Physical Applications of Stochastic Processes

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Lecture-22
Dichotomous diffusion

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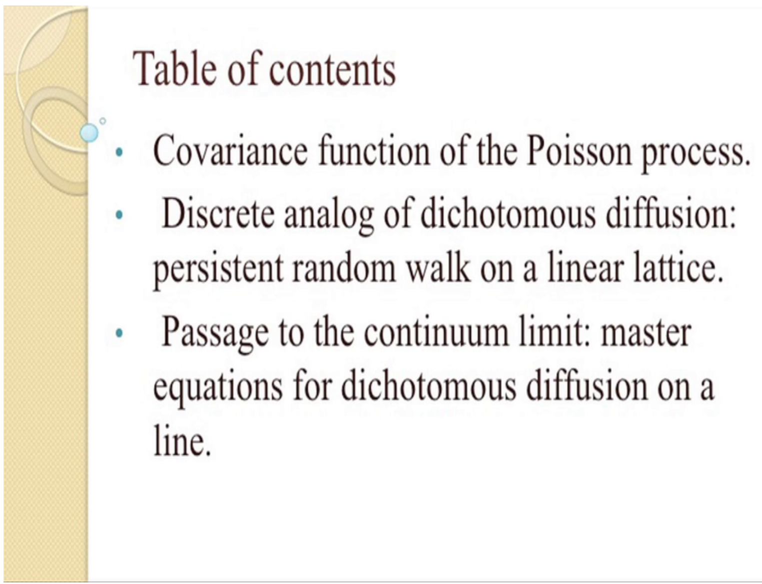

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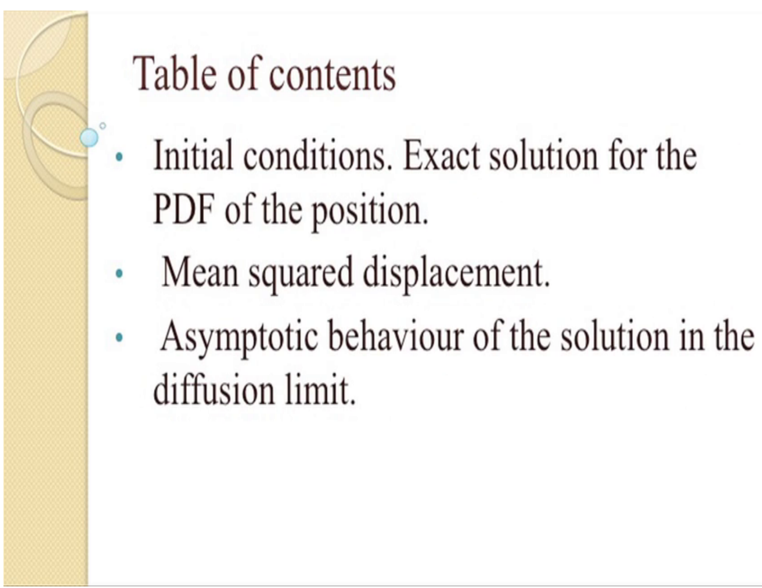

We started last time by introducing the idea of what I call persistent diffusion. So, we will take this up today we will look at this problem today. And I will point out what its connection to a dichotomous Markov processes and we will see how it differs from ordinary diffusion. So, we will get back to the diffusion limit and so on and so forth. It is a very, very popular model and a very useful model of continuous random process.

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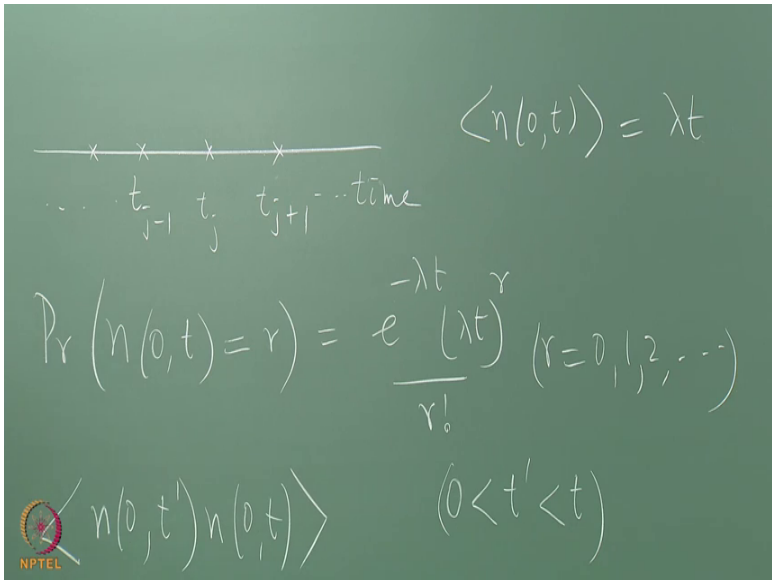

So, what is the problem it is as follows and I called it persistent diffusion but let me give its proper name dichotomous diffusion is a better name dichotomous or persistent and I will explain why it is called persistent we have in mind a particle diffusing along the x axis exactly as in the normal diffusion problem. But this time I specify the velocity of the particle to be fixed to be some finite value but capable of reversing in direction.

So, it either moves to the right or to the left with some fixed velocities C in some speed C. So, if I call the process X X dot of t is = a dichotomous Markov process in the sense that it takes values + C or - C so as Zi of t and this fellow is a dichotomous Markov process taking values + C or - C and the question is what are the statistical properties of the process X itself which is the integral if you like offer dichotomous Markov process.

Remember that the normal the usual diffusion problem we had X dot = square root of 2D times Eta of t where this fellow here was a Gaussian white noise and therefore this X became a Wiener process and this process was not differentiable anywhere it had all these strange properties it satisfied the probability density functions satisfied the normal diffusion equation. But now the question is what happens if I integrate a dichotomous noise here with these values.

What answer would I get what sort of probability density would I get and so on. We can actually sketch this function without too much difficulty because you can see that schematically if this is a dichotomous Markov process what it does is to flip between + C and - C at random intervals of time so this is C and this is - C this is what the process X dot does or Zi of t. And when it is + C X is uniformly increasing with time linearly and when it is - C its decreasing with time in the other direction.

So, it is immediately clear that if I plotted t versus X of t not in this case but in the other case then if it is going up in time it does that the function of time and then it decreases with speed - C then goes up again and decreases and so, so it does something like this. It is a lot more smooth looking except for these points of direction reversal it is a lot more smooth looking than a very jagged curve that Brownian motion actually is or Wiener processes.

So in that sense this is expected to be a milder process in a certain sense more handleable more tractable and we would like to see what its statistical properties are okay. Now you can get to this in several ways you can get to the probability density for this guy in several ways. But let me just write down so this implies that X of t = integral 0 to is = X of t - X of 0 is = integral 0 to t dt prime Zi of t.

So, in some sense you want the integral of a dichotomous process we can find out what the correlation of this is etc it is if I will do that shortly. But the process itself in a typical realization would look like this in this fashion. So, as you can see everywhere there is this basic idea of a Poisson sequence sitting everywhere. These points where this thing undergoes reversal are supposed to form an uncorrelated Poisson sequence of points.

And we have already studied what a Poisson pulse process does okay we know what it is correlation is and so on and so forth just as a digression let me remind you of what this whole thing is. If you took the several ways of defining a Poisson sequence of points but the way one convenient way is to say the following is to say that on the time axis you have a whole lot of epochs or instants of time which are completely uncorrelated with each other.

So, this is tj, tj + 1 you know this is tj - 1 no not completely independent of each other and now let us the statement is that in any finite interval of time the probability that you have n of these prosses is Poisson distributed with some mean rate. So, if you say for example if you start with the number in 0 to time t n of 0 t that is a random variable the number of such epochs occurring in a given time interval 0 to t is a random variable.

You can take on any integer value any non-negative integer value and one can ask what is the distribution of this n here? So, the probability that this is = some integer r some non-negative integer r is a Poisson distribution so this guy is = e to the - lambda t lambda t to the power r over r factorial and r = 0, 1, 2 etc that is it. This lambda is called the intensity of the process it is the mean rate at which these crosses occur okay.

And these t's are independent of each other so we can actually define a Poisson sequence in another way which is to say that the mean that the gap between two successive crosses is distributed exponentially. The probability that if you have a cross at t = 0 the probability that you do not have a cross till time t it goes like e to the - lambda t that is it. So, several ways of defining this we also defined it yet another way by saying in any interval delta t any infinitesimal interval delta t.

There are only two possibilities either there is a cross in it with probability lambda delta t or there is no cross in it with probability 1 - lambda delta t. The probability of two or more crosses occurring in a delta t is a higher order and infinitesimal. So, the several equivalent ways of handling this yet another way is to say that this is Poisson distributed and different non overlapping intervals are independent of each other completely independent of each other.

So, n of 0 t and n of 2t, 3t for example they are not overlapping intervals these are independent random variables completely. What would the correlation of this n be what are the autocorrelation of this n of 0 t be. So, let us let us call this fellow let us leave it like that so what is n of 0 t prime n of 0 t what would this be let us be very definite let us say 0 less than t prime less than t.

What is this expectation value likely to be this autocorrelation do you think it is a function of t - t prime remember the statistics of these t's the sequence t's this a sequence tj it is a stationary sequence in the sense that the mean rate lambda is independent of time whether you look at this interval as 0 to t or whether you look at it as t naught to t naught + t whatever t naught is it does not make a difference.

You get exactly the same distribution, so in that sense it is stationary so my question now is do you think n of 0 t prime which is a random variable is independent of n of 0 t, if that were true then I can remove this average I can write this average of this product as a product of averages incidentally. What is this = what is this guy = exactly it is lambda t its lambda t it is a Poisson distribution and the average value is lambda t.

So, of course it says that if the interval increases the number of points in it will increase with the mean rate lambda so it is proportional to lambda t. What is the variance of n of 0, t also lambda t right. Now let us look at what this fellow is do you think first of all it is a function of t - t Prime first of all do you think that these two are random variables which are independent of each other no, because they does not overlap.

There is certainly an overlap t is bigger than t prime n of 0,t prime is sitting inside 0, t, so, these are not independent random variables therefore you cannot factor it out immediately and there is a non trivial autocorrelation.

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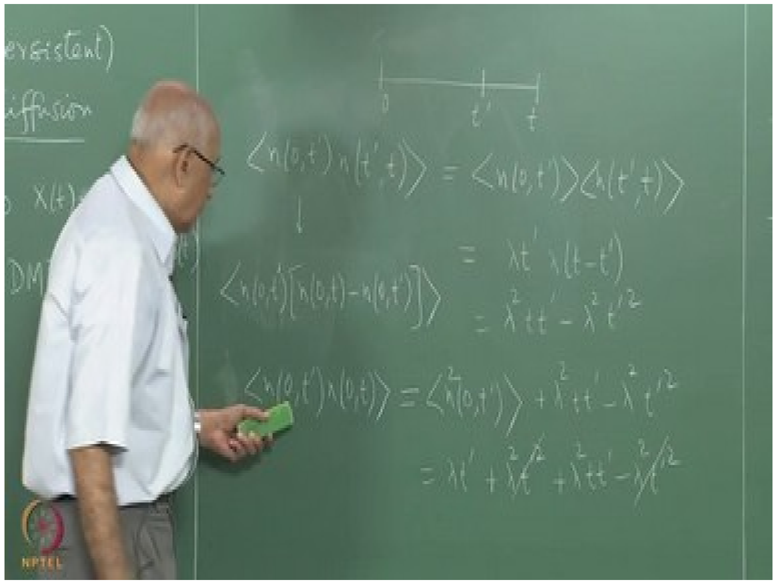

So, one way to do this would be to say let us look at it like this let us look at n of 0 t prime n of t prime to t, let us look at this one here now is it just for reference so you have 0 t prime and t on the x-axis on the t axis. So, let us look at this quantity and ask its expectation value are these independent random variables yes because they are not overlapping. So, they are definitely independent random variables.

So, this is = n of 0 t prime n of t prime t and what is n of 0 t prime lambda t prime and what is this lambda of t - t prime so this is = lambda squared t t prime - lambda squared t prime squared that is certainly true right. But you can also write this as n of 0 comma t times n of 0 t - n of 0 t prime and then take the average by definition n of t prime t is this difference. So, that is = sorry n of 0 t prime and then this so this is = n of 0 t prime n of 0 t - n of 0 t prime squared and then the average n squared of this fellow here.

So, I want to find this quantity that is the correlation I want to find autocorrelation, so I move this to the right hand side and I have shown that this is already = that out there so this quantity therefore this implies this is = this fellow + lambda squared t t prime - lambda squared t prime squared. But what is this quantity because we already have this other result we have this result it says variance of n of 0 t = lambda t.

Because for a Poisson the variance is the same as the mean right that means that n squared average is lambda t + the square of the average lambda squared t squared right, so if you put that in this is = lambda t prime + lambda squared t prime squared + lambda squared t3 prime - lambda squared t prime squared right and this cancels this guy cancels. So, it says this autocorrelation is lambda t prime + lambda squared t t prime not a function of t - t prime.

But the average is not 0 here n of 0 t or n of 0, t prime does not have a 0 average so the autocorrelation should be defined as the covariance by subtracting out the mean.

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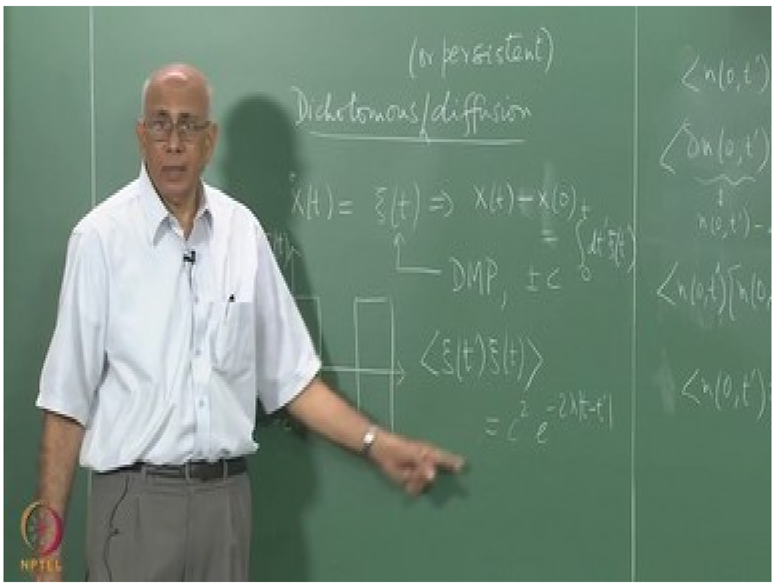

So, what we need is the following what we need so let us write these results down so what we have is expectation n of 0 t prime n of 0 t this fellow is lambda t prime + lambda squared t t prime t but what we need is delta n of 0 t prime delta n of 0 t expectation and this stands by the way for n of 0 t prime - expectation n of 0 t prime that is the definition of this Delta. So, what is this quantity = all you have to do is to take this guy here and subtract out the product of the means.

And what is the product of the means lambda t prime times lambda t prime t so this is = this lambda t prime okay and we took this to be t prime to be less than t what happens if I took t prime to be bigger than t place so in general this guy here is = lambda times min it is not stationary. So, this n of 0, t I mean it has a correlation this random variable has a correlation so the number of such epochs such instance of time in a Poisson sequence in any interval of time the correlation function of that is given by this.

And of course when this is = that t prime = t this is = t lambda t as you expected it is a variance that is it. Now this t this linearity is at the origin of the linear dependence with t in all these diffusive processes this word is finally going to finally leads to this whole business of linear dependence. So, it comes straight from here from this processing because now if I attach a process to this by saying there is a velocity process which goes either + C or - C at these instants of time.

The statistics of that carries over signatures from this and so on but remember that this process here how is it correlated now it is the Zi process what is the correlation of Zi this is a stationary Markov process definitely a stationary Markov process is a jump process. And what is this = Zi of t prime Zi of t what is this = we looked at the symmetric case where is a + C and a - C naught C 1 under C 2 and the other symmetry is that which I said that it is going to reverse with a mean rate which is the same whether it is down - up or up – down.

If it is some lambda then what is the correlation here it is exponential it is C squared e to the - 2 lambda 2 lambda actually mod t - t prime it is stationary. So, you must distinguish between these processes they look very similar to each other they are very closely related to each other but this is in time this is just a set of points in time uncorrelated that is got its own statistics. But now if I attach to it a process which goes up or down in this fashion this becomes a Markov process and then it is exponentially correlated in this fashion.

Now we are asking the next question what is the integral of this process going to look like what is this that is our dichotomous diffusion all right and now this X in the X space is or the whole of the real axis see the whole of the x axis we would like to know what its statistics are and it is a continuous process. This is a jump process but the X itself the integral is a continuous process although it has those kinks whenever the velocity reverses and we are looking at the statistics of that.

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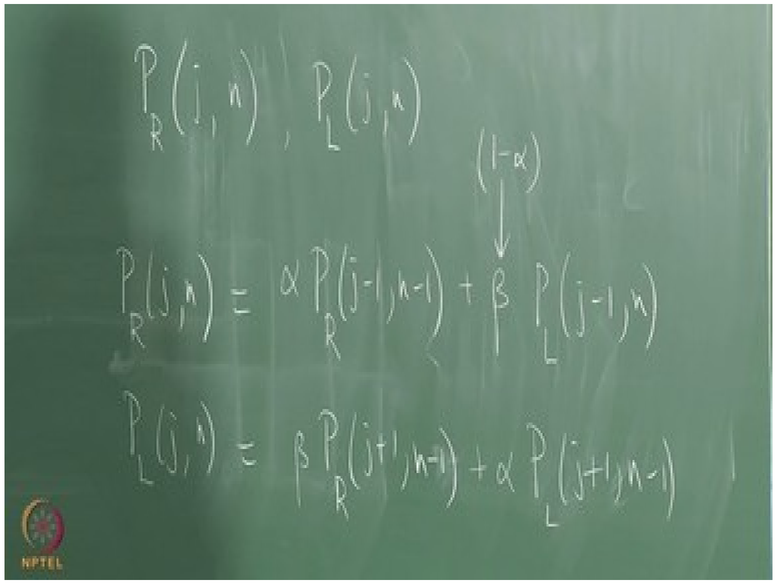

Now you could ask can I get this process so let us keep this aside for a moment and we can ask can I get this process from a discrete random walk after all what is being remembered is the velocity namely the direction of motion. So, can I get this from a linear lattice by doing a random walk on it with the following kinds of rules? So, once again I have a lattice whose sites are labeled by the integer j this is j + 1, j - 1 and I asked for the probability that you are a j at time step n.

So, I take steps of tau time step tau and there is a lattice constant A which I will introduce subsequently but for the moment we will just label the lattice points by the integers on an infinite linear lattice and I want to know P of j, n but with the following proviso just as earlier I said in the case of the DMP I said we would like to remember the velocity whether it is + C or - C the analogue here would be to say is it moving to the right or is it moving to the left.

So, I really have two different probabilities I have a PR and at PL, so it is at this point at time and moving right words or moving left words and there is a certain rate at which or a certain probability with which it will reverse direction. So, let us put a bias in it in the following sense instead of saying I toss a coin with probability alpha I move to the right 1 - alpha I move to the left if it is a biased coin.

I now use the same biased coin and I say with probability alpha I continue in the same direction as a previous step and with probability 1 - alpha I reverse my direction. Then what are the rate equations for this PR of j n must be = clearly alpha times PR of j - 1 at time n – 1, so I have come here and I'm moving right and I continued rightward so I am in the right it is not a contribute to PR + beta which is 1 - alpha PL of j - 1 namely.

I am here at time n - 1 but now I reverse direction and move towards the right and therefore I am here in the right moving state those are the only two contributions. What about PL but this is clearly beta times PR of j + 1 n - 1 because I am here at time n - 1 and I want to contribute to the left moving guy whereas I am moving right there, so I am a switch and then move left and the probability is beta + alpha times PL of j + 1.

So, I am here moving left and I jump here with probability alpha and I continue to move left and therefore I contribute up here. So, these are the rate equations and the point is to solve these now for any given alpha between 0 and 1 okay. So, the solution is not that trivial as you can see they are two coupled unknowns here PR and PL but in principle this can be solved this that can be solved.

What we are interested in doing is taking a continuum limit of it. So, that we get some diffusion type equations and how would you take this limit.

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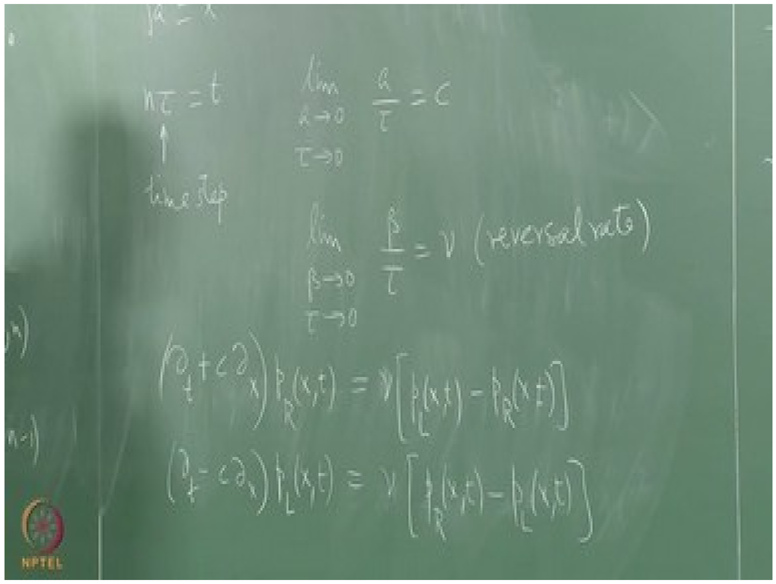

Well the first thing you do is to say that the lattice constant should go to 0, so let us put j a = X and this is the lattice spacing n tau = t this is the time step and it is clear yeah no you are asking what is the contribution to one of these say PR to be in the right moving state at this point at time n right, it can only come from here or here because we have allowed only nearest neighbor jumps. But then it could have come from different states and these are the only states from which you can feed in to this point here that is it okay.

We are not writing a rate equation we are writing an actual difference equation for finite n right. So, we have a time step and we have space step a and clearly the velocity C. So, you must write limit a tends to 0 tau tends to 0 such that a over tau = C that will tell me the speed. Now what about the rate of reversal well reversal is measured by the probability beta so, clearly you also want limit beta tends to 0 tau tends to 0 such that beta over tau = Nu this is the rate of reversals okay or lambda in the case of the dichotomous process the parameter I used.

I will use Nu because I want to distinguish this from that lambda do not want to confuse it with a dichotomous process all the time. So, let us call it Nu so I put those limits in here I put this in here I write j times a n times tau and then subtract from it the n - 1 part and so on and write differential equations now. And then with a little bit of manipulation exactly as we did in the case of the diffusion equation you end up with the following.

You end up with Delta over Delta t so let me call this let us use some shorthand notation del t + C del X these are partial derivatives with respect to X and t of PR of X, t this is a probability density function whereas these are probabilities this is the probability density the positional probability density function at time t to be at the point at the point X in the right moving state. What is this = what can this possibly be =.

Now comes the gain and loss terms because we are writing a rate equation Nu times PL of X , t - PR of X, t that is the only possibility remember this guy here is like the total derivative it is like the convective derivative this Delta over Delta X this is Delta over delta t so d over dt of this guy must be = this is the gain term that is the last one and the other one has a velocity - C so it is - C del X dl X , t Nu times now this is PR those are the two equations rate equations okay.

And they are coupled the couple to each other so we need to solve these equations and these are the solutions for dichotomous diffusion the total probability P of X, t where you have integrated or summed over the both the velocity states + C and - C that when you integrate over X must be = 1 at all times right.

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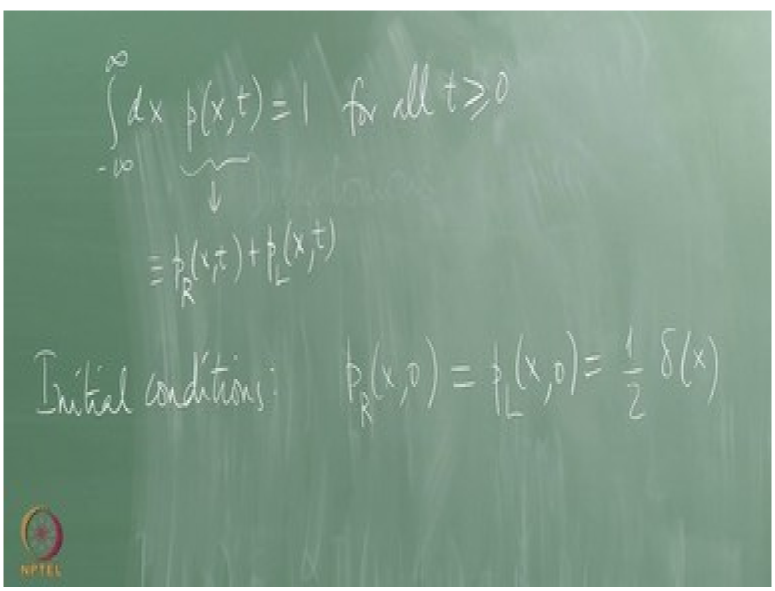

So, we want this condition normalization condition integral - infinity to infinity dx p of X t =1 for all where this quantity by definition is pR of Xt + pL that should be normalized. What are the initial conditions well this depends on what conditions you would like to specify. So, let us you do our usual kind of specification we say that the particle starts at X = 0 at t = 0 right. Now you could put whatever initial conditions you like but the most symmetric ones would be to say it starts from 0 and it starts with equal probability moving right or left you do not care right.

So, the initial conditions this would give us symmetric nice symmetric solutions incidentally you can solve this for any you can solve this set of equations for suitable initial conditions and boundary conditions could be quite general. But let us look at the simplest cases. Initial conditions would be pR of X 0 = pL of X 0 = half of them half a delta function okay half in the left state half in the right equally probable right.

So, when you add the two you get a delta of X and you integrate it you get one. But we need conditions on this because these are equations which involve derivatives you need conditions on Delta deltas as well right. Because you see what you can do to solve these equations is to eliminate one of them. You can eliminate pR or pL and get another equation for it right. So, let us do that and you see what the equation looks like.

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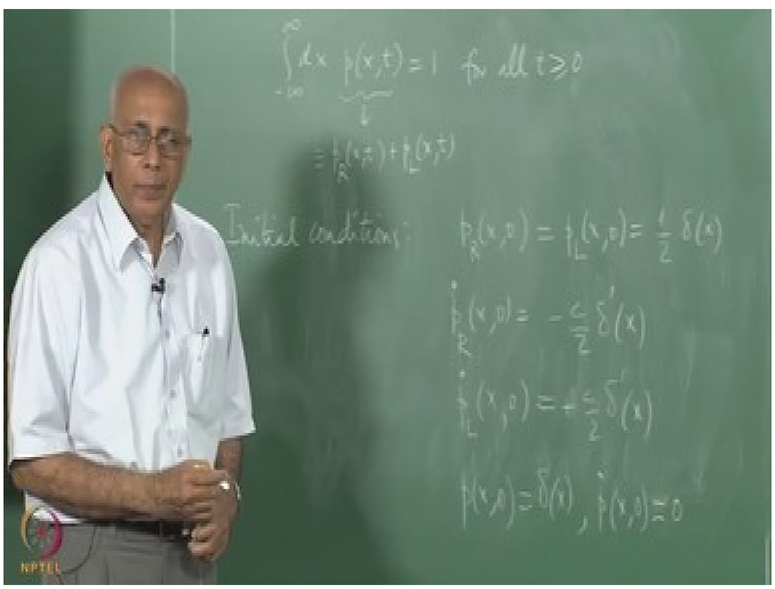

By the way this guy here starts looking suspiciously like the wave equation is going to emerge right. It looks like one factor of this wave equation so what should I do I want to eliminate pL so the thing to do is to I know what this operator on pL is so let us put this operator on both sides of this equation here. So, del t - C del X del t + C del X pR = Nu times this guy on pL that is this guy which is this.

So there is a Nu squared pR - Nu square pL that takes care of this fellow here and then I have to take this operator and act on this so that is = - Nu del t - C del X on pR right. Now what does this give you that tells you the second derivative partial derivatives and then there is a Delta C times del t del X and a del X del t but it does not matter which order you do this and those cross terms cancel and then you have C squared with a - sign del X X second derivative not the first derivative squared but second derivative of the derivative.

So, it says del t t - C square del X X of pR = Nu squared pR - Nu del t pR + Nu C del X pR - Nu squared pL now what should I do for that pL what should I do to get rid of that pL I want an equation to eliminate pl I use this, I use this equation for pL right. So, I have a formula for Nu times pL so let me write this to be = - Nu times whatever is there for Nu pL and that is = Del t + C del X pR + Nu pR and if there is any justice in the world this says new squared pR cancels against this.

Then a Nu C del X pR cancels this fellow cancels against this and you get this 2Nu del t pR. So, we finally get an equation it says Delta tt - + + 2 Nu Del t - C square del X X = 0 you get a second order hyperbolic equation now this looks like the wave equation except it is got that reversal term Nu. What would be the equation for pL, pL differs from PR by very symmetric thing the C becomes a - C in that state right.

But this equation here is square in C so what would you expect is the equation for pL the same equation I expect exactly and therefore for p and therefore for this guy too I expect them to obey the same equation how do then differentiate this between these solutions but the initial condition is the same for both of them the initial condition is the same that looks like it says pR = pL and that is not true that is not true right.

I agree with you that this is going to be pR , L = 0 exactly the same equation but me why does it give me two solutions no once I solve that equation I get solution is unique once I specify the initial conditions what is the order of the differential equation in time it is second order so for either pR or for pL I need one more initial condition on the derivative the time derivative right. I need to know what is pR dot and 0.

How do I find that, how will I find that how will I find that initial condition I have I have all the data here I have this here feed it into the differential equation feed it into the differential equation at t = 0 after all the differential equation should start be satisfied even at t = 0 right. So, feed it in and what do you get you get pR dot of X, 0 + C del X and this is half Delta of X and each of these is half Delta of X or t = 0 they cancel.

So, what is pR dot at X, 0 move this to the right hand side. So, - C over 2 Delta prime of X why not, I define, I I am not defining the square of a delta function you find its derivative it is a very singular object yes but I define it by integration by parts always after all how do I define an integral of a delta function I select the value or the Delta function fires for a derivative I do integration by parts on the test function think of it physically.

What is the Delta function it is a function which looks like this goes up very sharply and comes down in the limit what does the derivative of this function look like? This is an even function so the derivative is got to be an odd function. The slope at the origin is 0 for this so the derivative vanishes at the origin right and what does it look like. Well the slope is increasing very rapidly going to 0 and then decreasing very rapidly and coming back to 0.

So, this function looks like this very sharp what would the second derivative look like goes up and down a few more times than they lose to 0. So, it is getting more and more singular I agree but it is defined. After all one way of defining a delta function or any distribution is through its Fourier transform. So, the inverse Fourier transform of a constant is a delta function and differentiation is = multiplication by k.

So, the inverse Fourier transform of ik is the Delta prime apart from some sign and then ik squared is the second derivative and so on. So, I can define those things so it is - C over 2 Delta prime X on this side and what is pL dot of X, 0 what is this = + C over 2 so I expect the solutions to be different pR and pL because the initial conditions are different but they both obey this equation and the initial condition.

On p of X t itself is even more different right, so p of X, 0 = Delta of X p dot of X, 0 = 0 it is not going anywhere these two add up so these are three different solutions to the same differential equation. Now you are faced with a thing like this what would you do next it is not the wave equation because there is this guy sitting here. Now what is the effect of this Nu what is it physically doing actually.

What is this new comes from the dichotomous Markov process the velocity is reversing all the time right. So, it is preventing this guy from becoming ballistic motion every time it is moving fastest is reversed it is not getting anywhere right. So, it is acting like some kind of dissipation in that sense. So, you would expect that if I get rid of that dissipative factor then you would have something like the wave equation possibly.

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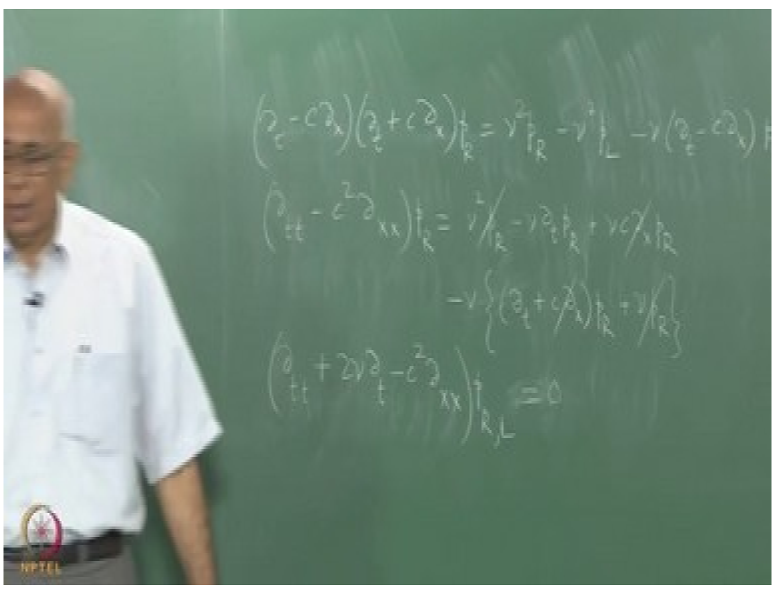

Now that can be easily arranged incidentally does this start looking like the diffusion equation at all is it like the diffusion equation ever. Well if you get rid of this it is exactly like the diffusion equation so we keep that in the back of our mind it looks like if you let C tending to infinity which is very reasonable because I remember I said that the formal velocity of a Brownian particle is actually infinite.

And C came from a over tau but the diffusion constant and the diffusion equation came from the limit a squared over tau a over tau is infinite and a square you put in one more a which goes to 0 and then you get a finite limit right. So, it is very reasonable that C tending to infinity and Nu tending to infinity such that C squared over 2 Nu tends to diffusion constant D gives you the diffusion equation back again.

So, you should keep this in the back of our mind that the moment you let this limit you take this limit in the solution to that equation we should be able to recover the diffusion equation solution the Gaussian solution. But right now the solution to this the envelope p is or R and L these are not like the Gaussian at all. But the central limit theorem still operates and you will in the long time limit you should be able to get back the diffusion equation.

So, we will come back we will come back to this meanwhile since we have a guess that this Nu is creating this problem this term here and there is dissipation the problem.

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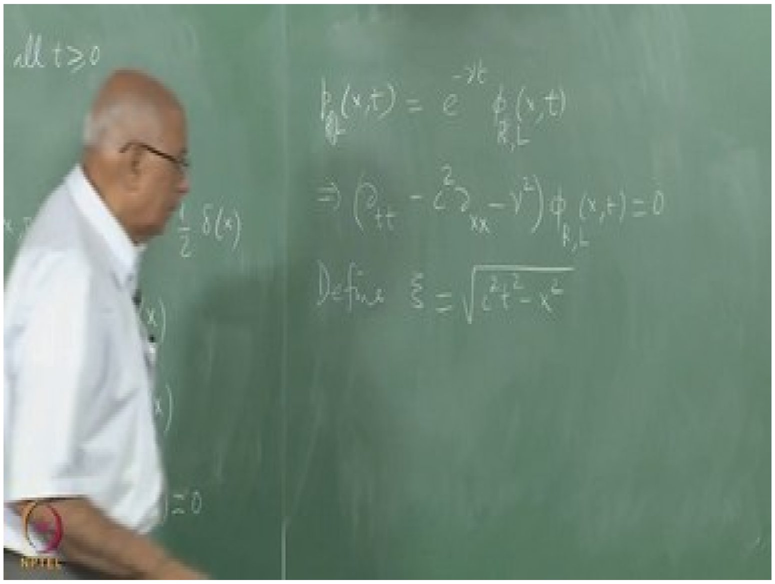

So, let us set P R of XT = e to the - Nu t some Phi R X and t do the same thing for L there is got to be some damping out so let us remove that and then see what happens right. Now this is not hard to show if you plug that in that this term will get cancelled out this goes away. But you will have terms which come from the second derivative of acting on this guy here and that pulls down a factor Nu squared.

So, this is a matter of algebra all you have to do is to put this in here and see what happens to this then let us write the answer down and then this will imply that del tt - C squared del XX - Nu squared ends up in that. It does not change the initial conditions because as you can see this. This guy here I can put and find out what are the new initial conditions etc what does this equation remind you of it is not the wave equation but it is the Klein-Gordon equation.

It is exactly like the Klein-Gordon this is like the box operator in three 3 + one dimensions it is a wave or operator and then there is a constant squared. So, if you can solve the Klein-Gordon equation you can write the solution for this. You have to specify moment to give me anything this operator acting on some function of X t = 0 and if you tell me what the function does at t = 0 and its first derivative does at t = 0 arbitrary functions of X.

I can write the explicit solution now okay. But we need it in this very special case these things here let me write the solution down the solutions look like this. So, let us define sorry to use this variable but it is a standard symbol defines Zi to be = square root of C squared t squared – X. You expect that factor to appear all the time because this is this operator has appeared.

So, this is going to appear now we immediately can see intuitively that since the velocity is never gone is always the speed is always C it is clear that if you start at t = 0 at X = 0 you are never going to go beyond the point C t in time t. Similarly you are never going to go to the left of the point - C t unlike the normal diffusion equation where the velocity was infinite and you started with a delta function at the origin at t = 0.

And it immediately spread out into a Gaussian which had an infinite extent for arbitrary small t very narrow Gaussian it is true but still it was arbitrarily extended here that is not the case the envelope or the diffusion envelope will end at + - C t.

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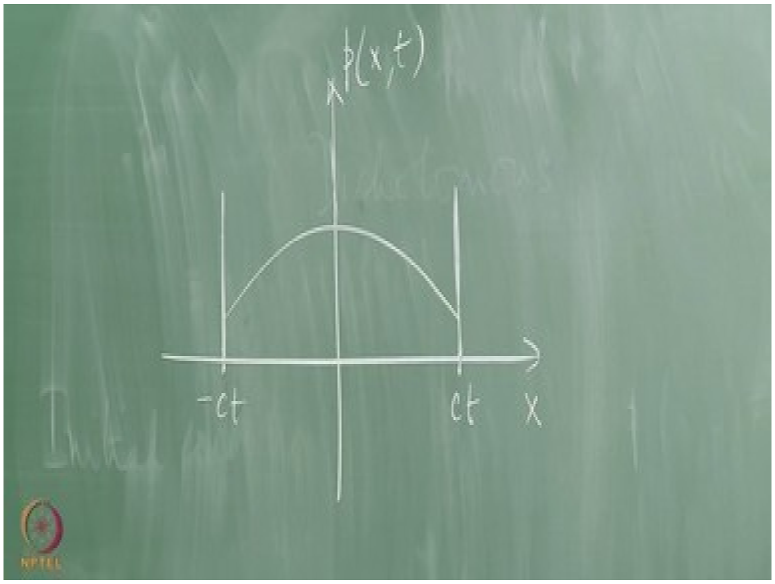

So, the graph will actually look like this so you start with the Delta function here but then as a function of X for a fixed t p of X, t pR and pL will have a symmetry in the following sense it is not hard to see that pR of - X t = pL of X t you have that symmetry at every instant of time with those symmetric initial conditions. Then it is cutoff between - C t and + C t and what you really have is in pR you have a delta function spike.

In pL you have a delta function spike and in between you have some function which does this kind of thing which we will write down.

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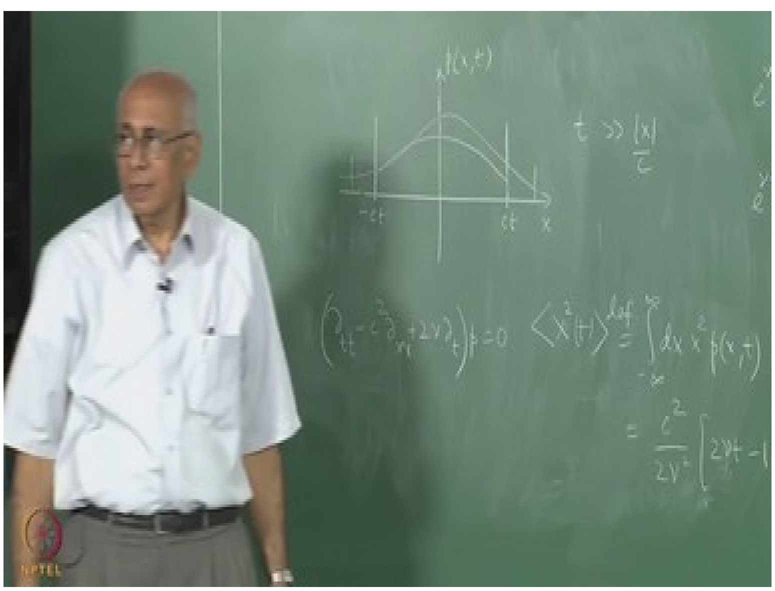

So, the solution it turns out looks like this let me write the solution down the half remains except this Delta function which was half of Delta of X at t = 0 is really Delta of X - Ct as you would expect it propagates this is the ballistic part propagates right that is the contribution which comes from no reversal of sine just keeps going. And then there is the portion which has the rest of it which is + let me see if I remember this right.

An envelope which is theta of X + Ct - theta X - Ct this restricts you to this region here there is a square pulse of unit height which tells you that this solution is restricted to that region times Nu over 4C of i naught of Nu Zi over C what you need is a dimensionless quantity here and because of the symmetry of the differential operator remember that Zi was defined to be = C squared t squared – X squared to the power half positive square root of that.

This has dimensions of t inverse that has dimensions of length. So, it is a velocity and it cancels here + CT - X over Zi 1 of Nu Zi over C these are modified Bessel functions which we have seen many of very often appear very often and random or problems we know what their properties are their entire functions of their arguments and so. And similarly pL of X t is 1 half Delta X + Ct that is this Delta function here + the same thing theta of X + Ct X - Ct Nu over 4C I naught of C the CT + X Zi these are the solutions which satisfy the initial conditions on the two functions pR and pL.

And of course the whole thing is appears is some of and the whole thing is multiplied. So, e to the Nu t times that e to the Nu t times that because there is that damping factor sitting there all the time. So, what happens when you sum the two is that this fellow becomes Nu over 2C times I naught but then this X and that - X cancel out and then you have 2Ct, the C cancels here and you have Nu t over C its approach.

And you are guaranteed due to the presence of these quantities these theta functions here that it is restricted to this region here and of course p of X, t is a symmetric function of X because it does not distinguish between left and right. Now we can check if it will go to the correct diffusion limit or not, that is not very hard to do. But before that we need to also find out does this diffuse what is the mean square displacement going to look like etc.

We do not know what the mean square displacement looks like we need to compute that explicitly so we need to compute X squared average but it is sensible to compute X squared average by not by solving the differential equation but remember only this is an exercise by solving a much simpler equation you know what each of them does pR and pL and what p does we know the differential equation for p right.

So, all you have to do is to argue that x squared of t by the way what is the average value of X of t this is = integral - infinity to infinity dx X squared p of X, t by definition and it is obvious that in this case this fellow vanishes outside + or - Ct on either side so the integral runs from - Ct + Ct times some Bessel functions it is too messy to do it that way. What you should do is to say I start with the differential equation.

So, I have del tt - C square del X X + 2 Nu del t p = 0 multiplied by X squared and integrate from - infinity to infinity or from - Ct + Ct integrate this equation here get rid of this by integrating their parts and then you get a differential equation for X squared average as a function of t and initial value is 0 right. So, you can compute what the actual average is explicitly. So, I leave it to show that this quantity turns out to be C squared over 2 Nu squared 2 Nu t - 1 + e to the - 2 show right.

Differential equation for this guy in time ordinary differential equation and solve it with appropriate boundary conditions to get this solution it is linear in t at long times. What would you expect it will do it as t tends to 0 it should be ballistic it should go like C squared t squared finished and indeed it does because you can see that the one cancels the Nu t cancels the next term is of order Nu squared t squared the Nu squared cancels and you have C squared t squared over two factorial the two cancels and so on.

So, it is precisely C squared t squared for very small t for very large T what do you expect what is meant by large T Nu t much, much bigger than one right the velocity correlation time what is the velocity correlation time in this problem. In the original diffusion problem there was a gamma there and I said the velocity correlation time was gamma inverse what is the velocity correlation time here because we know this is a dichotomous Markov process.

We know it is exponentially correlated what is the velocity correlation time 2Nu inverse 1 over 2 Nu. So, when t is much, much bigger than Nu, Nu t much much bigger than 1 you can drop this you can drop this you are left with this at C squared over this goes away so this goes to C squared. Remember in the diffusion limit we said that the D the diffusion constant was C squared over 2 Nu.

So, this says that it is = 2dt which is exactly what the diffusion limit is so this thing here what do you think happens to it when t becomes very, very large. We are really saying that mod X is much, much less than Ct or t becomes very, very large team comes much, much larger than mod X over C right. So, the condition the diffusion limit is that t much, much bigger than mod X over C but when C becomes infinite this is simply saying that t is positive.

That is what we are looking at in any case right positivity what happens to this profile it moves further and further right but it is multiplied by e to the - nu t because this guy goes to the right hand side so after a longer time it likes like this finally in the diffusion limit this goes away and you end up with a Gaussian. You are back to that and we should be able to see that explicitly.

So, let us see we can see that explicitly you see you look at so you agree that if I write pR + pL and I put the a to the - Nu t on the right hand side and let Nu t tend to infinity the e to the - Nu t the Delta function goes away goes to 0 height and I have only got to worry about these guys. So, let us see what that does I have to make t very, very large in it.

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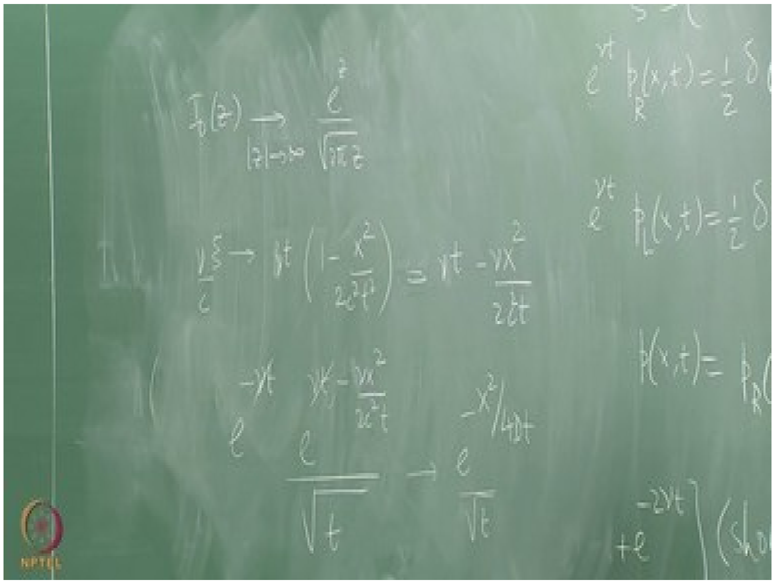

So, I have an e to the - Nu t multiplied by I naught or I1 does not matter and what is it going to look like look at this function here for instance look at I naught what does the argument t is becoming very large that is the only thing that Zi is getting very large. So, as Zi is getting very large because I square root of C squared t squared etc and what happens to I naught when the argument becomes very large.

So, let us write this out properly I naught of z goes when mod z becomes very large e to the z over square root of 2 pi z in fact ij of z does that for all positive integers you know negative integers and i - j is the same as ij and what does Zi do as t becomes very large Ct you take it out and then you have 1 - X squared over C squared t squared to the power half. So, it is roughly this guy here right sorry yeah so this is = Ct - X squared over 2 Ct =.

So, Zi goes to this for very, very large C, so it says Nu Zi over C becomes beauty Nu t right. So, our solution looks like e to the - Nu t and then e to this guy so it is e to the power Nu t - Nu X squared over 2C squared t divided by square root of whatever it is this is the leading term Nu t becomes very large so apart from some constants it goes like this. This cancels against that and you are back to the Gaussian solution e to the - Nu X squared over 2C squared C.

But remember that C squared over 2 Nu was the diffusion constant so this is twice the diffusion constant here and there is another two here so it is 4 dt so this guy really is going to e to the - X squared over 4 dp over square root of 2 apart from some constructs. So, that is why I said that that envelope becomes a Gaussian goes to the Gaussian. So, we have a very fairly complicated process but the fact remains that it finally at the end of the day reduces to the diffusion equation.

In the diffusion limit you know well defined limit right but it is a much smoother process than Brownian motion is much more tractable and so on.

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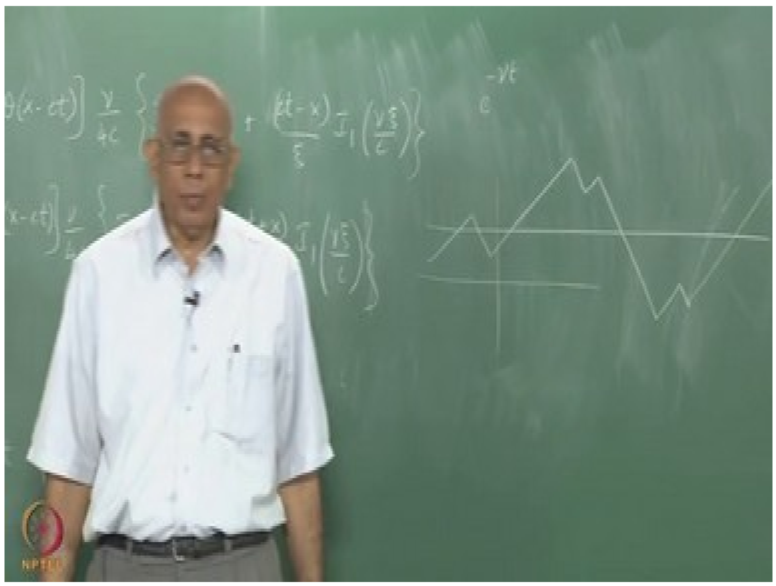

In particular we can now answer the following question we have this thing going up and down we have it is a sample space its sample paths look like this and then the same slope over again and we really have when we found pR and pL we have actually found the Joint Distribution in position and velocity because the velocity is either + or – C. So, we have really found P of X ,V ,t and once we have done that and is not a Gaussian.

There is some crazy Bessel functions right once we have done that we know that the variances of these guys are finite at any time t. So, we can find what is the rate of level crossing. We can ask if I put a threshold here how often does it cross what is the mean rate. In particular how often does it cross 0 itself and all those formulas we derive for level crossing can be applied immediately to this right. So, do that and we will write down the solution next time?

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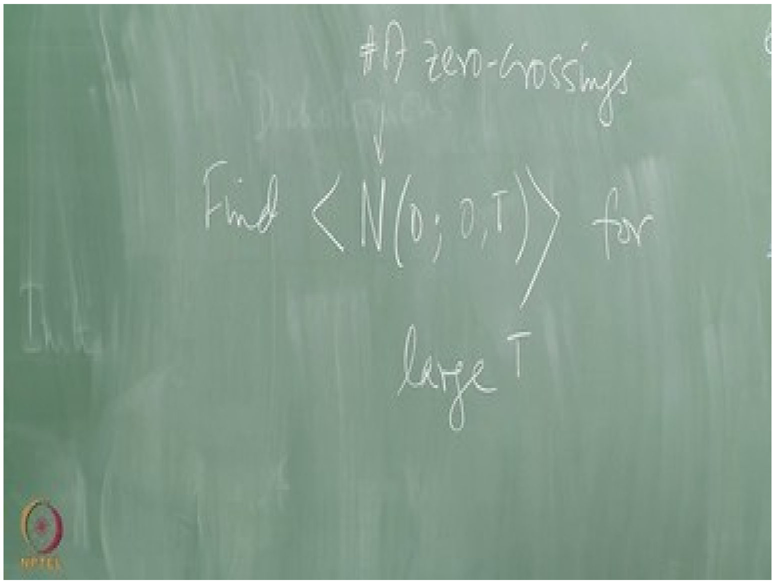

So, find let us take a simpler case let us take the simplest instance so find the average value of the number of times it crosses the threshold 0 in a time interval 0 to t, number of 0 crossings. So, if you are looking at pR it is going to give you the up crossings and pL will give you the down crossings and you can compute it explicitly you can compute what is the exact contribution in this case.

I put X = 0 as a threshold because it simplifies the formulas because you can see that if you put X = 0 this is Zi becomes just Ct and then it is much easier to handle in this case. So, use those level crossing formulas to discover what this is and then we will be able to say on the average how many crossings will there be in a time interval t how will it go with t. What is your guess so the question I am asking is if this was a 0 if this was a 0 level in X here as a function of t.

The question asked is in a time interval 0 to capital T how many on the average crossings are there of this up and down put together it will of course be an increasing function of capital T the question is how is it going to increase what would be your guess we will go like t will go like e to the power t will go like square root of t what would you think. So, work this out work this out of course as t becomes very large it makes a huge difference.

What power of t's is a crossing but it is explicit here you can actually get an explicit formula and we want the large t limit find this thing as a matter of fact you can even find the variance of this number that is not really very much harder to do even compute the variance of this number although that is a little more intricate. We did not talk about it in level crossing but you can compute even the variance in this case.

So, I want you to find out what is the power law this increase is like a power law no clearly not exponential but increases like the power of t then the question is what power is it going to be. So, that is a useful thing to know because whenever you model something by a Dichotomous Markov defuse your process of this kind by the way this random walk is only a paradigm I mean this X is not really does not have to be positioned.

It could be anything could be a large number of things just like a Poisson sequence will modify our model all kinds of things models the rate at which telephone calls and I rate at which people arrive in a public place and it is a very large number of possible applications. Similarly here this process this negative integral of the dichotomous process it is also the Dichotomous Markov model is a large number of random phenomena.

And it will be interesting to find out what are the threshold processes what does it look like so do this and then we will take it from here and I want to emphasize again that p X in this case is not a Markov process it is definitely not a Markov process. On the other hand X, t together is a Markov process so it is jointly a Markov process and we alone is a Markov process it is the dichotomous Markov process.

So, this is exactly as it was in the case of ordinary diffusion as well but there X was also the integral of white noise was also a Markov process in this case the integral of a dichotomous Markov noise process is not a Markov process. But together the to form a vector process definitely is all right. So, let me stop here by the way I forgot to mention that the dichotomous Markov process is also called a random Telegraph signal with some modifications of rates and things like that.

And these two equations we wrote down the coupled equations for pR and pL they call the telegraphers equations because they also appear in the equations for all transmission lines we are not interested in that aspect of it here. But you might see in the literature the telegraph was the equation see the equation with the 2 Nu del t in between that is the telegraphers equation ok all right. So, let me stop here today.