + of course dot, dot + F of k n - 1 j, so, I hit it for the first time at time n - 1 and after that all I have to do is to remain there okay. And there is one more possibility which is that I hit k for the first time at time n itself and then it is finished right. And that exhausts all the possibilities.

If you like I can put P of k 0 k but by definition this is identically = 1 by definition the probability that I am in k if I start with k at time 0 is of course by definition 1 so I do not need. This now this is the discrete analogue of the renewal equation that I wrote down earlier right and now of course the obvious thing to do is to define a generating function.

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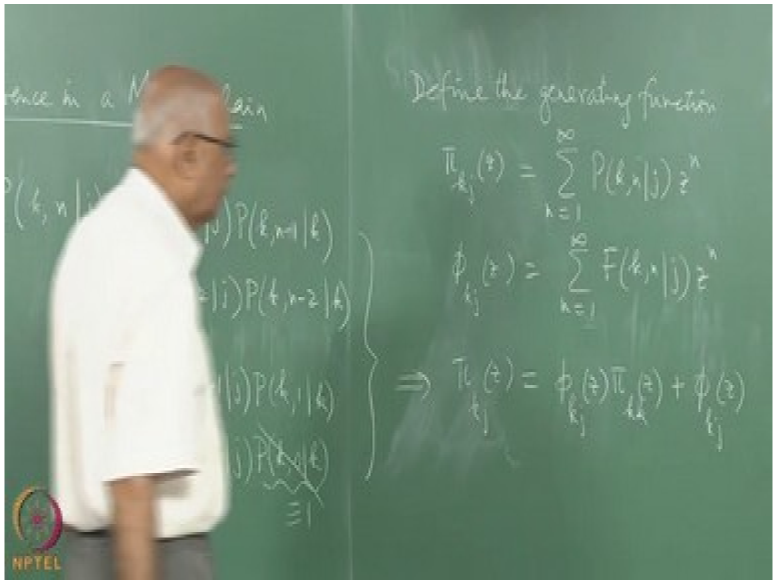

So, if I define the generating function for each of these guys so let us define a generating function for this for instance this is = Pi kj of z that is = a summation from n = 1 to infinity P of k and j is there to the power n. And similarly let us define 1 for this first passage time first passage probability Phi k j z = summation n = 1 to infinity F of k from 1 to infinity I sum and multiply by z to the n and some and that is the usual way to define a generating function.

Remember that the notation is such that this is the initial state and that is the final state I have retained that same order here so this always stands for the final state and that stands for the initial state. Then this relation here will imply the following relation Pi kj of z is = F kj of z, I am sorry Phi Phi kt times Pi k k of z + Phi this fellow does not have a P here. So, there is only an F you are summing from 1 to infinity.

Whereas in the continuum I took Laplace transforms and use the convolution theorem because time was continuous here time is discrete so I just use the generating function.

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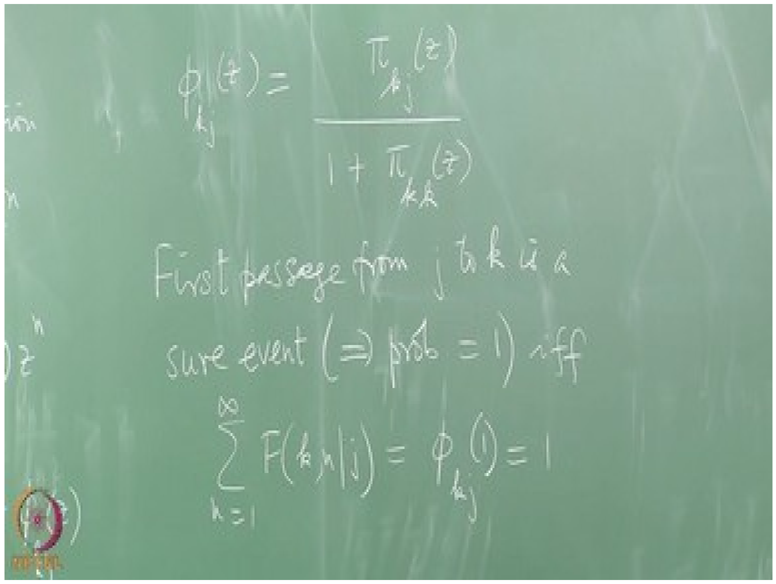

And that gives us an expression which says Phi kj of z is = Pi divided by 1 + Pi kk a relation between the generating functions. So, this means that if you solve the Markov chain equation for instance for this conditional probability and find its generating function then you found the generating function for the first passage from j to k you found the distribution itself. Now when can you assert that first passage from j to k is a sure event?

Remember that these are all mutually exclusive events not so here not so here at all different ends these are not mutually exclusive events there will be lots of overlaps but here the statement that you go from j to k for the first time at time n is distinct from the statement that you go for the first time at time n prime. So, the different ends are all distinct events they are mutually exclusive and I use that in writing this equation okay.

So, when can I assert that first passage from j to k is a sure event, I do not care when but it is a sure event? This means the sum over all these F's over n must be = 1 so at some time or the other it hits it for the first time right. So, first passage from j to k is a sure event implies probability = 1 if and only if summation n = 1 to infinity f of k n j = Phi k j of 1, so, you put z = 1 and you have to get 1 then it is a sure event otherwise it is not a proper random variable.

So, this n is not a proper random variable it is not normalized to unity but if this is true then it is a sure event okay. Now in these Markov chains in which every system every state is connected to every other system and there are no traps or cyclic subsystems and so on it will always happen that you wait long enough you sum over all n it will definitely hit it okay. So, we will come back to that. But now let us ask what happens for recurrence?

I want to start with an initial state and I want to say do I ever come back to this state or not. So, recurrence would imply that this k is = j.

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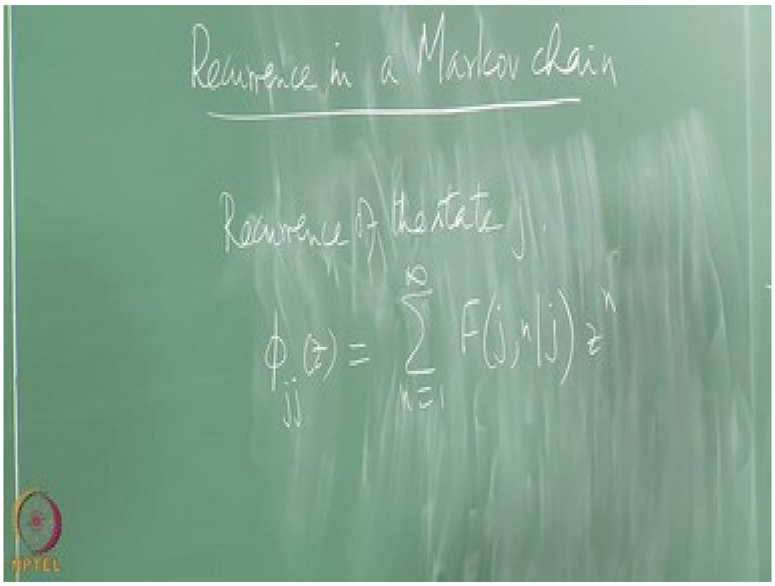

So, recurrence of the state j recurrence of the distribution now the moment generating function is Phi j j of z this is = a summation n = 1 to infinity f of j n j and this should be unity if you put z equal z = 1 because this itself is z to the power n here and we know that Phi jj of z = Pi jj of z divided by 1 + Pi jj z these are all probabilities your summed over probabilities multiplied by z to the n. So, if z is x which is a positive number for instance.

These guys are all positive numbers and this clearly looks like a fraction here but if its recurrence is a sure event Phi jj of 1 must be = 1.

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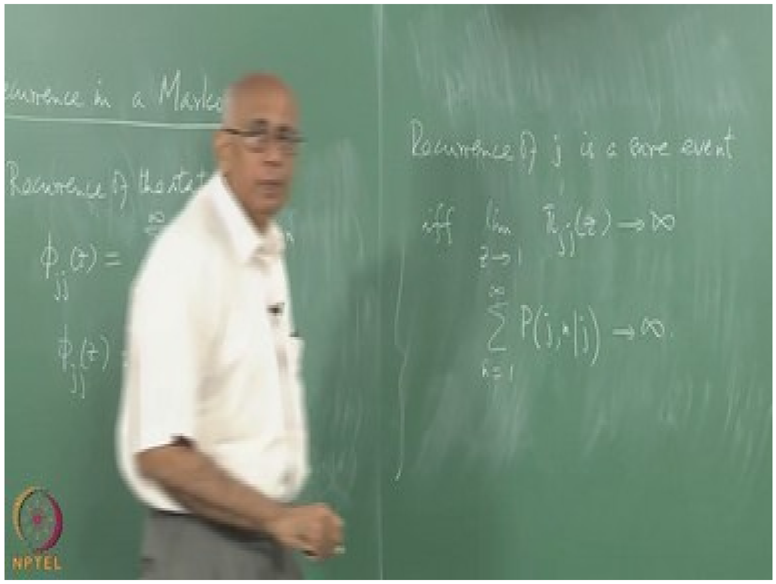

So, this means recurrence of j is a sure event if and only if how can this fellow be = 1 only if Pi jj of 1 tends to infinity, if a diverges that is the only possibility if and only if limit z tends to 1 by jj of z tends to infinity it should diverge in other words summation n = 1 to infinity P j n j should tend to infinity only then will it happen. Let us check this out let us put let us say let the Markov chain be a random walk on a linear lattice it is just completely translation invariant process infinite linear lattice. So, we can call any site the site j we will call it the origin okay.

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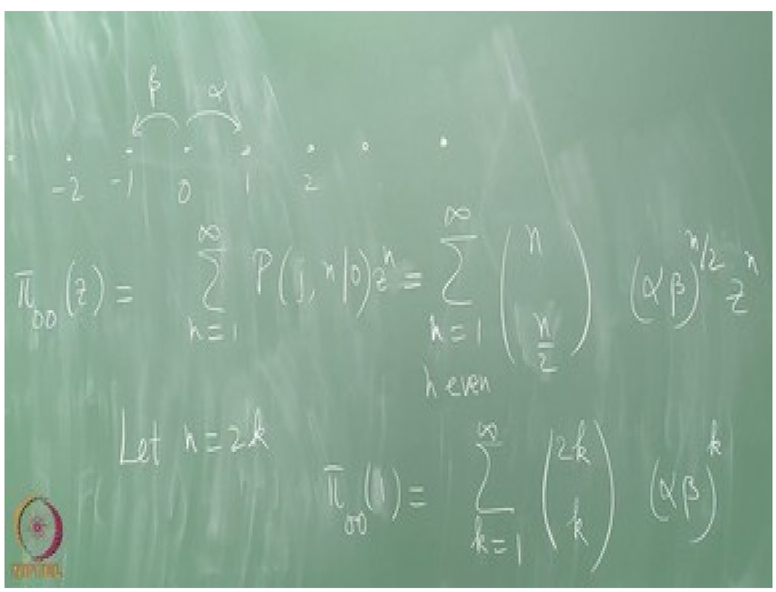

So, let us look at return to the origin in a linear lattice. So, you have all these sites etc this is the origin this is – 1, 1, – 2, 2 and so on. And let us look at a biased walk the more general case so the probability of a jump to the right is alpha and the probability of a jump to the left is beta where alpha + beta is 1 yeah this is the first yes starting with j at t = 0 the question is when do I come back to j for the first time.

Now if I remain at that point then at time 1 I am still there so that is F1, n = 1, n = 0 is a starting point so, n run 0, 1, 2, 3 etcetera okay. So, I started an origin at the end of every second I jump now I do not know one second I may not have jumped in that case I am still there and that contributes that would correspond to F j 1 j. Now if I want to look at 2 j 2 and I come back I go out on the first step and come back in the second step.

So, that would be j n = 2 it is clear that on this lattice when I look at the lattice case it is clear that I can come back to 0 only on an even step I cannot come back and an odd step. So, that should emerge that F would not contribute at all F would be 0 if n is 3 and j is 0 it would not be able to come back and 3 steps either comes back in 2 steps or 4 steps or 6 steps and so on and so forth okay. So, notice that it is a good point notice I am summing from 1 to infinity because that is the meaning of recurrence, I have to let time go on okay.

Similarly for P, I have kept it from 1 to infinity and we will see the significance of that okay. So, we need to see if this is true and under what conditions it is true if it is finite if this series is finite then it is clear that in this ratio this is a finite number this is a finite number and this is not = 1 if you have F of z = 1. On the other hand if it diverges we can be sure that it is going to be a recurrent event. So, all we need to do is to compute this number and ask what it is.

So, let us look at what Pi jj is for the linear for the biased random walk 5 Pi is 0 0 without loss of generality I take j = 0 here in each side at all of z this is = what it is = a summation n = 1 to infinity P of 0 n 0, it is the probability that at time step n, I am at the origin, having started from the origin at time 0 right. So, this is = a summation n = 1 to infinity n binomial sum symbol and then you had n - J over 2 to go to site j if you are going to end up at site j.

But that is going to be site 0 so it is n over 2 on this side and then you have alpha to the power n + j over 2 and beta to the power n - j over 2 to hit the site j at time n but we want to hit the site 0 once again. So, this is = alpha beta to the n over 2 moreover this probability this combinatorial thing the binomial distribution is true only if the point j is starting from the origin the point j has the same parity as n so n - j must be even.

So, what are the conditions for if you had Pj n and you wrote n in + j over 2 alpha to the n + k over 2 beta to the n - c over 2 the statement is that mod j less than = n because you cannot go beyond that in n steps and n - j even but if you set j = 0 it only says n is greater than = 0 which we know and n is even and that is the answer there. But this summation runs from 1 to infinity this thing runs from 1 to infinity right.

So, what does it are; there is also a z to the power whatever so z to the power and n must be even. So, let us put let n = 2k and k runs from 1 to 3 etcetera because n runs here from 2, 3, 2, 4, 6 etc so it says Pi 0 0 of z = summation k = 1 to infinity 2k and then there is a k here and let us put you know of 1 to see what happens, alpha beta to the power k and that is all we have to sum this series and find out when it converges and if and when it diverges right.

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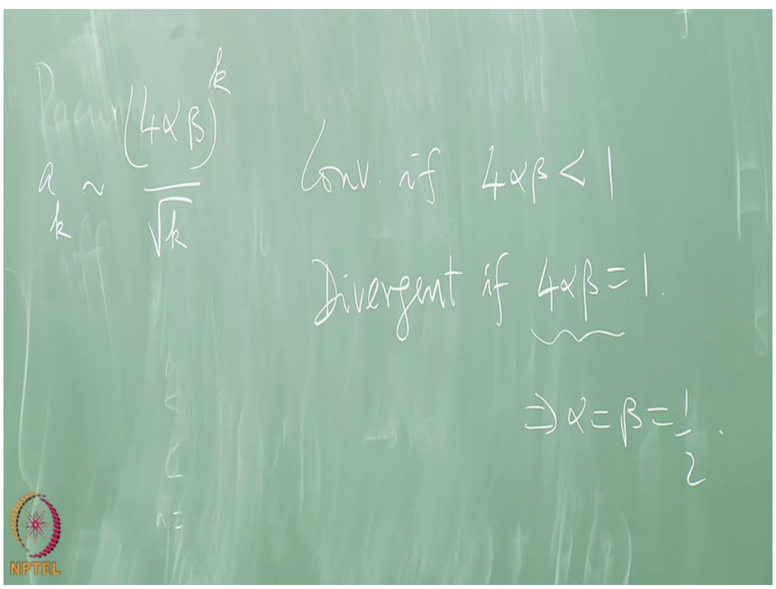

How would you do that well a simple way to do that is to write down what these factorials are. So, what does the 2k k alpha beta to the power k, what does this fellow do well it is clear that this is = 2k factorial over k factorial squared and then there is an alpha beta to the power k and we need to know what the large k behavior of this coefficient is and then do a ratio test or something like that right. But a large k you do sterling, so this goes to 2 k to the power 2k.

So, let us write it as 2k to the 2k k to the power 2k e to the - 2k and then square root of 2 Pi times whatever it is so square root of 4 Pi k that sterling use there and in the denominator you have k to the k e to the - k square root of 2 Pi k squared where there are two of them and then there is an alpha beta to the power k and as you can see k to the 2k cancels e to the - 2k cancels you have a 1 over square root of k from these two and then you have a 2 to the 2k if I combine it here it is 4 alpha beta to the k right.

So, this coefficient so the term like any typical term goes like the k term for example goes like 4 alpha beta to the k over square root of k this is what the kth term goes like finally and I do a ratio test take the n + 1 to the nth term and then of course square root of k + well k over k + 1 tends to 1 so the power of k does not do anything and you are left with just for alpha beta right. So, it says convergent if 4 alpha beta is less than 1 and divergent certainly divergent = 1.

Divergent because this will go away and then you just have a 1 over square root of k which if you sum over k becomes infinity it will diverge like the square root of the cutoff. Now what is the largest value that 4 alpha beta can have 1 because beta is 1 – alpha. So, the largest value it can have is 1, so this will imply alpha = beta = half unbiased. So, if it is unbiased you are guaranteed that the return to the origin is guaranteed.

But if it is biased either to the right or left if alpha is bigger than beta and less than beta it does not matter 4 alpha beta is less than 1 and then the series converges so the probability of a return to the origin is less than 1. We should like to find out what exactly it is like you know in that case what is it which we will do in a minute?

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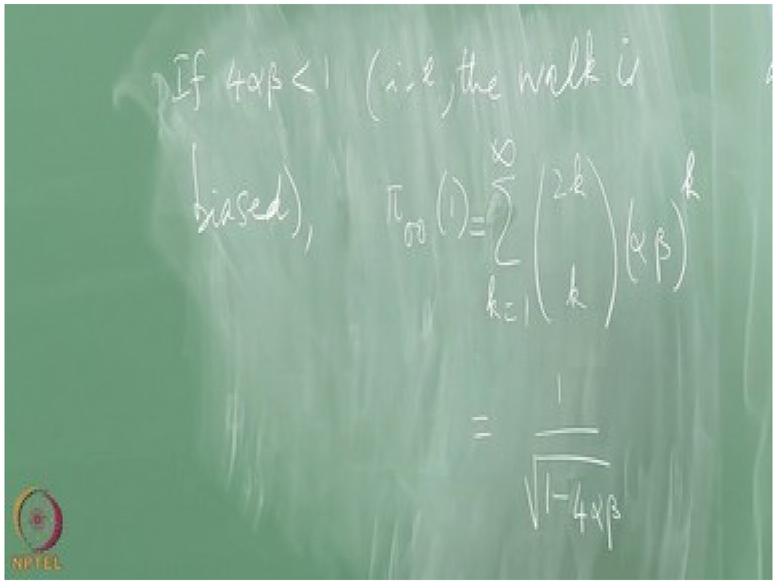

So, let us see so let us do the following let us first look at this case where the return to the origin is not probable with probability 1 probability less than 1 find out exactly what this probability is and then we will come to the case where we have a return to the origin which is sure with probability 1 and try to find out what is the mean time that it takes to do so. So, let us first consider 4 alpha beta less than 1 it says if less than 1 ie the walk is biased.

Then we still need to know what is Pi naught of z naught naught of 1 rather of 1 this is = this guy 2kk alpha beta to the k, summation k = 1 to infinity does anyone remember what the series is it is a binomial series we can play around with this but it is a binomial series I will just write the answer down this turns out to be = 1 over square root of 1 - 4 alpha beta. There are several ways of doing this you can remember it in many, many different ways.

It is useful to remember what the general binomial theorem is in a form which is easy to apply to these series because what happens when you have negative index negative in non integer index etcetera is that a lot of - signs cancel out etc.

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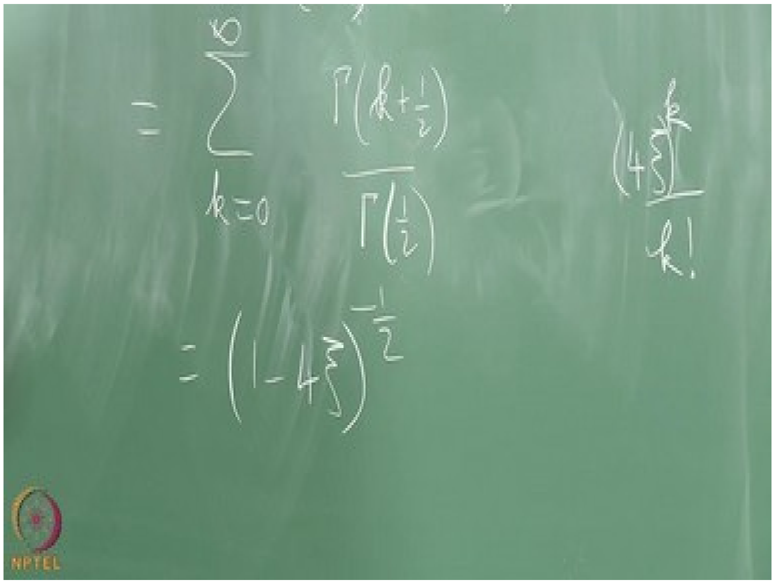

But the way I remember it is slightly different than this what one way to remember it is like this write this as summation consider k = 0 to infinity by the way the k = 0 term has been left out here so this is this - 1 but let me show you how this is remembered oh wait how I remember it. So, if you have 2kk and then something let us call it is Zi to the power k thing like this then this is = a summation k = 0 to infinity 2k factorial that is gamma of 2k + 1.

So, this is gamma of 2k + 1 over gamma of k + 1 that is k factorial another again another k factorial and then Zi to the k. So, let us write this I to the k over k separately that do not separate. Now I remember what is called the duplication formula for the gamma function gamma 2z can be written as gamma of z times gamma of z + half times a 2 to the power 2z – 1.

Which is 2 to the 2k over gamma of half which is root Pi gamma of k + half gamma of k + 1 over gamma of k + 1 Zi to the k over k factorial and this factor cancels out. So, you have 2 to the 2k so that becomes 4 Zi to the k that is right. So, it is 4 Zi to the k and then there is a gamma of k + half over gamma of and this is a binomial series. It is a binomial series for 1 - 4 Zi to the power – half. So, if you had an alpha here some constant alpha it is 1 - this whatever is the argument to the power – alpha.

So, that is an easy way to remember the binomial series and if you apply that here it is 1 4th I put a 4 here and then 1 - because there is a 2k here it gave me a half it is get 1 over square root u – 1. So, that is Pi naught naught of 1.

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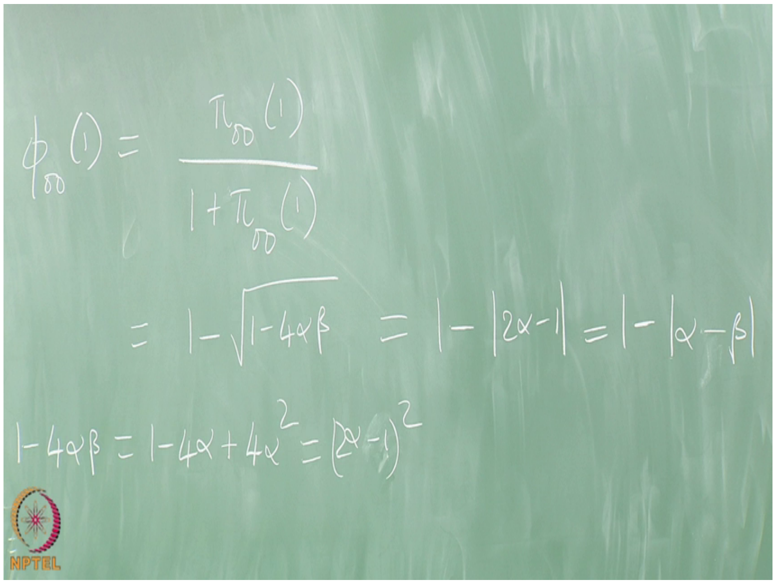

Therefore we know what Phi naught naught of 1 is so it says Phi 0 0 of 1 = Pi 0 0 1 over 1 + Pi 0 0 this is = 1 - the square root divided by the square root and it cancels out so this is = 1 - square root of right. But what is this fellow we can simplify it 1-4 alpha beta we go 1-4 alpha times 1 – alpha, so it is + 4 alpha squared = 2 alpha + 1 - 1 the whole square. so, this is = 1 - the square root of 2 alpha - 1 squared what is that mod 2 alpha - 1 mod which is one - mod alpha - beta it is a useful way to remove it.

So, if you got a bias and alpha is not = beta then the total probability of a return to the origin is this number summed over all walks and that is a number between 0 and 1 but if alpha is = beta the probability is 1. So, this right away tells us that summing over all possible walks if you have a bias in either direction because you have got an infinite lattice on both sides the system is less than 1 probability of returning depending on how biased it is.

Of course as you can see if either alpha or beta is 0 so that it is an unidirectional motion you are never going to come back the probability is 0. Now the question is when it does return when alpha is = beta = half what then is the actual distribution of this return time.

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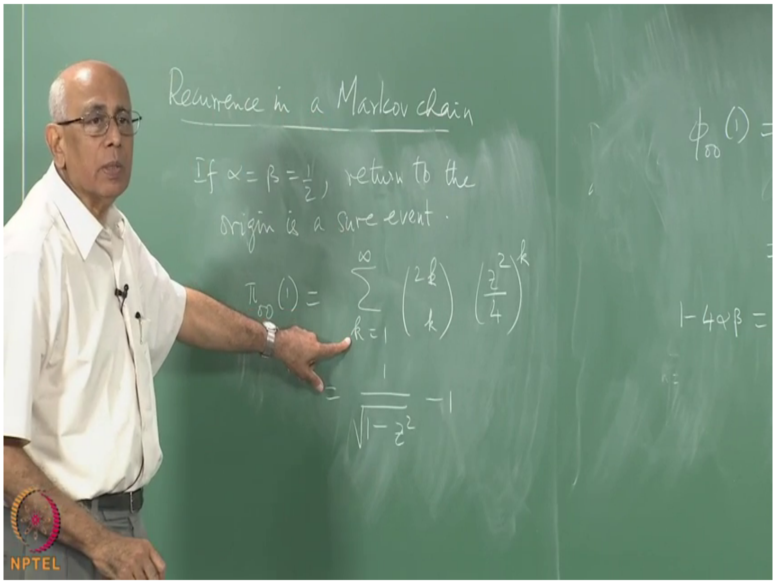

Now what we have to do is to compute this number so if alpha = beta = half recurrence or return to the origin is a sure event but we need to find out what is the distribution of the time. As I said you can return to the origin only on an even time step. So, it is clear that the probability distribution will our support only when n = 2, 4, 6 etcetera. So, let us see what this probability distribution actually is now we need to compute these numbers.

So, let us compute Pi 0 0 of 1 this is = summation k = 1 to infinity again 2k and a k that is this alpha = beta = 1 = half in this case so there was a one fourth here and inside you also had a z to the power n but I put n = 2k, so this says z squared over 4 to the power k and by the rule which we just had this is = 1 over square root of 1 - 4 times this guy which is z squared - 1 because I had left out the k = 0 term here.

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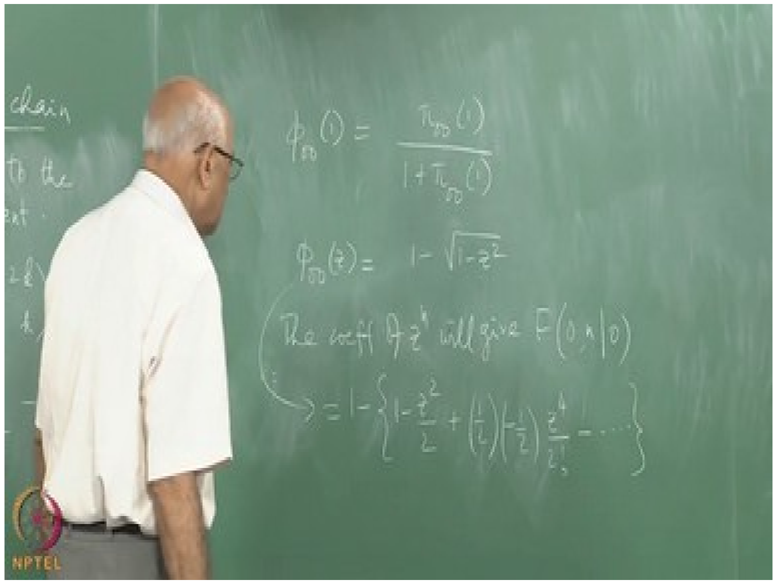

So, that gives us an explicit formula here in this case it says Phi 0 0 of z = 1 - square root of 1 - z squared divided by the square root and the whole thing divided by 1 + this guy which cancels the square root therefore that is it that is it. If you now expand this in powers of n z in powers of z it will give you the coefficients will give you the first passage first recurrence time probabilities at those ends okay.

So, the coefficient of z to the n will give f 0 n 0 will give you the distribution of this time and of course n = 0 is not a recurrence, so that term cancels out as you can see if you put z = 0 the 1 cancels out and let us do this is a power series and see what happens so this thing here is = 1 - and then 1 - z square to the power half, so 1 - z square over 2 the next term is + half into half - 1 which is - half that square whole square that is z power 4 over 2 factorial – dot, dot, dot.

So, you will see that all the terms will remain positive the next one will have half into - half into half - two and so on and so forth – 3 halves etc but it will appear with a – sign.

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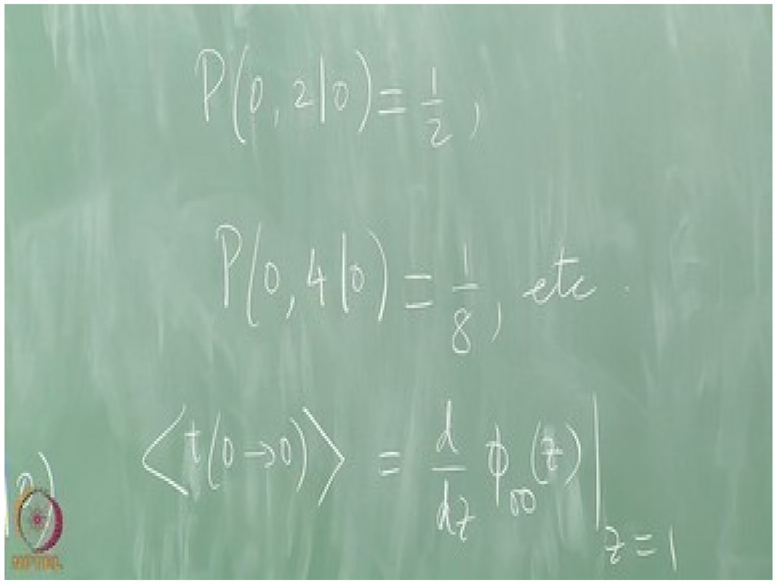

So, it will again give you a positive contribution this will tell you for instance that peak 0 to 0 = the coefficient of z squared which is half P 0 4 0 = we can compute what this number is it is going to be 1 8th that is it. So, you can compute what the probability of return is at the end of every even time step and all the odd time steps there is no return to the origin and you can check that it is normalized to unity. I do not have to check it independently all I have to do is to put z = 1 and show that the answer is 1 and then the probability is summed automatically.

So, it is clear that it is normalized to unity in this case what is the mean time it takes to do this what is the mean time it takes it is the derivative of this at z = 1 right. So, the mean time t to go from 0 to 0 to recur to 0 averaged over all over this distribution must be = d over dz Phi 00 at z = 1 and what is that = it is infinity because if I differentiate this I get a 1 over square root 1 - that squared and I put z = 1 it is gone right.

So, this is = infinity, so that checks out we knew that this process is null recurrent it is recurrent but it is null recurrent in the sense that the mean time to have for it to happen is infinity the first moment diverges but it does happen with probability 1. We could also ask as time increases how will this diverge how will this mean time diverge. Remember that in the continuum case we discovered that the first passage time from any point to any other point went like 1 over t to the 3 halves for the diffusion equation.

And I multiplied by t I ended up with a thing like integral dt over t to the half so over a long time if I integrate this guy here goes like t to the power half and it diverges in this fashion right. So, the question is what is that going to do is that a similar thing is there a 1 over n to the 3 halves or whatever where does this come from if so where does that come from the answer is very straightforward.

All we have to do is to look at what the general coefficient here does after all it came from the expansion of this function here.

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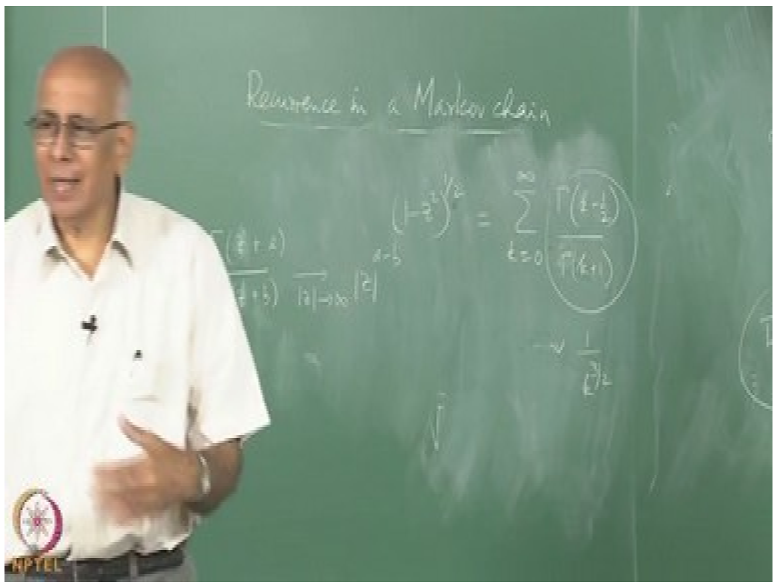

Now we work this backwards in the other direction and then you discover I use this identity here but now I got 1 - z squared to the power + half, so it is clear that this must come from summing k = 0 to infinity – 2k k this guy here in this fashion or in terms of the gamma functions gamma of, I have a + half here so this must have been a gamma might k - half over gamma of - half which is - 2 root Pi, so that is irrelevant here some constant and then there was also a gamma of k + 1.

This is what the coefficient looked like right times this z squared or whatever it is sitting here now what does this fellow look like for large k what does this look like from sterling what is what is this how going to do gamma of k + a divided by gamma of k + b, z + a z + b what does this look like when mod z becomes very large. You have to use sterling once again remember this will go like z + a to the power z + a this will go like z + b to the power z + b.

And if a and b are finite and z is large this has z to the power a on top and z to the power b below so it is clear that this is going to go like this mod z to the power a – b, so what is this ratio going to do it is k to the power - half – 1, so it is going like 1 over k to the 3 halves that is precisely your 1 over t to the 3 halves behavior in the continued, k is like time, 2k was n as you can see and now you are going to multiply this by a k and sum.

You are going to get coefficients which go like 1 over square root of k and when you integrate that you are going to get the cutoff to the power + half on top little diverge. So, it is the same divergence exactly the same power log divergence such as we had in the continuum case. But this is an exact solution this tells you exactly when it converges and when it does not okay now I made the statement.

So, the; to summarize in one dimension on a linear lattice if you have a biased random walk a constant bias either to the right or to the left then return to any particular point is not a sure event it is an event which happens with probability less than one no matter how long you wait. On the other hand if there is no bias in the walk then a return to any point is a sure event as this first passage to any point from any point is a sure event.

But the mean time for it is infinite such a process is said to be recurrent but null recurrent when the mean time is infinite okay. Now you can ask what happens in higher dimensions what happens in 2, 3, 4 etc and the answer is known and I will do this next time because we need a little bit of information about the green function. It will turn out that in dimensions 1 and 2 first passage in the case of unbiased walks.

First passage from any point to any other point is a sure event but the mean time is infinite. In dimensions greater than 2, 3,s 4 etc it is not a sure event and the total probability of a return to any point is less than 1 our first passage to any point is less than 1. Even if there is no bias in the walk and we can establish this directly it is not very difficult to do so. So, we will do that next time.

But again notice that the Markov property has been used very clearly because without that then all these statements corridor the window and you need to re-examine the case over again. But because of that renewal equation we were able to make this very powerful statement. Now I might mention you could ask what happens on that Sierpinski graph if I now go to an infinite generation I generate this so that the number of points increases exponentially.

What happens on such a graph well if it is a finite graph it is a gothic in the sense that every point is visited infinitely often unless you have tracks? Then of course it ends but if you do not have traps and if you have an infinite graph then is it guaranteed that if you start at any one point you hit all the other points any arbitrary point this has to do again with a dimensionality being less than 2 or greater than 2.

But it is not the walk dimension or the fractal dimension but a certain ratio of these two which is called the spectral dimension and that has to be less than or = 2 for the walk to be recurrent otherwise it is transient I will state what that the dimensionality is next time.