Physical Applications of Stochastic Processes

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Lecture-21
Random pulse sequences

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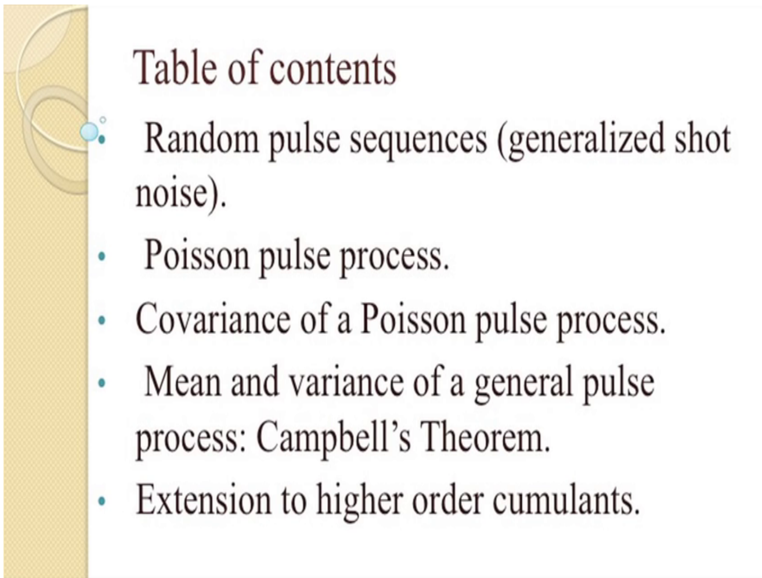

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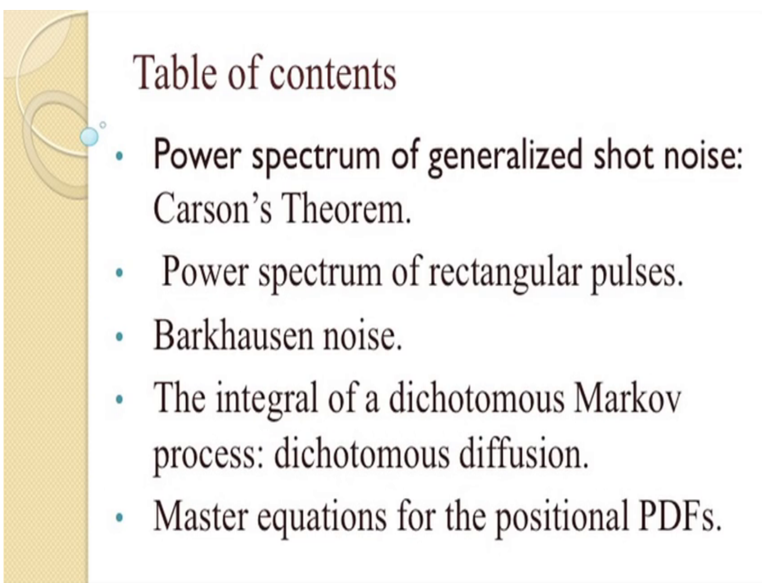

Today let us look at some properties of noise in general, in particular I want to look at the phenomenon called shot noise and we wouldd like to study the statistics of this because of the very large number of applications that such a process has. Now to recall a few things to you we looked at what a Poisson sequence did Poisson pulse sequence namely we said if you have a completely uncorrelated set of epochs or instants of time distributed according to a Poisson distribution they form of personal sequence.

And then some properties of this Poisson sequence we studied in connection with statistics. But now let us look at a more general version of this and that is the following.

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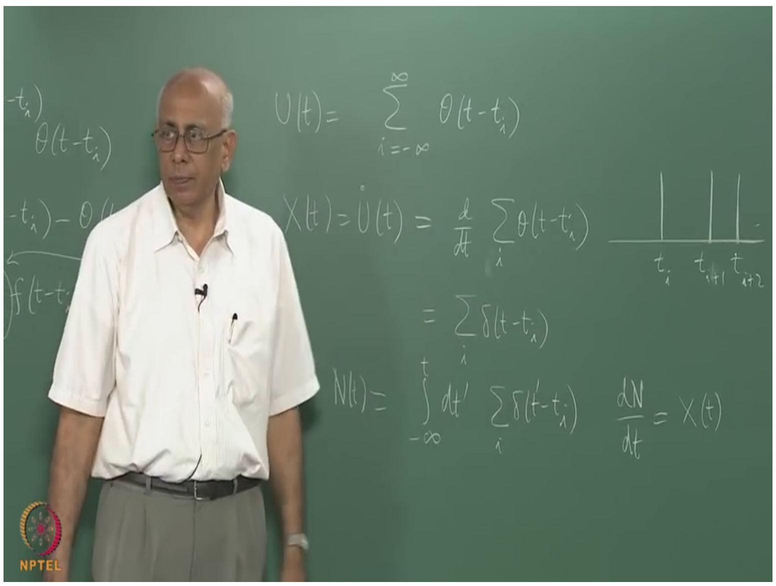

So, as a function of time let us suppose that at random instants of time you have an event occurring, a pulse occur like neuron all spikes from the brain activity or radio signals from outer space or whatever some kind of random pulses appearing at random instants of time. The only proviso being for the moment that they see the times at which these pulses are completely uncorrelated with each other.

So, in general you might have a signal that perhaps looks like this there is sudden spike and then any case and then a little later there is another one etcetera this fashion. And these pulses occur at random instants of time let us call it ti, ti + 1, ti + 2 and so on and it is assumed that this has been an ongoing process for a long time okay. Then the question one would like to ask is what is the statistics what kind of statistics or does the stochastic process have corresponding to these signals that you detect.

And let us suppose that these pulses are cut at times t sub i and let me call the shape of each pulse let me call it f of t - ti that is the shape of the i'th pulse so it starts at time t, ti and then goes on decay's could be a finite duration could go on forever could exponentially decay it is a ti we do not care so the pulses the shapes could actually overlap with each other we do not care about that but the starting instances instants are completely uncorrelated with each other.

So, the probability so this sequence ti is a Poisson pulse we will assume that each pulse has a shape given by this function t – ti, so this f of t is not 0 could be taken to be 0 for t less than ti starts at ti and then it goes off in some fashion compact or otherwise typical examples would be for instance if this is an exponentially decaying pulse with time it could be of the form some e to the - lambda t - ti multiplied by some height or whatever but it is also multiplied by a step function t – ti.

That is one typical shape or it could be something which is a set of pulses which looks like this and so on rectangular pulses of some kind. So, perhaps it is of the form theta of t - ti - theta of t - ti – tau, if tau is the duration of the pulse this quantity is 0 everywhere except between this and that the pulse and that is it is given by this difference of theta functions step functions. So, we have in mind a signal so Zi of t this is what is detected and what you have is all these pulses getting detected.

So, this is = a summation from i which could be from - infinity to infinity you might have an infinite train of these pulses times f of t - ti and they could be of different heights we do not care so let us put a height variable there let us put H some Pi and the H sub i could themselves be random drawn from some independent distribution. So, these are random pulse heights you could generalize it.

Even further and say even the shapes of these pulses could change over the pulses we are assuming here for simplicity that the pulse is exactly the same shape each pulse is of the same shape we have allowed for a little variation by changing its height for instance. In the case of these rectangular pulses for example you could say even the duration of the height pulse could be different from that of the j pulse so, perhaps you have this right and so on.

So, a lot of general possibilities occur and the question is what is the statistics of this is Zi this process Zi. What is the first order statistics what is the power spectrum what is the correlation function and so on this is what we would like to find out. Now if you go back a few steps you realize that the way to get this at this process is to say all right let us first look at simply the Poisson process itself.

Let us just look at something which is a every time there is a ti you add 1 to it that is it. So, that process let me call it U of t = summation from i = - infinity to infinity theta function of t – ti, now what does this process look like well it is just a step every time there is a it crosses one of the ti is it adds one to it and keeps going so it is a staircase function that is increasing. So, the first thing to do is to find the statistics of this and then we find the statistics of successively of its derivative and so on and so forth.

Now what is its derivative look like let us let us call it X so let us call X of t = U dot of t and what does this look like this is = d over dt of these theta functions over i time t - ti that is = summation over i delta of t – ti, so you have a spike of unit strength a delta function of unit strength at every instant ti. So, what does this look like this function this thing simply looks like this so every time there is a ti there is a spike etcetera.

So this is ti, ti + 1, ti + 2 and so on, you could ask what is the number of pulses up to time t this number would be n of t would be the integral of this from - infinity up to time t because every time there is a delta function it counts 1, so this is simply = integral - infinity up to time t dt prime this guy. So, summation i right and that counts precisely this. What is the rate at which these pulses are occurring well that is the rate at which the original Poisson process is occurring this pulse sequence?

So, let us give it a name let us say this is a possible sequence with mean rate Nu, so the rate at which the pulses are occurring that itself then the number of pulses per unit time is itself a random variable the mean rate of course is given to be Nu, so that is clear from here that the n over dt the derivative is precisely delta of t - ti this is precisely or X of t because if I differentiate under the integral sign this gives me a 1 and I put t prime = t and I get just X of t.

What is the average value of X of t that is the rate at which the pulses are occurring so the average value is Nu that is immediately clear right.

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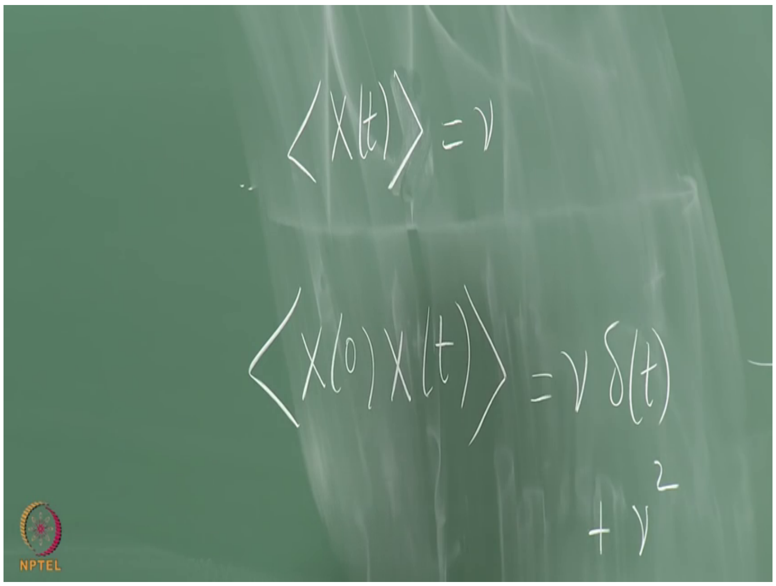

So, let us write that down so it says X of t = Nu is X of t a stationary stochastic process, is the stationary because remember the original pulse sequence is a personal process and Nu is independent of time that rate does not change with time. So, the question the way you decide whether a stoker a random variable is stationary stochastic process or not is to ask whether X of t and X of t + constant have exactly the same statistics or not.

And it is clear if I just add a constant to X of t the statistics does not change at all at any order right. So, X of t is a stationary random process strictly stationary random process because we have random processes which are strictly stationary which means the statistical properties simply do not change none of the distributions changes when you have constant added to the random variable or wide sense stationary which means that the mean value is independent of time and the two-point correlation is a function of the time difference alone but higher order correlations might not be.

So, this is strictly stationary this process is strictly stationary here and this quantity X of 0 X of t this guy here can be computed I am going to leave this as an exercise to you and tell you how to do this give you a hint and tell you how to do this you can actually compute the autocorrelation of this process and this turns out to be Nu times Delta function at t + Nu square. Now a word about how you actually compute auto correlations of these, from of stationary processes. First of all if a process is stationary its derivatives are stationary and so on and so forth.

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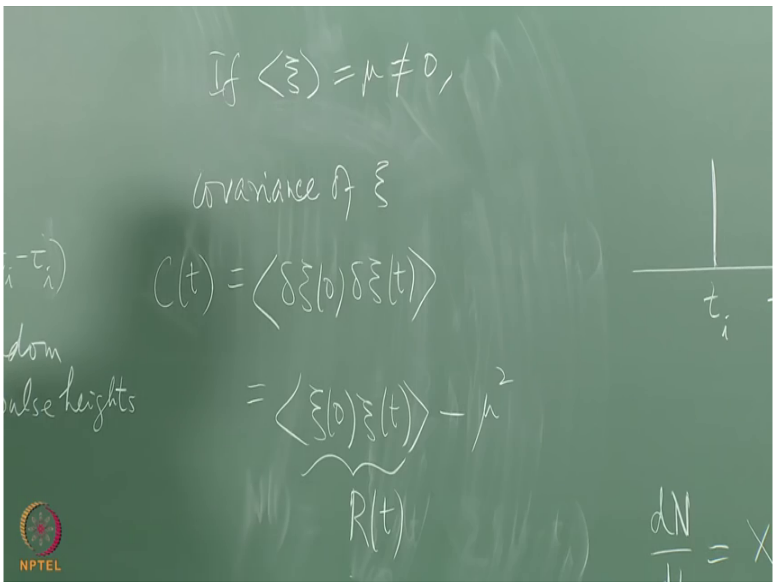

Now the way to do this is if you give me a call let me just call the random variables i for the moment so that Zi of t has a correlation function. Let us call this correlation function I use a symbol for it right you see for it C, but you got to be careful there could be a mean value if this is I does not have a non 0 if it does not have a 0 mean but as a non 0 mean then the correct way to define the covariance of this object is deltas i of 0 Delta's i of t as you know right.

So, let me let me write this down if average value of Zi = Mu is not = 0 then the correlation a covariance it is called the covariance = Delta is Zi of 0 Zi of t which is = this quantity is Zi of 0 Zi of t - Mu square the average value we know that or lower again we have gone through this and we define the power spectrum as the Fourier transform of this fellow here. That is not the same as the, this is the correlation function and that is the covariance function.

So I need a slight change of notation or to be a little careful so let us call this C of t and let us call this R of t that is the standard notation books on random processes. So, C of t and R of t a slightly different objects they just differ by this guy here always.

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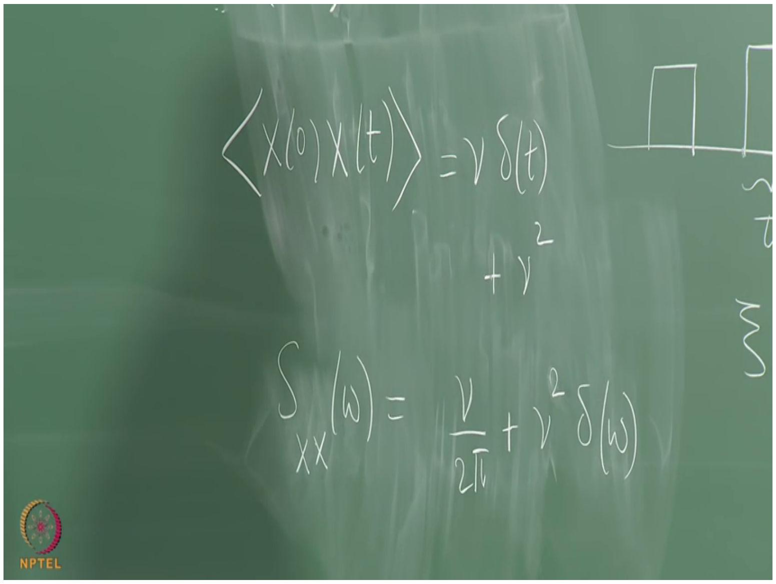

Now the way to do this is the following if you have a stationary process then R of Zi well let us be even more careful this is R Zi, Zi of t that means is the correlation of Zi times 0 which i at time t right. Then the first step is Zi i dot so the correlation of Zi of 0 which Zi dot of t this guy here is = - d over dt or Zi Zi and R Zi dot Zi dot - d2 over dt2 R Zi Zi, so I got to differentiate each time which is why when you do the power spectrum you multiplied by Omega by i Omega each time.

So, let us not get into this thing here but let me just point out that to find this result for the Delta function sum of Delta functions start with the theta functions and find its autocorrelation and that is easier to do. And once you do that twice differentiating it will give you this. So, I am going to leave this to you as an exercise you must remember that the derivative of a delta of theta function is a delta function you need to remember that that is where this comes will turn out to come from.

So, whatever let us take this result it is easy enough to establish? So, let us take this as our basic result and then what do we want to add on to this you see now I am going to simply motivate it heuristicaly we can go through the derivations but it is not particularly enlightening me the result is very clear. We want the autocorrelation we want Zi of 0 sorry this guy here, what this implies but before that let us write the power spectrum of this fellow out.

So, this is SX again let us use proper notation both are X this guy here is a Fourier transform it is simply 1 over 2 pi integral - infinity to infinity dt e to the i Omega t of this. So, this fellow will give me Nu over 2 Pi + what does this give me yeah it gives me a delta function of Omega right because it is just 1 over 2 Pi e to the i Omega t - infinity to infinity and that is a delta function that is the power spectrum we have the autocorrelation.

Now the question is what is the autocorrelation of Zi given that of X and one can sort of guess what it is going to be in fact let us guess.

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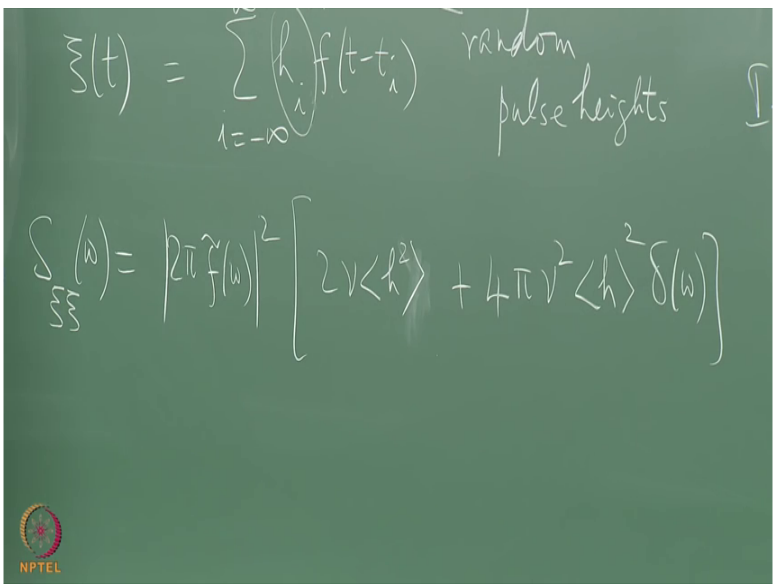

What the let us guess what the average value of Zi of t is going to be I defines Zi of t in this fashion. You want to know what is its average going to be this is an independent random variable so this average is going to come out when you take averages and then you need to take the average over the random variable ti. So, it is clear that somewhere along the line you are going to get an integral of this because we want the average value of this F.

But now we appeal to Ergodicity namely you take the time average of F and that is going to be the ensemble average okay. So, this definitely going to be and then it is multiplied by the rate at which these fellows have happen because the average for the process X of t is itself Nu. So, there is definitely going to be a Nu and then there is going to be an average of h in this multiplied by - infinity to infinity dt f of t.

Now of course remember that f of t might be 0 up to t = some given value some say 0 and then it drops down in this fashion so this integral is formal is not going to run up to - infinity because the pulse starts at some point and I said you could define f to be 0 for negative values of its argument and that is it that is the average value of Zi. What is the variance of Zi going to be, this is Zi squared and it is square and this is = now this is a non trivial statement it is definitely going to involve the rate Nu.

Because it is all controlled by this guy by this process and then we want the square of this fellow here and then you want this function squared inside. So, this is = and this is a non-trivial result its Nu times h squared times integral - infinity to infinity dt f of t and it is squared inside this requires proving this is not a sphere trivial statement and together these statements are called Campbell's okay.

But physically it is very clear where these terms come from so in particular the fact that Nu appears here multiplicatively outside in all of them. In fact there is an extension of Campbell's theorem to the case when this function f itself has to my ensemble averaged the shape itself might be different and so on. And as a further, there is another extension of this even in this case to the higher cumulants.

It turns out that f Kn is the nth cumulant of Zi then kn turns out to be Nu times h to the n inside there and then - infinity to infinity dt f of e to the power n it is quite a remarkable thing that that is all it is you can write down full statistics of this Zi. We still not come to the point of what is its correlation it is a stationary process so all these averages are time independent explicitly. We are also in real time averages, this is independent of time.

So, we have complete first order statistics here by Campbell's theorem and its extension. On the other hand the fact that you know what this fellow is what the power spectrum is for X also tells you what the power spectrum is for Zi itself and this implies that the power spectrum of Zi and let me write it as a Zi Zi of Omega this is = now depends on the Fourier transform conventions but let me be careful here this is the 2 Pi f tilde of Omega mod squared.

You would expect that because this f is going to appear everywhere and it is got to be quadratic because it is a power spectrum going to be two of these fellows sighs multiplying each other times 2Nu average value of h that is what this is going to give me and it comes from this term + 4 Pi Nu squared I sorry this is average value of X squared and then this is h squared Delta Omega that comes from here and this goes by another theorem which is familiar to electrical engineers what is this called?

Well the one for the mean and the variance are called Campbell theorem and this one is called Carson's. This term as you can explicitly C arises there is a wait here at Delta this there is a delta function at Omega = 0 there is a DC contribution coming entirely from the fact that the average value of h may not be 0 that is 0 this is gone. By the way what is the power spectrum for the process which is again controlled by a pulse process, for some pulse process?

But where the variable takes has two values say + 1 or - 1 or + C and - C the dichotomous Markov process what is the power spectrum going to be for that.

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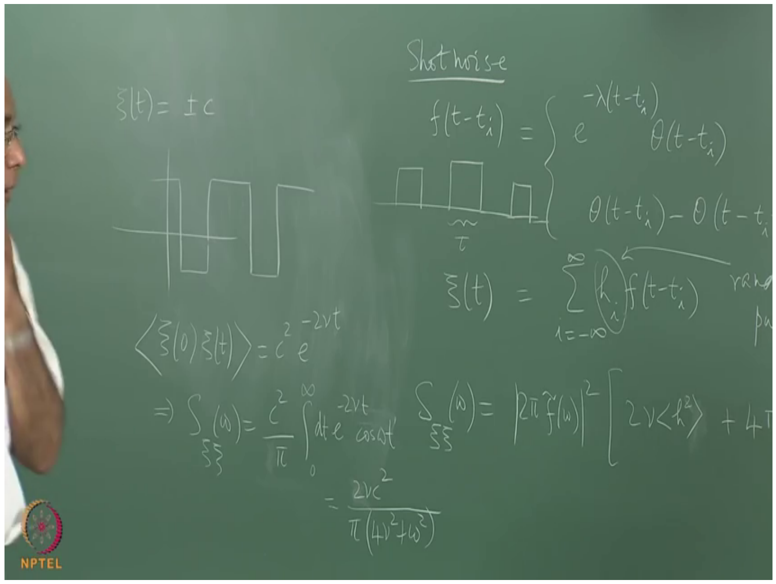

So, in that case you have a process Zi of t = + or - C and it goes like this that is with the rate which is got to be prescribed to you in some fashion the rate we use a symbol lambda for it but let us use Nu right now because these things here are controlled by a Poisson process assuming that both the down to up and up to down plot transitions have controlled by the same Poisson process okay.

So, we know that Zi of 0 is Zi of t = C squared e to the - 2Nu t 2Nu modulus t, so for t greater than 0 it is exponentially correlated what is the power spectrum therefore C squared over 2 Pi and then the Fourier transform e to the i Omega t etcetera. So, there was a twice you could do the cos Omega t by integrating 0 to infinity, so 0 to infinity dt e to the - 2 Nu t cos Omega t and what does that give you this is = 2Nu C squared it is Laurentz's here, this Lorentz's in this case. So, we can plug in all kinds of functional forms here and see what this looks like the power spectrum of this noise looks like.

Now once you have this they are in place that is it an image, let us look at the simplest instance there is a short noise the original short noise right. Now the whole thing I should have titled the whole thing short noise because it is general short noise in general that means a set of pulses of some kind uncorrelated with each other that was our assumption. But the original short noise was the noise which was seen when you had these electrons coming out of a cathode in some electronic tube.

You collected the current and you discovered the current went up in spikes every time an electron hit the collector you got a spike right it was just pure delta function spikes in the simplest case so it is essentially our process X of t decision except that each time you had a charge and therefore there was a current.

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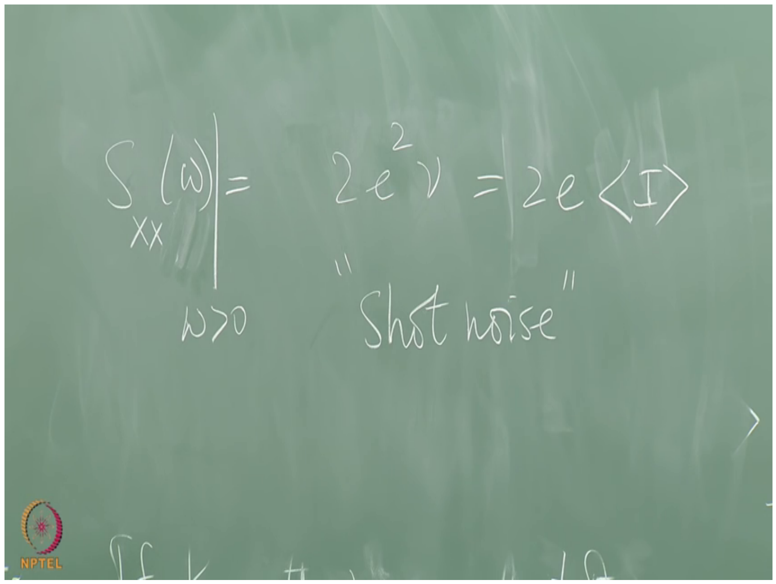

And what is the power spectrum of this going to be in this case. So, if your rate of at which things were happening was your X of t process SXX of t of Omega was = we wrote this down somewhere there was a Nu apart from the 2Pi factor etcetera and then there was a + Nu square delta of Omega right. So, let us look at it for omega greater than 0 it was essentially this and then if you put in there was a factor 2 in the engineering convention of doing things and h was 1 in that case.

So it was basically 2Nu because this fellow here was 1 shape was 1 inside so it became 2Nu and then therefore was 2e squared Nu that is it but this is = twice e times e Nu but that is the average current because the rate at which these electrons are coming in is Nu many times Nu is the rate is a current average current this was the original short noise proportional to e and not proportional to the temperature.

Whereas we saw that for white noise you had a temperature factor so this was the way in which liquid could distinguish the two kinds of noise this was noise was independent of temperature. What happens if you have a square pulse?

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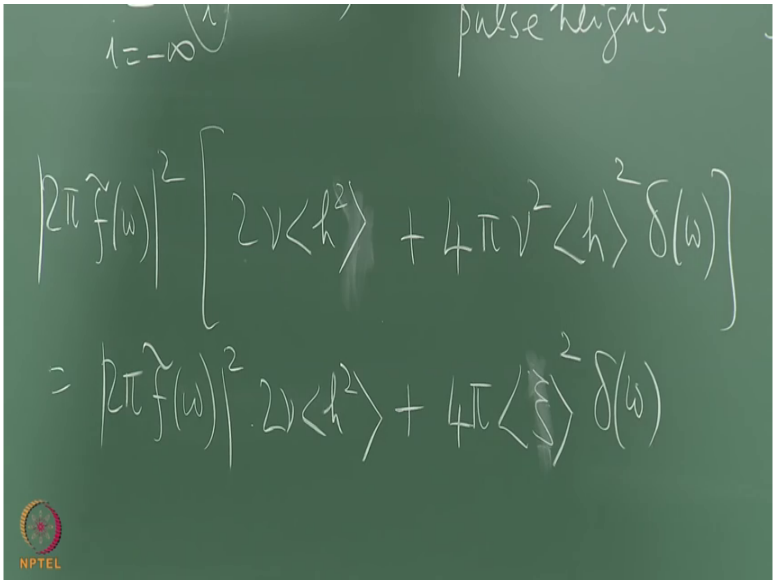

So, let us do that incidentally one can write this formula a little more simply in the following way so let us write that down a little more simply. We will use the fact that Zi squared average sorry Zi average = Nu times average of h times integral - infinity to infinity dt dt f of t but we know that 2 Pi f tilde of omega was the integral e to the i Omega t times f of T so f tilde of 0 is this integral.

So, we have Zi = Nu average h sine 2pi Nu average h f tilde or 0 so this squared is let us write this as 2 Pi f tilde of 0 whole square Nu square h square so this thing here becomes 2 pi f tilde of 0 omega whole squared and then there is a 2 Nu squared + 2 Pi by the way once I multiplied by delta omega I could write this as f tilde of 0 right. So, we have 2 Pi f tilde of 0 whole squared and then 4Pi squared.

So, the Nu squared is there h squared is there and then 2 Pi this guy is there right. So, this is = 4Pi average Zi square little more convenient to write it like this in terms of the average of Zi itself.

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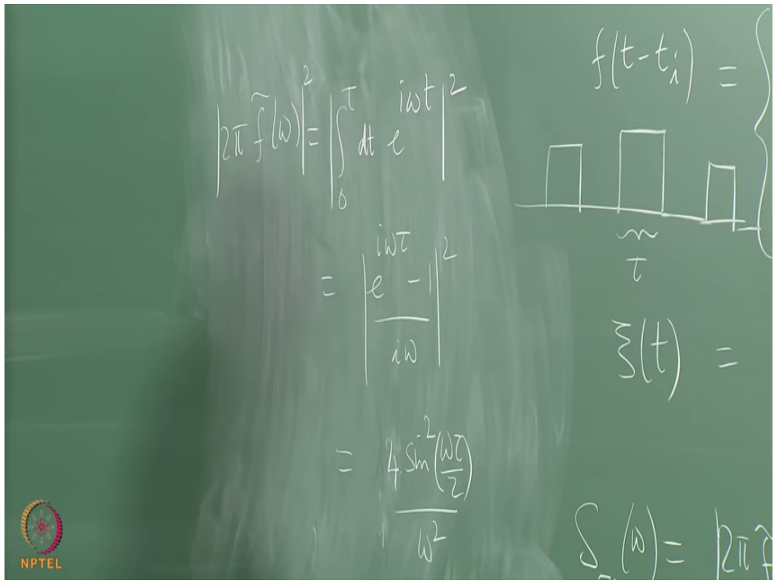

Look at what happens for a square pulse is very, very common situation so f of f of t - ti is this guy and let us take the pulses to have the same. So, what is f tilde of Omega is 1 over 2Pi integral - infinity to infinity or let us find 2 Pi f tau 2Pi this guy = an integral from 0 to tau dt e to the i omega t times 1 also unit height a theta function right. So, this is = e to the i Omega tau - 1 over i. So, we need this fellow here mod squared mod squared we need mod square of this is =, I pull out an e to the i Omega tau over 2.

And then I get mod squared that goes away and then I am going to get 2 sine Omega tau over 2 over Omega, so this becomes 4sine squared Omega tau over 2 squared it is the sync function squared and you plug that in here.

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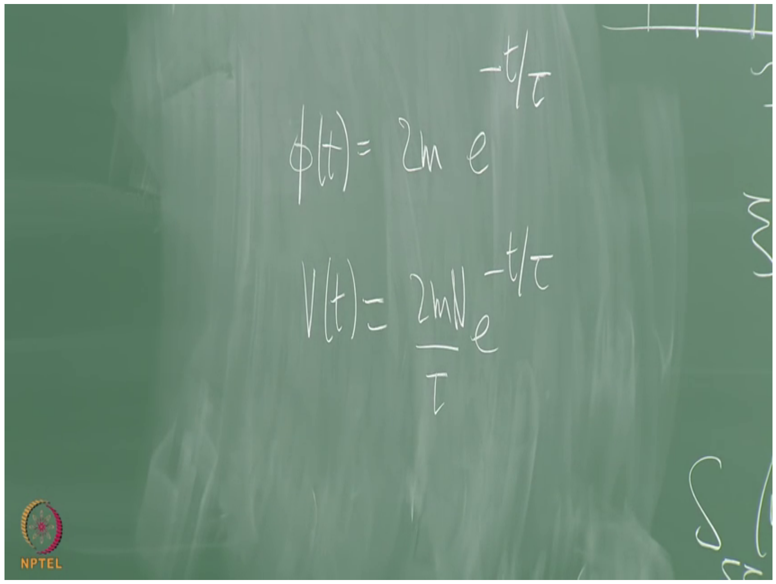

So, for the case of square pulses S Zi Zi of Omega for square pulses rectangular pulses is = apart from various constants and so on. For Omega greater than 0 let us forget the DC contribution it is proportional to Mu times sine squared Omega tau over 2 by Omega square. So, that is a very, very typical result what happens if the tau's themselves varied randomly why do we, in gentle of course it is Nu times h squared here.

What happens if the tau's themselves varied randomly and independently the only placed our peers is here. So, it just get average time in this for the ravaged over whatever distribution you care to prescribe generally with cut off between some lower and upper range of some window of tau's continuous window distributed in some fashion and you can find what its averages. So, let us look at another example a very simple example of physical one if you take a piece of iron and you magnetize it you apply a magnetic field when you jack up the field.

As you go along then you discover that domains inside this piece of iron start orienting in the direction of the field and they do sort random. So, every time it does so the magnetization changes by a small amount. So, in the simplest instance let us assume the magnets were pointing in one direction and under the field and the upward direction they orient upwards right.

So, the change in magnetization is going to be perform twice so if it runs from there is a little domain with magnetization m pointing down and then it appoints up the same domain points up then the change in the flux magnetic magnetization is 2m and this change in flux will die down exponentially. So, there is some 2m e to the - t over tau this is what the flux will look like and now if you detect this, this flux will induce a current in a coil.

And then that current can be used to drive a loudspeaker and you can actually hear it is called Barkhausen and noise. So, what is the rate of change of flux and that is what the voltage is going to be the random voltage. so, V of t is going to be 2m the derivative of this which is going to be tau e to the - t over tau with a - sign that is what Faraday's lord of law of induction says and if you got a coil with n turns there is going to be an n.

So, that is going to be the random voltage that occurs if there is a change of domain at time t = 0 this is your f of t the pulse shape right.

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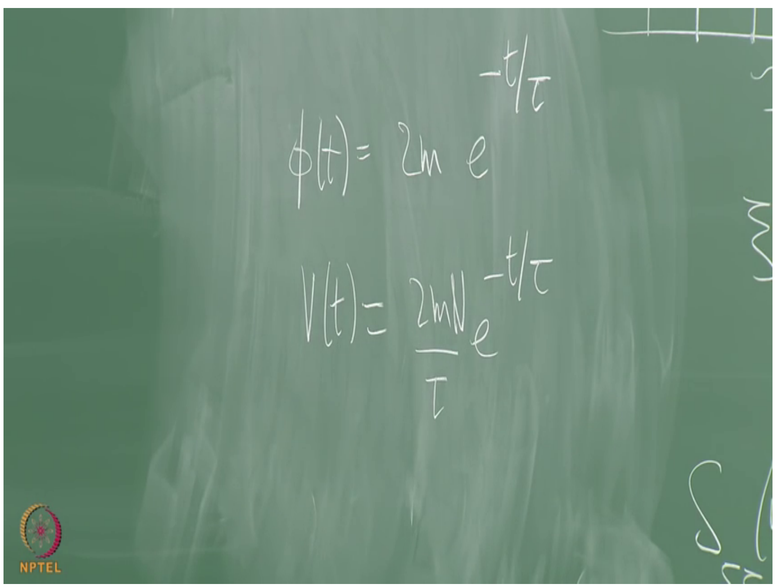

So, what is the correlation going to be now what is the power spectrum going to be, it is essentially like you have to take this quantity and take its transform right that is it f tilde of Omega. So, what I have f tilde are going to be for this guy this is for t greater than 0, so what is f tilde of Omega going to be and what is its model is going to look like Lorentz's same again it is just going to be Lorentz's.

So, this will imply that SVV S Zi Zi of Omega is proportional to once again Nu is going to sit there and then there is going to be Nu over 1 + Omega squared tau square apart from some constants just going to be Lorentz's shape and you can measure it explicitly. So, this formalism these two theorems Campbell's theorem and Carson's theorem actually very, very powerful as long as the pulse sequence is uncorrelated you can get fairly simple closed form answers for the power spectrum assuming the process is stationary and of course the original uncorrelated assumption.

Now it gets more and more intricate when things get correlated to each other and some results are available for the case where the height is say the correlated with the width of the pulse and so on but that is the second secondary problem there is a separate problem altogether. So, I thought I would mention this simply to round off the fact that in practical situations you measure the power spectrum directly and there are these two theorems which help you understand what or no pulse sequence will do.

The interesting problems arise in correlated pulses when you have policies which are not independent of each other in other words you do not have a Poisson process then you have a much more complicated situation and I will make some comments about that little later. So, let us now move on to a class of processes which go beyond the Markov process let us look at some non-Markovian classicism.

And what is the simplest such process we should look at well there is several ways to approach this but since we have been doing a lot of diffusion let us take a problem which involves diffusion itself and ask what happens if I have a position variable for a diffusing particle which will obey a non Markov process which will be a non-Markovian process okay.

And here is the way one does you know that the simplest model of diffusion for a particle on a line such that its probability density obeys the diffusion equation was in fact as the integral of white noise.

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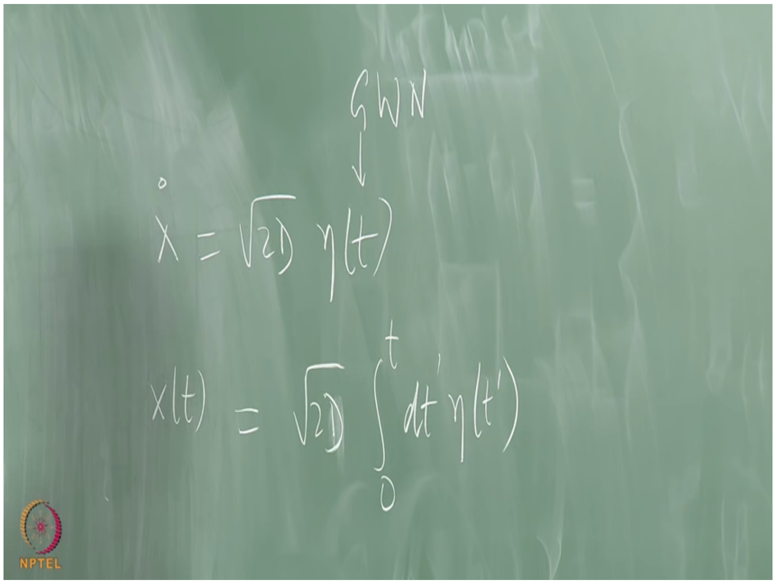

So, we said X dot = square root of 2D times this is Gaussian white noise and then immediately we got a process which was Markovian but not stationary we discovered that X of t the displacement starting from 0 say is 2D integral 0 to t dt prime Eta of t prime and this was the Wiener process of you we call that the Wiener process okay. And we found out what the correlation of this was the average value we found X of t, X of t prime average of 2D and so on.

So, this diffusion process the position was the integral of white noise now I say all right and we also saw that white noise came about I mean we saw that this process came about as the continuum limit of a particle undergoing a random walk on a line in which the step length went to 0 and the time between steps went to the time step also went to 0 such that a squared over 2 tau went to the finite limit d.

So, it was a limit of a simple random walk where the walker toss a coin and with equal probability took a step to the right or a step to the left and we took a limit. We said that the step length so this is a continuum limit of a simple random walk in which a squared over tau when two twice d we also accommodated bias diffusion here by saying that there is a probability of moving to the right or left for different p and q etcetera.

That led to a diffusion equation with a drift term present which took into account the fact that you had a constant force such as gravity right. So, that should be clear that the simple random walk in the unbiased case gave you unbiased diffusion give you this unbiased diffusion limit and the biased random walk gave you a diffusion with drift as if you are a constant force present right. But in both cases the X process was the integral of white noise Gaussian white noise.

Now you could say well suppose this particle should also have some memory of its momentum we are now looking at the process saying that well the position alone is not enough to describe the state of the particle. You should know its position and velocity at any instant of time or position and momentum. Now what would be the discreet way of looking at implementing this is to say well the walker moves to the right or left with different probabilities depending on the direction in which he was moving in the previous step.

So, with probability p he continues in the same direction that he was moving either to the right or left and with probability q he reverses direction okay. Now what is the effect of this it means there is memory in the process the walker is remembering where he came from the previous step? So, it immediately tells you this cannot be a Markov process at once there is some memory right. Earlier the person was just tossing a coin and saying p with probability p I moved to the right probability q I move to the left hand.

You do this at each site and now you say all right I now move to the right or left with proper I now continue in the same direction or reverse direction with probabilities p and q respectively. So, I remember where I came from in the previous step I can write difference equations for this process also and what would happen in the continuum limit it would be correspond to saying that the Walker remembers the direction of motion.

And now if you say that the drift velocity is constant in a suitable limit we can show immediately that the velocity of the particle is constant then it is equivalent to saying that a particle is diffusing with velocity either + C or - C and there is a certain average rate at which the particle changes direction right. Then let us write down what the master equations could be for this process.

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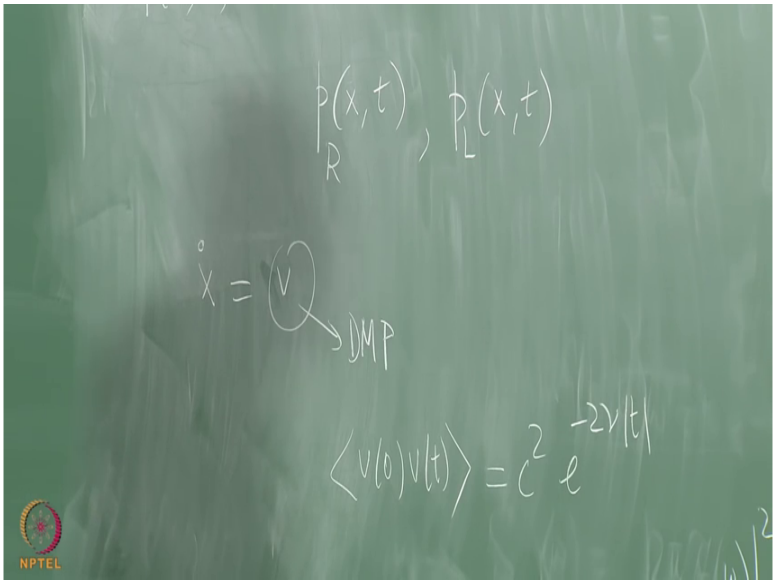

So, I am going to skip going through the discrete and going to the continuum will write down the continuum immediately. So, now instead of saying p of X, t is the probability density of this particular and any given time t right I should also tell you the state of the particle by saying what is the direction of the velocity is it moving towards the right or is it moving towards the left right. So, there is a pR and a pL of X and t.

I need to write down rate equations for pR and pL together and that is done very simply there are only two states R and L for each X so this is like saying I am writing down probability densities for pX V t except that I am saying V is + or - C that is all I am doing right and what do you call a process in which you have a + C and a - C and that is a Markov process it is a Dichotomous Markov process.

So, it is essentially saying that this velocity process is a Dichotomous Markov process so this is DMP and the position is the integral of the velocity right. So, we are saying X dot = V and this fellow here is a DMP with some correlation time right. So, if the reversals are occurring with a mean rate Nu for instance then this process we are guaranteed that V of 0 V of t in this problem this can only have the values + or - C and it is a dichotomous Markov process.

The mean value of the velocity is 0 so this must be = C squared e to the - 2 Nu mod p and now we are asking a hard question we are saying what about the integral of a Dichotomous Markov process just as we looked at the integral of white noise and you got a Wiener process now we are asking what is the integral of a Dichotomous Markov process which has a finite correlation time. So, what kind of correlation time does this have?

It turns out the problem is now non-Markovian because X this process X is not a Markov process on the other hand the combined process X together with V is a two-dimensional Markov process.

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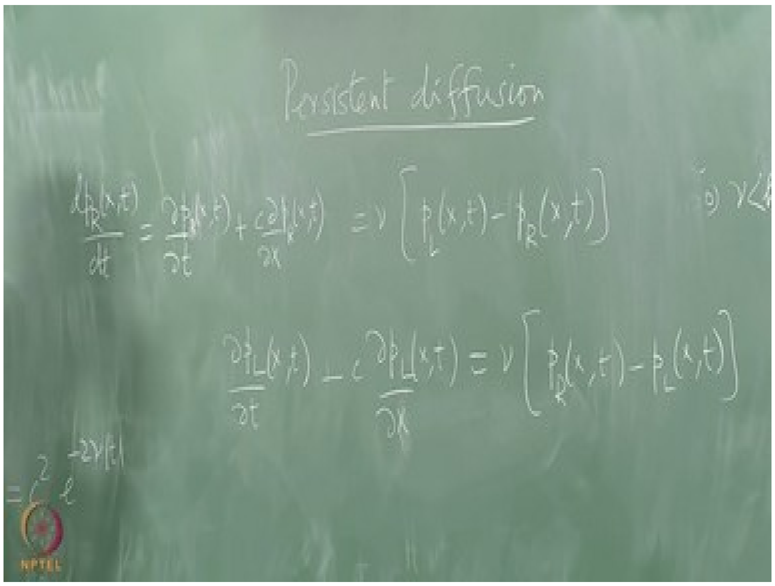

Let us write the equations down for this and you immediately see where this is taking us what is dp R / dt of x = what can this be = whether there is a gain term and a lost term right. So, if at rate Nu you change from L to R so there is going to be Nu times p L of X t but you are also going to lose the fact that you are going to switch out to the other state so this is - pR X t, you should actually write this as pR of X t + delta t and put the delta t is outside there and so on.

So with probability Nu delta t you are going to actually reverse direction and that is going to if you go from L to R that is again for R if you go from R to L that is a loss for R and this is pL of Xt over dt = Nu times pL sorry pR of Xt and that is it but these are derivatives with respect to time total derivatives right. So, what is the partial derivative = Xt + the convective derivative C dot del but V is + C in this state and that is = so that is your equation.

And similarly this equation now becomes delta pL delta t Xt - C delta pLX these are the equations for this problem and this is called persistent diffusion or correlated diffusion. So, our task now is to solve these equations what would you expect will happen if I eliminate pR or pL in favor of the other probability and write an equation for pR alone you get a second order equation in time.

Which means that your initial condition has to be specified for both pR and pL that is equivalent to saying you are specifying the initial position as well as the velocity whether it is + C or - C so it is non Markov the X process alone is non Markov. But the combined process is marked off the X , V processes marker when they satisfy these two equations. So, next task is to solve these two guys which we will do write down the solution.

They look reasonably trivial but it is not all that trivial but they got many interesting properties we will see what this process does not detail but this is the physical motivation the original physical motivation this was introduced by GI Taylor and his original motivation was to introduce momentum into the problem because he had the hope of understanding turbulence using this of course that is not true.

But this process persistent diffusion itself is very important in something called dispersion in fluid layers and we will talk about that subsequently. Now these two are also called the telegraphers equation for a reason I will explain. The first thing you observe is that if you add these two guys the right hand side vanishes. So, that is one simplification but then you have to be careful there is a - sign here.

So, it does not immediately give you the solution on the other hand you have a little bit of simplification their first order equations so we could perhaps use a matrix method to try to solve this. On the other hand you eliminate one of the variables you are going to get a second order equation and then you should have to deal with that but it will have the advantage that will be a second derivative in space and the second derivative in time.

So, it looks like you know you can use a little bit of homogeneity property here would you expect this to have diffusive behavior would you expect the mean square displacement to increase in like a second order displacement in both space and time so it is not immediately clear what is going to happen it does you do see that the mean square velocity is C, C squared. So, it looks like they might be ballistic motion.

So, the mean square displacement may actually increase like the square of the time. On the other hand it is not immediately obvious what will happen as t becomes very, very large because you do expect the central limit theorem to start off operating right. So, we will see what happens you.