26Physical Applications of Stochastic Processes

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Lecture-23
First passage time (Part 1)

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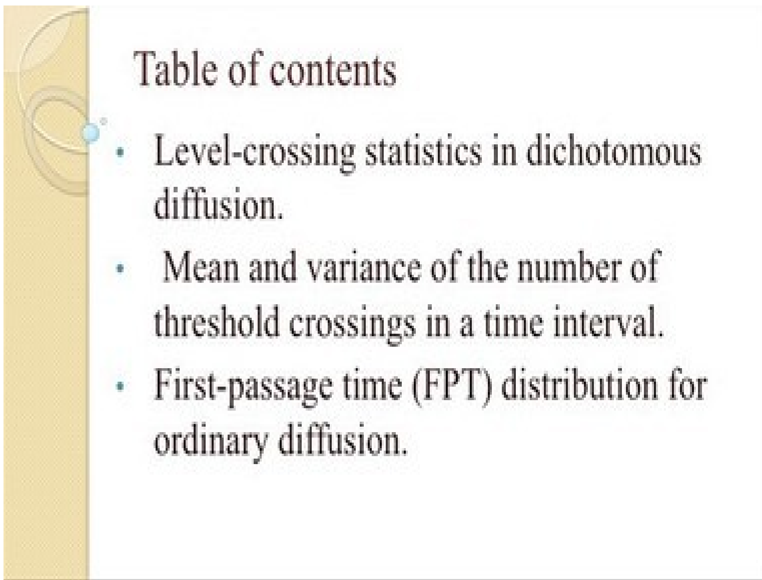

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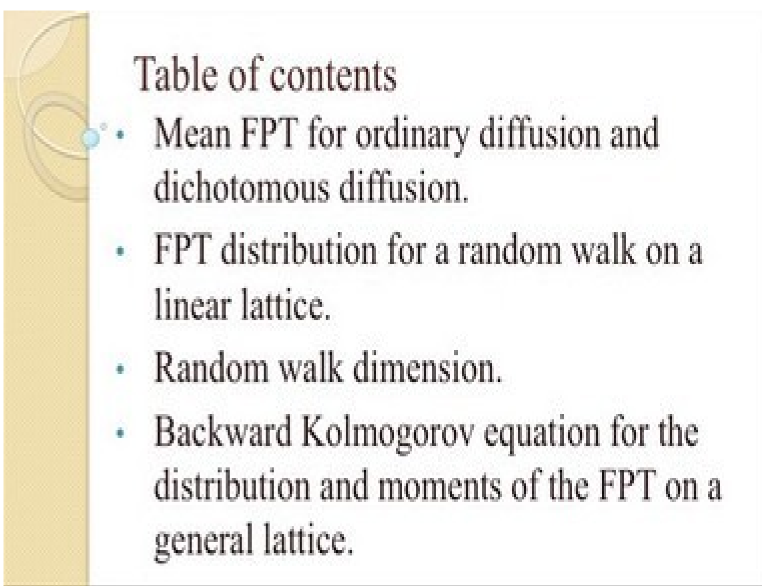

Alright last time I mentioned that we could use the formalism we developed for level crossings and apply to the case of dichotomous diffusion which has a finite velocity. So, let me show you explicitly how this is done and if you recall the problem is the following.

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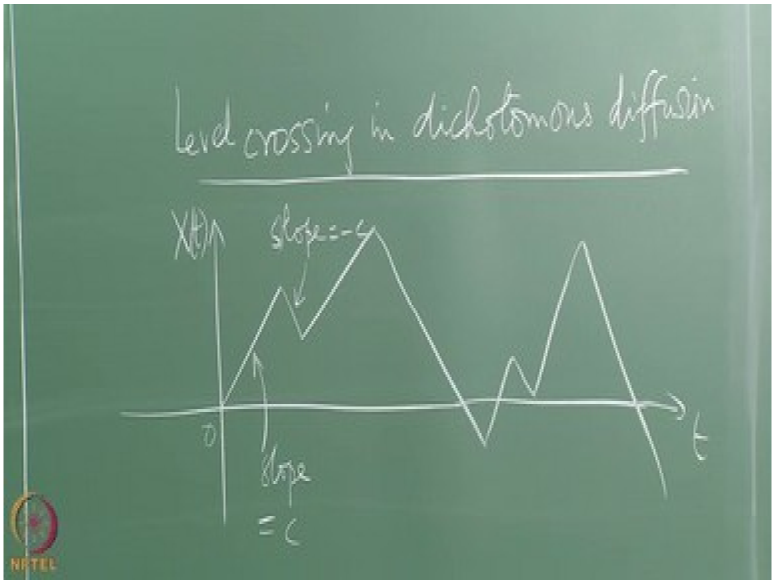

Its level crossing in dichotomies so recall what dichotomous diffusion looks like we have a system which a particle which is moving on the x axis with speed either + C or mind with velocity + C or -C reversing direction randomly and then a typical thing if it starts for instance from the origin in the + state in the +C state goes up like that and then reverses direction and keeps doing this kind of things.

And what we would like to find out is what the statistics of the crossing instants at which it crosses some threshold looks like without much for simplicity we could take the level to be 0 itself this is the origin and this is a function of time this is what the process looks like and here the slope is C and in this case the slope is -C and we want the statistics of this crossing. In particular we want to know what the average number of crossings is in a given time interval 0 to capital T say sufficiently long time interval.

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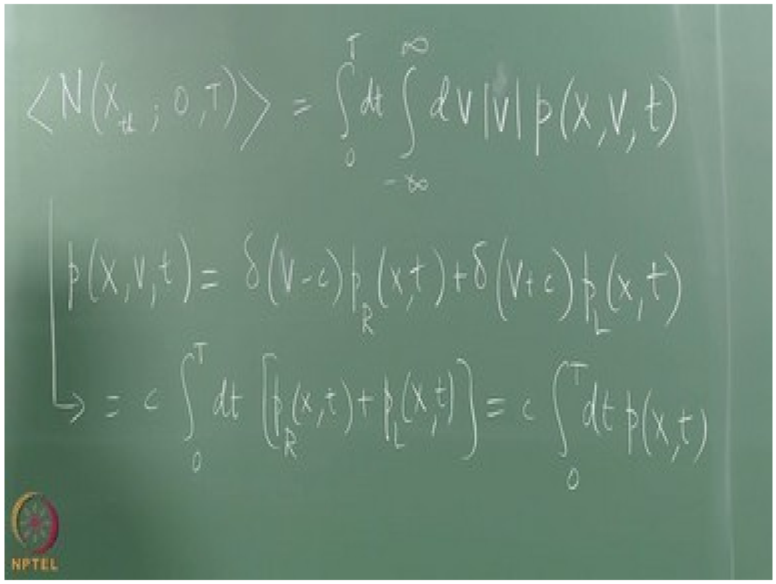

So, if you recall the formula that we wrote down for this level crossing for the mean number was like this we said the number of particles a number of crossings of some threshold X threshold in say a time interval from 0 to t this is what we computed the average value of this was = an integral from 0 to t dt and then an integral over X dot over the velocity V. So, let us just call it V dv and then V dot was here we wanted the number of total crossings namely up or down did not care about that.

So, this was, this we here and then after doing all the Delta function integers etc we ended up with the probability density of X V t and this integral ran from - infinity to infinity that was the general formula that we had. Now we want to apply this to the case at hand. Now let us be specific about the case that we are talking about we are looking at a system where the particle starts with equal probability in the right moving or left moving state for which we wrote down an explicit solution.

And then we computed this number we computed the left and p right separately and of course you can see that p of X V t is nothing but a delta function of V - C to show that it is the right moving state multiplied by pR of X t + Delta V + C p left of X and t it is obvious that this is the correct expression because either the velocity is + C or - C they listened V and therefore in the + state it was this was the probability density of the position in the - state it was this.

All we got to do is to plug this in and do this integral right. So, this quantity here wants to plug it in we can do the integral by the way not me talk to me those X dot the same as V this quantity is always C so it just comes out of the integral as you can see C times and then if you do this integration over V you pick up one from V - C and V + C, so this guy here becomes C times integral 0 to t dt and then all you have is pR of X t + pL of Xt which was the positional probability density itself for the same times integral 0 to p dt.

And we wrote a solution down for this p of X and t and all we have to do is to plug it in and compute this integral right.

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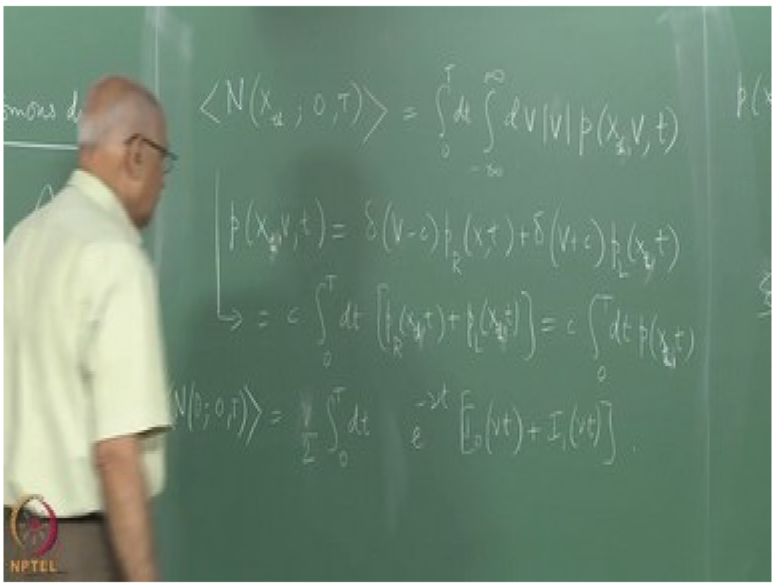

Now if you recall the solution for this p of X and t was the following with initial conditions corresponding to half Delta X in the R-state half Delta X and L state and corresponding things for the time derivatives this quantity p of X and t was = one half e to the - Nu k that is the reversal rate times Delta X - Ct + Delta X + Ct in this fashion okay + there was a Nu over 2C there was an e to the - Nu t always and then Nu over 2C I naught of Nu Zi over C + Nu t over 2Zi by 1 Nu Zi over C.

Whereas I was = C squared t squared - X squared that was the formula now when you want to apply that to level crossing we want to do this for crossings after for t greater than 0 within the interval 0 to t. So, we want to compute it in this interval here this open interval which means that you do not want to include the endpoints 0 and t in particular you do not want to count this fellow as a crossing it starts at 0 okay.

But you just look at this formula you realize that this is just the Delta function with which things start and as a function of t this goes on. So, this contribution is not to be included here for t greater than 0 this contribution is not included if you say the threshold is at X = 0. So, let us compute the statistics of these points and so on in up to some capital T. So, what we need to put in is this expression for p of X and t without this portion because that just corresponds to the initial Delta function which spreads out and count it from here onwards.

So, let us put without loss of generality sorry there is also a threshold sorry this is all at threshold t so this is X threshold t that is what we want to compute X threshold V and t. So, let us look at X threshold = 0. So, we want crossings of the origin itself on the X-axis now what happens to Zi when X is 0, it becomes just Ct okay. So, you got an I naught of Nu over C times Ct which is I naught of Nu t and you get a Ct here and the Nu cancels and again you get the t cancels and you get Nu over 2C.

So, what we need to write is this = C times an integral from 0 to t dt Nu over 2C e to the - Nu t so and then I naught, so what we have is N of 0 0 t average and then I naught of Nu t+ this also becomes the same thing I1 of Nu t and of course the C cancels and you get Nu over C that is it that is an exact formula for the number of average number of crossings of the origin of X = 0 and all you do I have to do is to compute this integral.

There is no approximation here it is an exact formula as it stands and you can compute this integral numerically okay. We would like to see what happens to it t very, very large over a very long period of time what is the number of crossings of the order of what is it going to be of the order of how does it increase what power of t does it increase like that is the question of interest that is not hard to answer.

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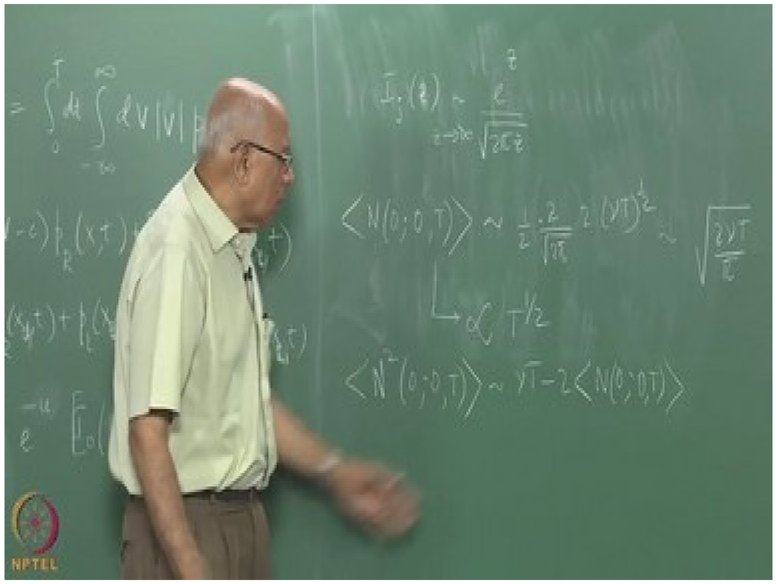

Because you know that if t becomes very large if the argument becomes very large I know that Ij of any z goes like e to the power z divided by a square root of 2Pi z as z tends to infinity that is the leading behavior independent of what j is. So, in this integral it is clear that when this becomes very large here then the contribution comes from very large values of t little t but then that is when this becomes e to the Nu t with a + sign that cancels against this.

And then you have this square root of 2 Pi t whatever it is so this tells you that in the leading behavior the asymptotic behavior is given by; Well let us first let us first do the following let us write this with Nu t as the integration variable let us call that = u or something like that then this becomes you e to the - u this Nu goes away so they are a half this becomes Nu t up there and then this is I naught of u I1 of u.

So, in this form it is much more transparent you want to see what happens when Nu t becomes very large. So, the contribution comes from the fact that this guy here also goes like e to the u but with a root 2 Pi u in the denominator right. So, this goes like one half and then there are 2 of these fellows so that is factor 2 goes away and then in integral up to Nu t du e to the u cancel and you get root 2 Pi and then u to the - half which is u to the half over half right.

So, this becomes Nu t and there is another factor 2 up there so this is = square root of 2 Nu t over Pi so that is the leading behavior and then there are corrections to it which will die off as capital

T becomes infinite. You can easily check that the next correction here it comes from the asymptotic behavior of this the next term will have a 1 over t to some power and that will not dominate this is the dominant part.

So, the summer substance is that this quantity is proportional to square root of T for very, very large T so it is a non-trivial statement that you have this particle going up and down on the x axis and it keeps reversing direction every now and then but moves of the same speed all the time then you ask how often does it cross its starting point in a large time t on the average it crosses I propose some a number times square root of T.

One could now ask what is the variance of this after all this T's and this number is a random variable you could ask what is the variance of this thing. That requires you to find N squared first and then take the average. So, it will involve two integrals of this kind and it will involve a joint probability density p of X t1 p of X prime t prime t2 which also you can write down and go through the same argument as before it turns out that.

In that case it turns out that N squared I am not going to prove this N of 0 0 T goes like Nu T - twice the average value of N this is not an obvious statement you have to actually work this out and it turns out it goes like this. Now what will that imply, it will immediately imply that the standard deviation wants you to subtract out this fellow here you get the variance. And you can see that in both cases both of them are going to go like square root of T.

So, the ratio of the 2 is going to go to a constant of some kind the relative fluctuation is going to go to some constant as T becomes very large. So, that is an interesting property of this dichotomous diffusion and you can generalize this to various other cases. One immediate generalization of this persistent diffusion or correlate dichotomous diffusion is the following. You could say well conceivably the rate of reversal when it goes from + to - C could be different from the rate of reversal from - to + C.

It is like having the integral of a dichotomous process in which the switching between the two different states occurs at different rates in general. So, you would have a sort of biased diffusion in that case the master equation will change they are a little more complicated but you can solve that problem to again write down closer expressions and so on. So, this is a non-trivial statistical property of this process.

The other non-trivial property is to ask what is the mean first passage time like what is the first passage time distribution like and so on.

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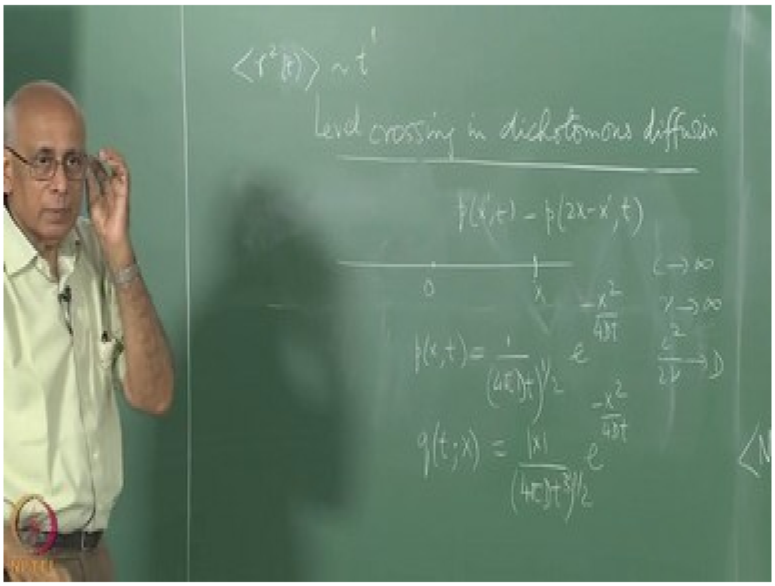

Namely you could say on the x-axis I start with the origin and there is some point XA and I would like to ask what is the distribution of the time of first passage through this point X? Now it is clear that this can only happen for t greater than X over C because this thing is ballistic with a finite velocity. Given that the question is what is the time of mean time of first pass what is the distribution of this time of first passage and how long does it take to do it.

Now in the case of ordinary diffusion just plain diffusion we know the answer we know that in that case p of X t is is = 1 over 4 Pi Dt to the half e to the - X squared over 4 Dt that was for ordinary diffusion plain diffusion and then we also discovered that the distribution the probability density function in time which I call q in time to start at the origin and hit a point X for the first time this guy here was = 4 positive X or negative X does not matter.

It was mod X over 4 Pi Dt cubed to the power half e to the - X squared over 4Dt this was a distribution levy distribution with exponent half in time defined for positive values of t. Now because for large T this goes like 1 over t to the 3 halves this follows harmless goes to unity it is clear from this that the mean time it takes to hit this point is infinite that was a property of normal diffusion. That the mean time to hit a point X from the origin on either side the average time is infinite.

Although the probability of hitting that point is one always, now you could ask what happens in the case of dichotomous diffusion. First of all I said that the time distribution itself this fellow here would start being non 0 only after greater t greater than mod X over C but we are interested in the asymptotic behavior for a long time what it does and so on. This can be solved because what you need now is to say what is the probability of the system being in this region to the left of this point X without being absorbed at the point X? Without having hit this point X.

So, you have to solve the problem of persistent diffusion in the presence of an absorbing variation with a suitable boundary condition. Now once you do that and that is not very hard to do by the method of images for example you would have a thing like if you have p with an absorber at X this would be = p without an absorber of X prime. So, if you ask what is this as a function of X prime t this is = p without an absorber X prime t.

This is not the way to write it but let me not get into this let me not write this out explicitly what you need to use is the method of images which is what we use for ordinary diffusion I can write this as a difference of two piece by reflecting the point X prime on this mirror here. So, I would have a p of X prime t – t 2X - X Prime and this is guaranteed that when X = X Prime hits the value of X you get 0 here things are absorbed.

So, we have to solve the diffusion problem on this left region use this solution in it and that tells me if I start from some point in this case the origin for example it will tell me the survival probability here when I integrate and then - the rate of change of that as a function of time is the cube the rate of absorption here that is it here. So, we can work this out now that we have an expression for p of X, t so laborious but this can be worked out and then you can ask what the mean time is.

Now what would you expect is the mean time would you expect it to be finite why not, well that does not immediately imply that because the fact that there is an infinite amount of space here does not imply that the time it takes to hit that the mean time is infinite it does not imply that at all because it depends on the other circumstances right for instance if I have a bias in that direction that is not true it will definitely hit it and in a finite time to.

So, the fact that you need the fact that you have an infinite expands to the left but would you expect this mean time in the case of dichotomous diffusion to be finite whereas it was infinite for ordinary diffusion. Well recall that this dichotomous diffusion in a suitable limit in the limit in which C tends to infinity and Nu tends to infinity such that C squared over 2 Nu tends to a finite value D goes over into ordinary diffusion.

So, for sufficiently long times this behaves like ordinary diffusion because these Delta functions will kill will die down the two Delta functions are + or - C t die down exponentially and this envelope becomes a Gaussian finally. We saw that explicitly last time right so at long times the process is ordinary diffusion essentially. And if that has an infinite mean time so will this pretty much the same reason it is still infinite at this point so although recurrence.

Although the passage to any finite point X is definite the mean time to do so is infinite certainly continues to be infinite. But it is a little more intricate to write down what the expression is for this q it is not it is not as simple as it is here it is not a levy distribution okay. So, this brings us to a point where we can take off in two different directions. One of them is to say alright this was an example of the X process was an example of a non Markov process.

This did not obey the X is not a Markovian random variable because this system remembers its velocity and it did not obey the ordinary diffusion equation. So, we can ask what are the further generalizations of these non-Markovian processes what do non Markov processes if they are controlled by non Markov processes what does the diffusion look like what does the behavior look like? We will address this question it will lead to something called anomalous diffusion which of some practical importance will address this question.

The other direction in which we can move is to say let us go back and look at first passage times the probability of hitting a particular point or a set of boundary points and asked what is the distribution of the first passage looked like what does a mean first passage time look like and so on because that is an complementary way of looking at diffusion itself. Now the main lesson we learned from an ordinary diffusion was that the variance of the displacement goes like time, linearly in the time.

That was our main lesson that was diffusive behavior to start with. We could ask the complimentary question we say all right you are saying that the mean squared displacement let us call it R squared assuming we start from the origin till time t goes like t to the power 1 that was the main lesson here. So, in a given time the mean square displacement is proportional to that time.

We could ask given a distance what is the mean time to reach that distance for the first time. That is the first passage time problem. So, here the random variable in this is the position you are giving me a certain interval of time and saying how far are you gone on the average in this. Now I could ask the other question I could say give you a certain interval of time and how far did you go was answered by this.

Now I ask I give you a certain distance to be traversed and ask how long will it take for the first time to get there that is the mean first passage time problem? So, let us do that in the case of a very simple problem namely a one-dimensional random walk on a linear lattice and see what this first passage time looks like because it will tell us a general way of looking at this problem. And that also will tell us how to characterize random walks on objects like fractals which is one of the things I would like to talk about.

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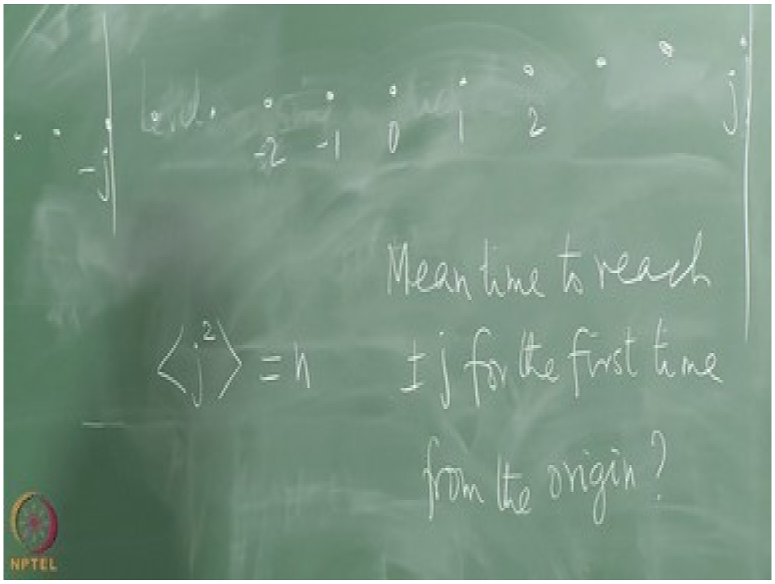

So, let us play this game for a while and see what general lessons we can get from it. So, I start with the linear lattice in this fashion and this is a site 0 this is - 1 this is 1 there is a site 2 this is site - 2 that is it I am interested in distances so let us keep it completely symmetrical and there is a certain site here which is j is a -j and it keeps going but I put a barrier here and the question I want to ask is I start from the origin what is the mean time to hit the point + or - j a distance gave away from the origin for the first time.

But before that we already know that if I say there is a random walk in which I take a step and at each time step I take a step to the right or left then we already know this that the mean distance that you reach in a certain time n is = n. Because the average value of j is 0 it is an unbiased walk therefore the mean square displacement j squared is n this we know already. Now I want to ask what is the mean time to reach + or - j for the first time that is the question I want to ask.

So, I imagine putting two barriers at + or - j and or detectors and the moment the particle hits either + C or - C I declare the end processes over then I start with another particle at the origin and go through it once again and calculate what the time is and take the arithmetic average of all these things that is the mean time so that is the problem we want to look at.

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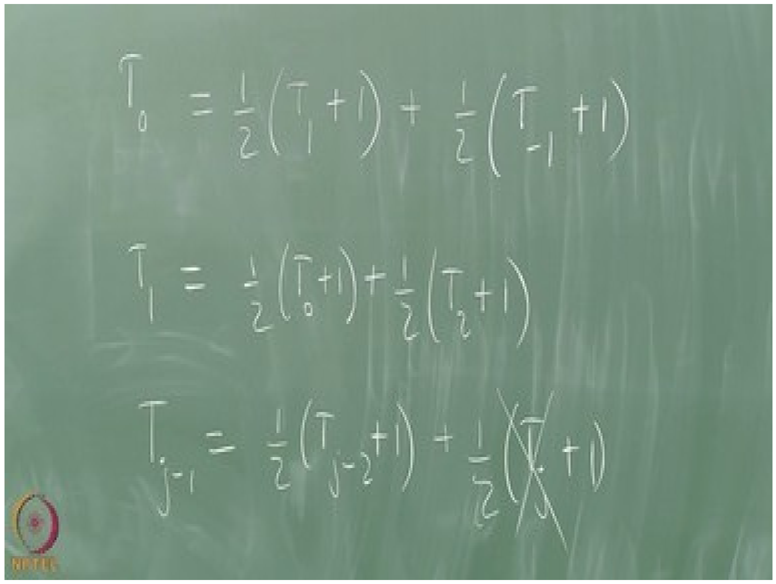

Now I know that this process is Markov and it is happening in discrete time. So, let us call Tk to be the mean time to hit the point + or - j for the first time starting from the lattice point k and k runs from -j to +j. What are the properties that you can think of for this so these are all mean first passage time problems. So is the question clear I start from a cite case some arbitrary sight k which is running between -j here and + j and I ask what is the mean time to hit either + j or – j.

This is what I would like to answer to start with right. Now we can do this in a number of ways but the simplest way to do it is to exploit the Markov property. So, I start by saying the following T naught is the mean time to hit +j or -j for the first time I am going to I am going to suppress all those indices I should really write Tk + or - j to show that you are hitting + or - j starting from the side etc but let us omit that it is understood.

So t naught is the mean time to start at 0 at t = 0 in discrete time and ask what is the time to hit this point or that point for the first time ok. It is clear that I cannot hit that point without crossing the point + 1 or - 1 and in the first time step with probability half I hit either 1 or -1 right. So, are the probabilities so this must be = 1 half times going to the point 1 and then the mean time to go from one to + or - in j and that is T1.

So, this must be = one half times T1 but I used up a time step in doing so, so I had a 1 + there was the possibility of going to - 1 so, probability of s1 T - 1 + 1 and I have used the Markov property in writing this additive property of the means okay. Is there any other possibility because at the end of a time step it has to be either at + 1 or - 1 you are not allowing for it to stay there that is it right. So, these are the only two possibilities right.

But you have exchanged one unknown for other unknowns. So, we have to write down what is T1. So, I am here at this point and then what happens with probability half I come to t0 with probability half I go to t2 right. So, this must be = 1 half t naught + 1 + one half t2 + 1 and I have to write the equation for all these fellows and all these fellows and there is a set of coupled equations which I have to solve in principle.

But some general things are emerging already what happens once I hit Tj - 1 what is the equation satisfied by T's j - 1 with probability half I might jump to Tj - 2 I hit j right I jump to the left I mean come here I am going back and forth. So, I could have come here right. So, this is = half Tj - 2 + 1. But with probability half I could have jumped there to that point and then the walk is over right. So, it is +1 half Tj + 1 but what is T's of j at 0 because it is the mean time to start at j and go to j.

I am already there and the walk is over right. So, Tj is 0 therefore you end up them. Similarly on this side and we have to now solve this equation the set of recursion relations has to be solved. But there are certain obvious symmetries in this problem would you say that if it is a walk is unbiased would you say that T1 is = T - 1 yeah there is nothing to distinguish between right and left this one is closer to that guy by one but it is further from here by one.

But this fellow is just reversed remember the boundaries are placed symmetrically about the origin right. So, it is clear that you have a general rule which says in this case Tk = T –k, so this thing can only be an even function of k. Because Tk must be = T - k now let us examine this fellow here a little bit this guy says that 1 half T1 + T- 1 - T naught = - 1 I keep this on this side I keep this on this side I bring this team out here and move the constant to this side is = that right and so on.

Each time I have this thing here now this is the arithmetic mean of the two neighbors with a half and then a - 1 as a coefficient. If I pull out this half what happens it is T1 + T - 1 - twice T naught what sort of difference is that it says f of X - epsilon + f of X + epsilon - twice f of X second difference second difference right. And the second difference is a constant. So, when the second difference is a constant.

What kind of function can it be of the variable quadratic it is got to be a quadratic function right. So, we already know that this Tk must be a quadratic function of k.

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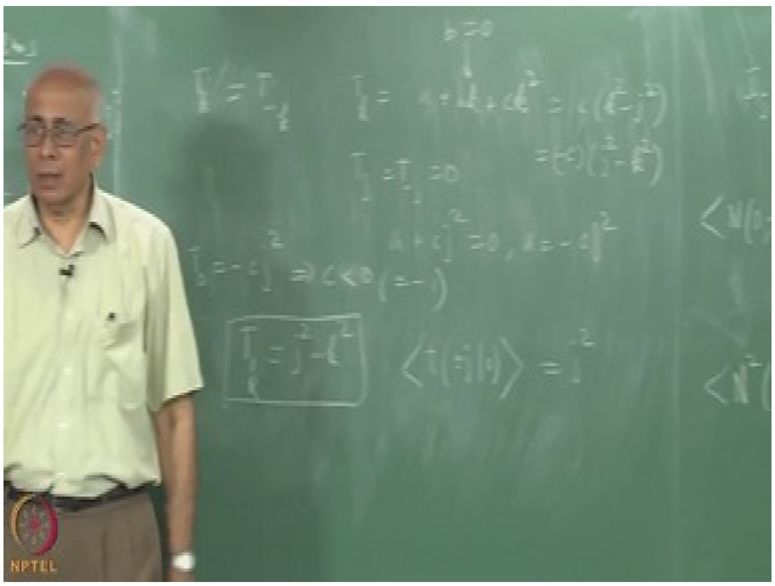

So, let us write Tk = some constant + bk + ck squared but this is 0 b = 0 because of this property. So, it is got to be of this form k + Tk square what is the boundary condition on this Tk, Tj = 0 right. So, it says a + C j squared = 0 all right, so this becomes = C times k squared - j squared all it leaves us though so that satisfies the boundary condition, not it leaves us is to find this value of the C constant TC, what is T0 if I put k = 0 I end up with - Cj squared but it cannot be negative the mean time to go from the origin to this point cannot be negative.

So, it means C must be negative as I have written in there so it must be over 4 - Cj squared - k squared and it must be independent of j this guy cannot depend on j. So, you can use any simple model to find what j is? To find what C is the constant C is how do you do this, what would you do? Pardon me but all you have to do is to take + or - 1 take any particular point and it will tell you the value of C immediately I leave you to work this out C it turn out to be - 1 okay.

So, that is very straightforward to establish, so it says Tk it is a simple exercise to show that this is = j squared - k square. So, it says the mean time to hit the point + or - j from the origin so the mean time t to hit the point + or - j starting from the origin this mean time is = j squared you see it is the perfect complement of this it is the perfect complement of this guy because it says the mean for a given the mean distance squared mean square displacement is proportional to the time.

And it is here it says that you give me the distance and the mean time is the square of that distance it is exactly the complement of it. So, you can define the diffusion constant in this manner we defined it earlier as saying the limit of X squared average divided by 2t was = D capital D you can define it the other way you can define it as the limit of in terms of the time you can define the mean time to go a certain distance.

And in this case in the discrete case to the continuum case the normalization is this 2D twice the diffusion constant. So, this is an equally convenient way of finding the diffusion constant and of retaining the basic property of diffusion namely that the square of the distance goes like the time in this case the time is a random variable the distance is fixed. So, now we generalize this for all sorts of structures.

So, I would start by saying I forget about putting these averages and so on so I say that r squared there are suitable averages depending on whether you want doing this or that this goes like asymptotically like t if it is normal diffusion but otherwise it goes like 2 divided by d walk. So, I introduced a dimensionality called the random walk dimension on any structure and I will explain what the rationale in doing this is as we will see this very shortly.

So, I introduced this dimensionality you need to compute what it is it says the mean square distance goes like t to this power or the time mean time goes like for a given distance in this fashion. For normal diffusion this is 2 because r squared goes like t in this sense or this sense okay. We will see that on fractal structures and so on this DW is different from 2 we will see precisely what this implies here.

So, let us do this in several steps first let us solve the problem of this first passage time on other lattices we did this for linear lattice. Let us see whether this gives us some general hints as to what to do and how to do this at all.

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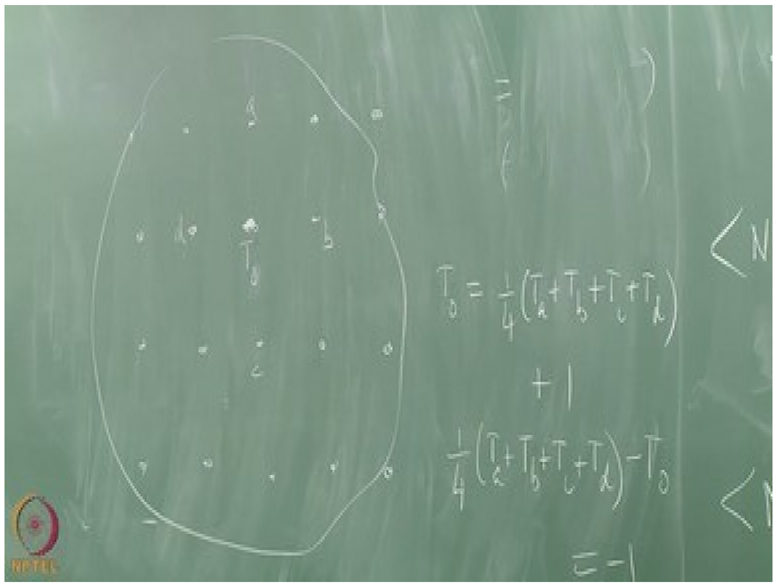

Suppose we did this on a square lattice for instance so you have this kind of thing etc and there is some boundaries and I want to compute what is the time mean time to start from here and do a random walk with equal probability of going to each of his nearest neighbors and hitting this boundary point, I need the mean time for instance. So, if this point is t naught or let us call this point the central point t naught.

And I have nearest neighbors and I assume nearest neighbor jumps always let us call this a, b, c, d for example the mean time to go from these points would be Ta, Tb, Tc, Td to this boundary they would be equal if the boundary is also symmetric has the same square symmetry that this lattice has but if it does not it it is not symmetric it does not matter does not matter in any case. But the moment I write an equation down t naught, you write see that t naught is = from here in the first step you go here, here, here or here with equal probability.

And what is that probability half so you are going to write this as 1 quarter Ta + Tb + Tc + Td + 1 because in each case you are going to add a 1 for the first step and 1 4th times 4 is = 1. In this fashion but I can rewrite this and then there is a whole set of equations. But I can rewrite this as 1 quarter Ta + Tb + Tc + Td - T naught is = - 1 and what is this fellow here on the left earlier it was one half times the second difference.

Now what is this thing what is this, this thing? Here it is not the second difference this is a very definite quantity it says 1 quarter is the coordination number of this lattice there are four nearest neighbor's right. When you are taking this, this you are summing this over all the nearest neighbors and subtracting this quantity here. This is the mean value of these 4 guys it is the mean value there are four of them and take one fourth it is the mean value right.

So, what is this object we take the mean value at any point – T naught what is that quantity it is the discreet Laplacian it is the Laplacian on a graph right. So, really writing the Laplacian and you see that del squared coming out here. So, it is a discrete Laplacian a in every case it will be the discreet Laplacian. So, what will the general case be if you have got an arbitrary lattice with all kinds of coordination numbers.

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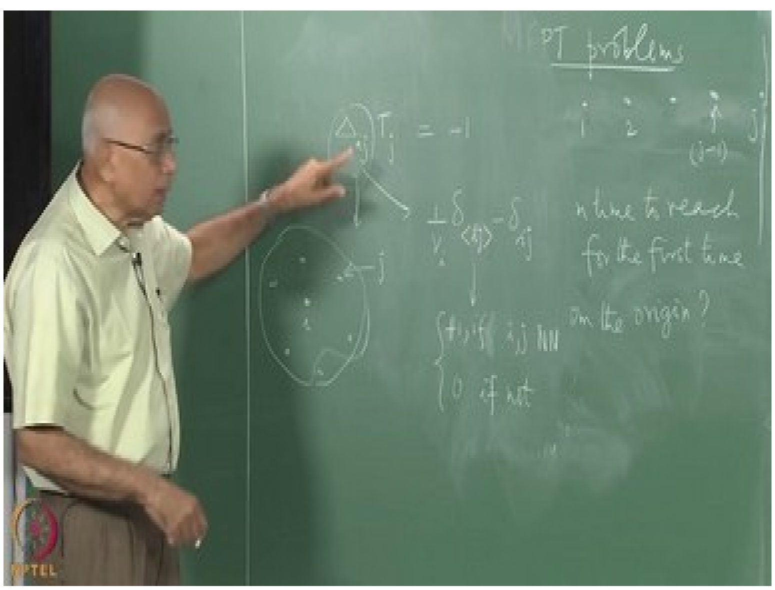

What you would have is the following you would have a delta ij Tj in this quantity = - 1 where I have to define what this fellow is this guy here for a given site i there are these nearest neighbors.

These are all the set this set of nearest neighbor this set of quantities here these are the nearest neighbors j right and what you have this quantity here is 1 over the coordination number of i because with equal probability you jump to all these other neighbors and let us call that Nu sub i.

4 if you have 4 nearest neighbor 6 if you have 6 nearest neighbors etc etcetera 1 over Nu i times a Kronecker Delta here provided i and j are nearest neighbors. This symbol stands for nearest neighbors it is = 1 if i and j are nearest neighbor 0 otherwise + 1 if ij our nearest neighbors 0 if not - Delta ij itself that is this quantity – t naught whatever it is. So, that is what the discrete Laplacian is weighted discrete Laplacian weighted with this coordination number.

And it says Delta Ij Tj = - 1 for every site and at the boundary sites the tj's are 0 whatever be the boundary sites okay. So, that is the equation you have got to solve in very compact form. Now what does this finally imply it implies that the solution for all the Tj's is just the matrix inverse of this the inverse of this matrix once you find the inverse of this matrix over the full lattice the matter is over that is it. And that is what we did for the linear lattice we did that by clever trick by observing that this is got to be a quadratic function and so on.

But in the more general case it is much harder to do but you really finally reduces to an inversion of a matrix. But if your structure has N lattice points then this matrix is of order N squared and it is a non-trivial point you really need some symmetries in the problem to solve it but in principle it is done. Now where did this whole thing come from it came from the following fact it is a special case of a more general relations and that is the following.

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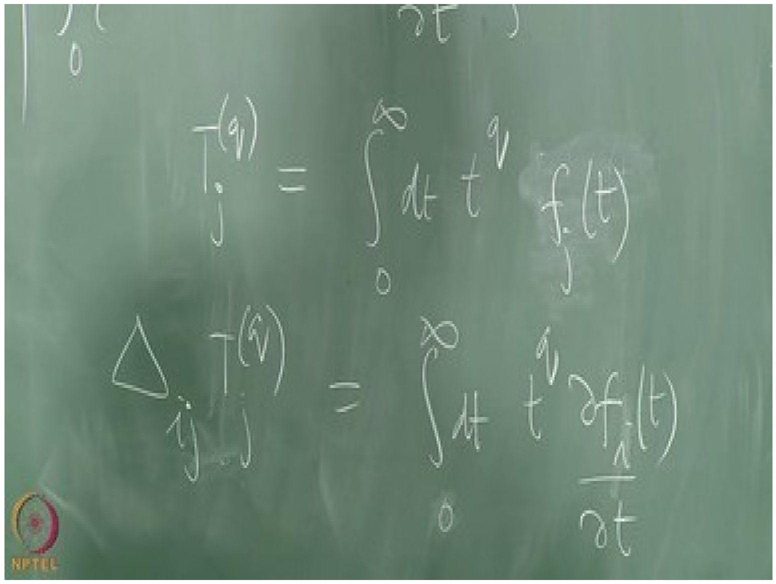

Suppose, I look at the diffusion problem on this lattice and instead of looking at the probability density at a particular point I look at the first passage time as a function of time to go from the site j to some specified set of boundary points or traps when the walk ends. Let us call that distribution in time fj of t it means it is the time fj of t is the probability density function in time that starting from the site j you are going to hit any of these boundary points for the first time in a time interval between t and t + dt that is what this thing means here.

Then it satisfies because this finally is reduced like in the earlier case we saw what the problem reduced to it reduces to solutions of the diffusion an equation of some kind right. And the discrete diffusion equation would look like this Delta ij on this side would be = the diffusion equation. So, what appears on the right-hand side this is summed over j, so what would appear on the right hand side?

It would be Delta fi of t over delta t because it is i is a free index that is the diffusion equation on a lattice yes okay. And that is fj is the first passage time density what is the mean first passage time the integral of this f multiplied by t so to go from Tj that is what the mean first passage time was but really I need not use the mean I could use the mean square mean cube mean any higher moments etc.

Let us call the cute moment this is the qth moment of the me; of the first passage time to go from j to whatever boundary points you want for the first time. What is the definition of this guy this is = an integral from 0 to infinity dt t to the power q Delta fj over delta t outside fj that is the first passage time density this is the qth moment of the first passage time density. What is the 0th moment of this first passage time density?

It is = 1 if and only if first passage to that point is a sure event then it is normalized to unity otherwise it is not even a sure event right. So, we first have to make sure that given enough time the system the particle will hit those boundary points in other words you have got to make sure the walk is recurrent and so on. That is the separate story all together on finite lattices it will always happen because it is a Markov chain with a finite number of sites no tracks.

It will hit it given enough time okay so that is a point which we have to worry about in the infinite media will come to that separately. But you agree that this is the notation is that starting from the point j to hit any tracks or whatever for the first time the qth moment of the mean time is of the time is this quadratically here. So, let us take this equation and multiply by T to the q and integrate over t on both sides.

Then I stay this multiplied by t to the power q and i do an integral over dt from 0 to infinity. This is spatial indices, so I pull that out and I get this relation which says Delta ij and what is inside is t to the qf this is Tjq the qth moment and that is = on the right hand side integral 0 to infinity dt T to the q Delta fj, I should be careful about notation fi by delta t. I can do this integration by parts out here.

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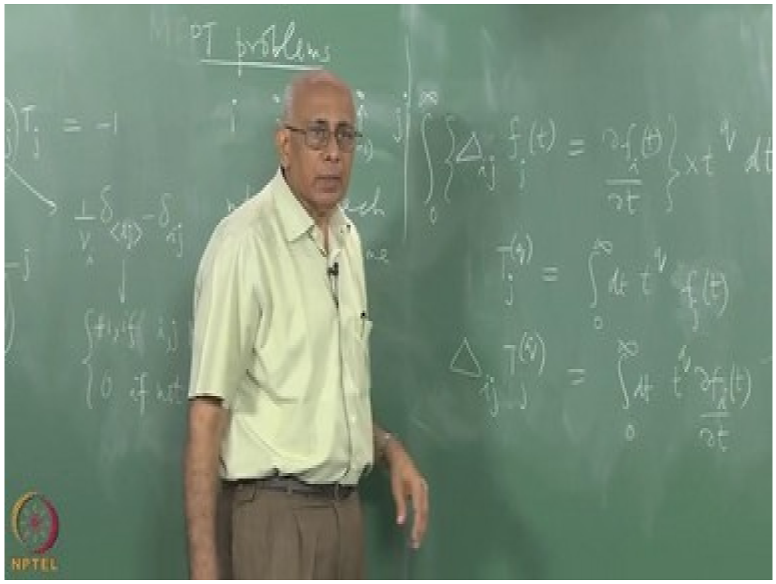

And then what is the first term give so this thing here is = t to the q fi of t from t = 0 to infinity - q times integral 0 to infinity dt e to the q - 1 fi by integrating by parts. Now this is the solution to the diffusion equation all moments of this quantity are finite implies that this goes to 0 faster than any power of t as t tends to infinity. And it is finite at the origin so this guy here is 0 this boundary term is 0.

So, we finally end up with an equation that says Delta ij Tj of q is = - q tq - 1q is greater than = 1, so that is the general relation by the way this equation is called the backward Kolmogorov equation I have not explicitly introduced it for continuous diffusive processes. But that is what it is we did this on a lattice and I motivated it by showing you what happens on a linear lattice and then simple generalizations of it.

What happens to that equation if you put q = 1 it says Delta ij Tj we call this the mean time the mean first passage time itself was = - 1 because this fellow T 0 is 1 that is exactly what we got earlier we found the discrete Laplacian acting on this set of mean first passage times gives you - 1 over right hand side. But that is a special case which follows from the backward Kolmogorov equation and you see now even without solving for this you can find all the moments.

Because you solve this set of equations you find all the Tj's you put that in there and then you get the second moment of all the first passage times put that on the right hand side and you get the third and so on in principle. So, now I need to explain to you what is meant by this walk dimension and what happens on lattices which are not regular lattices? So, once again we will take recourse to the simple linear lattice see what happens.

And then simple ordinary to the two dimensional lattices and then from there we will exploit this relationship will exploit this thing over and over again to see how to solve this problem on hierarchical structures. But I hope the logic is clear it follows from the diffusion equation finally it is just that on the discrete case it is a little easier to exhibit things explicitly like we saw for the linear lattice rather than write those messy Levine distributions and so on.

In the continuum case we can always go to the continuum but I think that the discrete case explains it much more clearly. I do not know what I do not know what word you use for this thing here for this object here must have some special name in electrical engineering because in a circuit this is exactly what you do you look connect up various things with resistors and then you I mean this is that Jason C matrix of this graph and so on and so forth.

But we have not made any specific assumptions about which points it jumps to and so on. I have just said for nearest neighbor jumps this is what it is but you can solve this problem even if you had longer jumps in. But to my mind is the simplest way of understanding this first passage time the behavior of this first passage time because as I have shown rather than looking at the distribution in position for a given structure it is easier to look at the distribution in time.

And ask how long does it take to go twice as far twice as far it is okay so next time you will see the explicit meaning of what this walk dimensionality is and how it is = 2 for regular structures but for anomalous structures like fractals and changes need not be an integer. So, we will do that next time.