Physical Applications of Stochastic Processes

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Lecture-20
Elements of linear response theory

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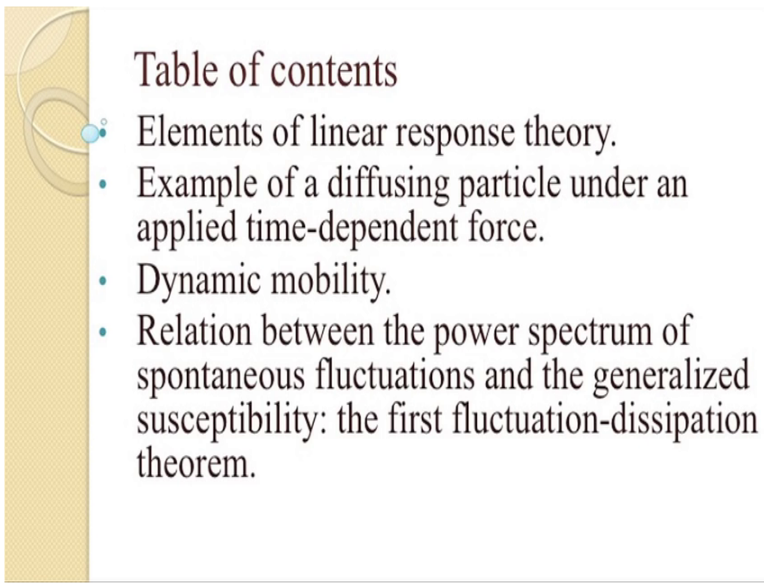

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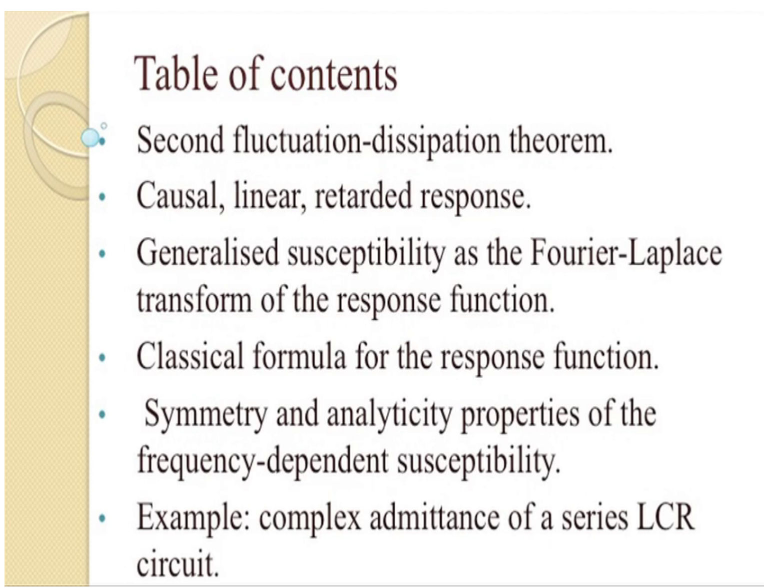

Before we go on to our next example of noise what I had like to do is to backtrack a little bit and put some of the things we have been discussing regarding the Langevin equation power spectra and so on in perspective and make contact with the more general formalism called that of linear response theory this is so that it gives you a complete picture of what exactly the generalized susceptibility is.

What the power spectrum is and how it is related and so on and again the most convenient or simplest model in which to do this is the example when looking or throughout namely the velocity one component of the velocity of fluid particle in fluid in equilibrium at temperature T. So, we wrote the Langevin model for it we extracted a lot of information from about it about the output process we proved that you had the on screen Hollenbeck process come out naturally and so on.

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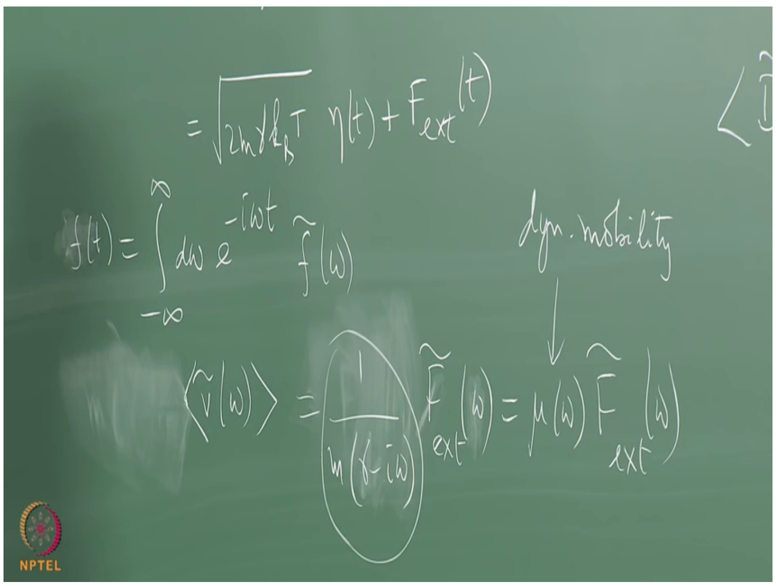

But let us put this in a slightly more general footing and see what happens just to refresh your memory. The Langevin equation that we wrote down was mV dot + m gamma V was = the force on the right hand side. Now this force we took to be a random fluctuating force in the absence of any external force but let us put an external force on the system and see what happens right.

So, there was this square root of gamma times Eta of t this was the fluctuating random force + suppose you apply some external force as a function of t this is what you would get and similarly exactly similarly this is we looked at the resistance model an LR circuit for example which had an equation like Li dot + RI was again = a fluctuating voltage we let us call it random me random of t + maybe some applied voltage we applied V of t in this fashion.

And there was a correspondence between these two things which came out as we went along there was a fluctuation dissipation theorem which related the strength of this noise to the dissipation here that is called the second fluctuation dissipation theorem and we will talk about this little more today. This turned out to be square root of 2m gamma k Boltzmann T times Eta of t + F external of t. And now if you took averages if you took statistical averages then this becomes m and then because the average value of beta is 0 out here.

It turns out that m times V average dot etcetera now let us do this directly in terms of Fourier components. So, I have in mind formally writing a function of trying as integral - infinity to infinity the omega e to the - i Omega t times F tilde of Omega and its inverse transform okay and this would immediately lead us to m times gamma - i Omega V tilled of Omega average is = this term vanishes upon averaging = whatever was a Fourier component there so F external of Omega in this fashion.

So, this says that the response that you have for each Fourier component of the velocity this quantity is = 1 over m times gamma - i Omega times F external or applied of Omega. The corresponding story here once again was that the current I tilled of Omega is = 1 over this produced a - i Omega so it is R - i omega l times V applied only here and of course this is what you call the complex admittance of the circuit.

So, this is some Y of Omega V applied so this quantity by definition is the complex admittance of this circuit it is reciprocal is a complex impedance course. Exactly similarly this quantity here is called the dynamic mobility because what it does is to measure what the average velocity is the component velocity component at the frequency Omega is per unit applied amplitude at the frequency Omega so this is by definition = some Mu of Omega times F external Omega and this quantity is called the dynamic mobility it is a complex number in general.

It is a complex function so this follows very straightforwardly from here but what is interesting is that we found a relation between the power spectrum of the input and the power spectrum of the output we discovered that the modulus squared of this Mu of Omega actually gave us the power spectrum in the absence of the external force okay.

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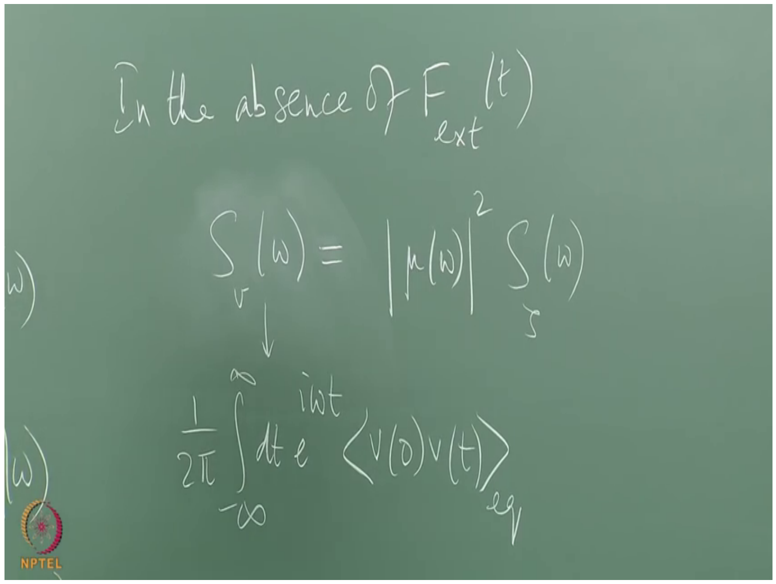

So, we actually discovered that in the absence of the external force in the absence of F external of t then you only have the random force the internal random force and it is fluctuating we discovered that the average value of the velocity was actually 0 because the fluctuating force had a 0 average unlike this case where you actually have a non0 average. But it turned out that you could find what the velocity correlation was and we discovered that this quantity S by the way we call this Zeta of t right.

We call this combination Zeta of t so that the in the correlation function of Zeta of t had already a gamma times the Delta function u so this quantity is Zeta the noise of Omega this thing here was related to the input out there and we found that in the other way about we found that S the velocity Omega was = mod Mu Omega squared times S Zeta of Omega for the power spectra and what is this fellow = this guy was = the Fourier transform.

So, because we use the symmetry property etcetera but in the absence of that 1 over 2Pi - infinity to infinity d to dt e to the i Omega t V of 0 V of t in the absence of the external force in equilibrium. So, this is a non-trivial relationship because after all what is the mobility measuring its measuring the response the average response to unit applied force at some given frequency.

On the other hand we also discover that the power spectrum of the output variable the velocity response variable due to thermal fluctuations is related to the power spectrum of the noise which is a driving force here through precisely the same Mu of Omega mod squared. So, this means there is a connection deep connection between response to an applied perturbation and spontaneous fluctuations in the absence of this perturbation.

This is a deep relationship it is the gist of it is at the bottom of linear response theory. Here this is crucial for stability because we already saw in a very soon this example itself we saw that if I did not have this term and I assume this to be delta correlated then it turned out that this quantity here the mean square value of this V in equilibrium increase with time linearly which is unphysical.

So, you needed the dissipation and at that point I said well this is sort of telling you that you cannot have uncontrolled fluctuations the more the system is thrown out of equilibrium the more it is brought back by the dissipation present in the system here right. So, stability is maintained and the consequence of that is that the power spectra are connected but this is a deep relationship because it is telling you that the average response in the presence of an external force is somehow related to the autocorrelation in the absence of this force okay.

So, this is not linear in B this is quadratic in be on the other hand the average is linear. So, it is a marvelous relationship it is a consistency condition which is essential for stability and we found that explicitly in this problem right. Now the same thing is true here too. So, there is a relation which will tell you the relationship between the fluctuations here and the situations here we saw what it was?

It is precisely this thing here with this translation of language m to l and m gamma is R so, its gamma is replaced by the characteristic time scale R over l inverse time scale. So, with this translation from one to the other these two models are essentially the same. So, we could write down similar things in that case too. So, there is a deep relationship between the fluctuations in the absence of the force and the average response in the presence of the first two first order in this external force which is why I keep saying linear response.

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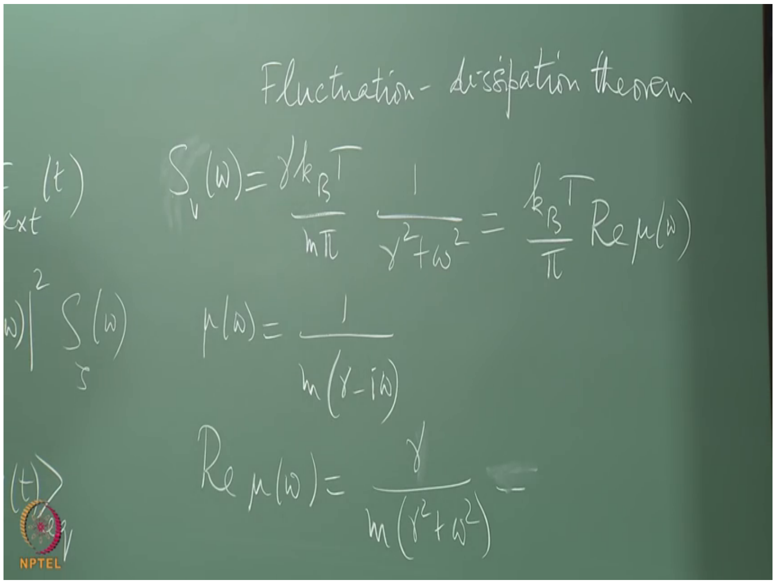

In fact we can go a step further we can see exactly what it is in this model it is not hard to see that if you took new of Omega but we already found out what this SV of Omega is right Sv of Omega was = this quantity here and by symmetry this was it was 1 over Pi and then there was a kT, so it is kBT over m Pi and then you had 0 to infinity e to the - gamma t so there was a gamma kt over 1 over gamma squared + Omega squared on this side.

And now if you ask what is the real part of Mu of Omega, so Mu of Omega is = 1 over m gamma - I Omega by the way the power spectrum here it is clear it is real because the way we have defined it sorry we should say this properly. We started by saying that this power spectrum was the Fourier transform 1 over 2 pi capital T 0 to capital T e to the i Omega T times this signal mod squared so it is a real number right.

So, it is fair to compare the real part of this Mu of Omega that gives you 1 over m gamma squared + omega square with a gamma on top. So, what does that tell you this is also = so this is = gamma over m can cancel, so kBT by Pi real power Mu of Omega you know for mu of Omega is the dynamic susceptibility it tells you something about the response of the system to an external force the average response to an external force.

And the real part of that susceptibility is directly = the power spectrum of the spontaneous fluctuations in the absence of this external force. So, there is one more way of writing this response relaxation relationship this side is a response and this side gives you the way the velocity correlations die out, so it is a relaxation and this thing is called a fluctuation dissipation theorem. It is actually called the first fluctuation dissipation theorem because there is a second theorem also which is this variable the driving variable.

This noise in the system is not related to the dissipation in the system yes in fact it is this the strength of this force here is directly related to this fellow here and that is very often called the second fluctuation dissipation theorem.

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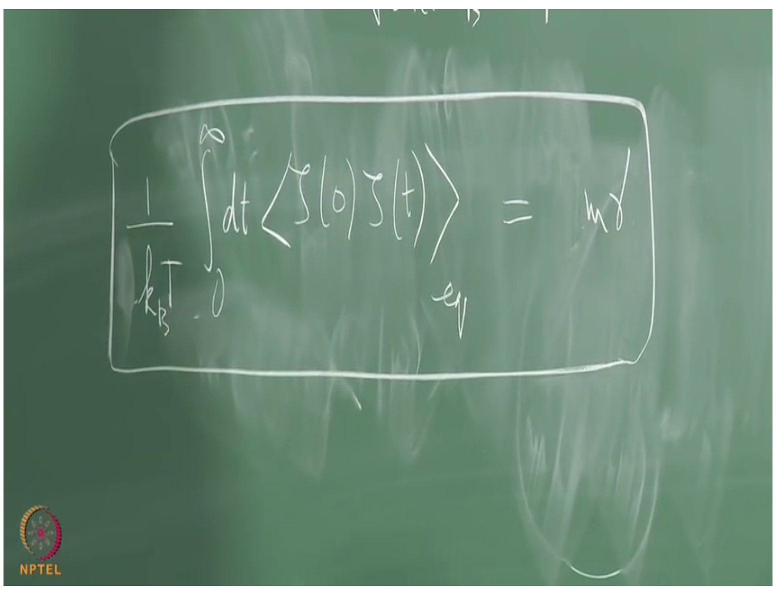

Let us write that down because I want to generalize that, so let us write that down so it I write this down in the following way Zeta of 0 Zeta of t in equilibrium let me put that just to show then there is no external force not that this is going to change because of that but let us for completeness put it here this guy here = 2 m gamma k Boltzmann T times Delta of t. So, if you integrate this from 0 from - infinity to infinity dt this guy here.

You end up with a 1 and if I write this as 1 over 2 k Boltzmann T that gives m gamma on that site and we know this is an even function of T, so I can write this as twice 0 to infinity of this is this camp and it is sometimes called the second fluctuation dissipation here because it tells you that the dissipation in the system is related to the spontaneous fluctuations in the system the noise in the system. This integral of this autocorrelation is that guided and there you have a similar relationship which again connects relaxation and response okay.

So, this term fluctuation dissipation theorem is sort of used interchangeably and we know that the two power spectra are connected through this relationship here now this quantity here is what would be called in engineering the transfer function I do not know what symbol you use for the transfer function H of Omega it is the mod square of this fellow 1 over R over R squared + Omega squared l squared is the transfer function for in LR said.

Now let us try to put this in a more general framework where this comes from what all this where all this comes from. And there is a small thing you have to notice which is slightly different and that is the following is it possible for me to write a formula for this mu of Omega directly in terms of this velocity autocorrelation function this is how we derive the answer is yes.

Because if you took this we will come to this formula I want to connect the susceptibility this dynamics as a mobility mu of Omega to directly to some integral over the fluctuate order correlation of the velocity. It is already implicit here in this but we will make it will make this relation look like that I want to make it look like that. We will see how to do this.

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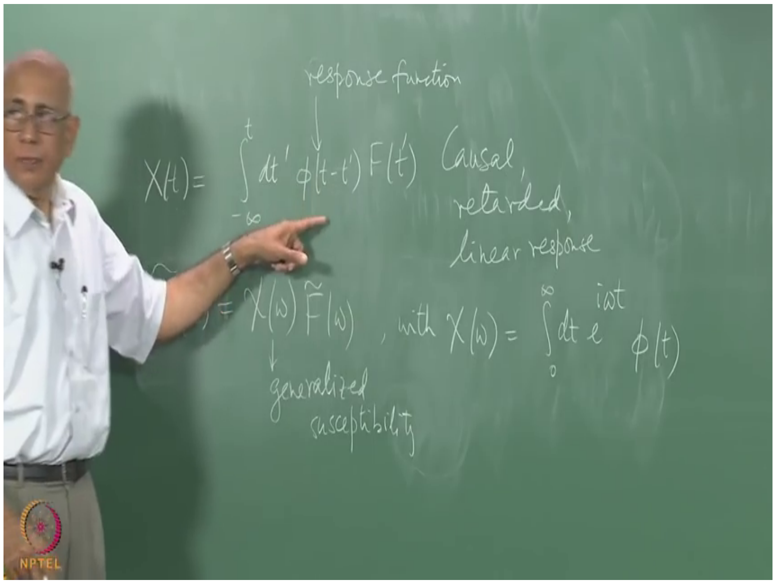

So, let us go back step back and try to cast this in a slightly more general language and see what all these correlations mean and where they come from first a few words about linear response okay. Suppose you have some force on a system some perturbation and the system you measure some observable system response through some observers which you measure and you want to ask what is this response light likely to be.

In the most general case where you assume just the following general principles, first you assume that the response is linear it is linear in the applied perturbation. So, it is got to be a small force in some specific strains. The second thing you assume is that it should be causal that is the effect should not take place before the cause okay. And the third thing is that it should be retarded namely the statistical properties of this system will always assume it to be in thermal equilibrium do not change everything is stationary and there is no aging or anything like that going on right.

Then if I apply in general terms if I apply some kind of force F of t to a system and I ask how does it respond and I measure some observable for want of a better word let us call that observable some X of t measure this quantity. This has got to be a superposition overall histories dt prime of this force time some response function in between some Phi which is a function of t - t Prime this is the most general linear functional that you can write down.

It is a sum over all histories of the applied force up to this time t, so there is no anticipatory response its linear in this F and it is retarded it is a function only of the elapsed time difference between the two every other application of any external force is a special case of this. Now once you have this you could ask general causal retarded linear response. Now of course if it is a vector or a tensor and this thing is matrix it does not matter we can put in all those indices later but this is the simplest instance.

Now if I formally make a Fourier transform on both sides I expand these things in Fourier transforms then it is a matter of very simple algebra to show that this quantity extent of Omega is related to the Fourier transform of this guy through a function so this is F tilde Omega multiplied in general by a function some Chi of Omega and this thing is called the generalized susceptibility. So, it tells you it is exactly the analog of the complex admittance or the dynamic mobility etcetera.

It tells you per unit applied force amplitude at a given frequency what is the response = at this stage there is no statistical mechanics or anything like that put in at all it is a general statement of causal linear response. This quantity here is called a response function and we cannot say anything more about this without knowing more about the system itself we need to put in more specific things.

So, now the question is can I write an expression for this Chi of Omega using just this fact here putting in the Fourier transform and the answer is yes it is immediate all you have to do is to put in the Fourier transform and change manipulate a bit and this will imply with Chi of Omega = an integral from 0 to infinity dt e to the i Omega t Phi of t. First of all note that this function Phi of t as it stands is only defined for positive values of the argument. Because you cut it off out here sometimes you write a green function you would say that.

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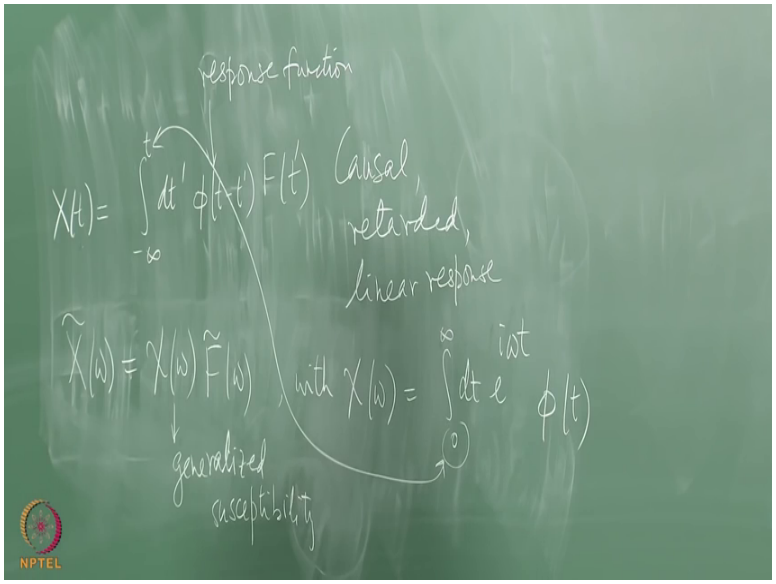

You now so just for analogy and so that you can make connection to that sometimes you have a problem in which you have some differential operator dx dt with respect to t say acting on a function X of t = a given function F of t sometimes you are given that kind of state right and then you are asked to solve for this X of t for a given F of t right with some initial conditions and so on and so forth.

So, what is the formal solution to this, this is X of t = dt inverse on F of t this is some differential operator involving derivatives with respect to time functions of time and so on so forth. We do not care what kind of operator it is and you have to find its inverse. Now it is reasonable that the inverse of a differential operator is some kind of integral operator.

So, in general the solution would look like this is = integral dt prime G of t and t prime F of t this guy is just a representation of the inverse operator in the explicit form. So, it is some integral operator with a kernel of this kind. Now if this operator is time translation invariant and so on so forth under suitable assumptions this will turn out to be a function of t - t Prime in this fashion the integral runs from - infinity to infinity.

And if it is causal it will say that this cuts off for negative values of the argument which would be equivalent to saying that this is of the form some Phi of t - t prime times a step function t – t, so, that the integral gets cut off. So, this is the connection between the causal brain function and the response function. I put this dt here explicitly so I did not write G otherwise that I built in G just to make connection with the normal right way of writing the green function.

So we are not going to use this but what we have here is a statement that you take this response function which to start with is defined for positive values non-negative values of its argument integrated with this weight factor e to the i Omega t 0 to infinity it is not a Laplace transform and it is not a Fourier transform either because it is one sided. It is 0 to infinity now this infinity comes from here though this manipulation.

But this 0 comes from here from this thing here from causality. So, that is why it is cut off this guy here is directly connected with this limit here and this is important to note. Now you might say maybe this integral does not converge you have to worry about convergence and so on of such integrals but the fact is that if it converged without this factor it would certainly converge with it because there is an oscillatory factor and there are places where it becomes negative and so on.

At a formal level if this is posed to you as an initial value problem from t = 0 upwards etcetera then what you do is to take Laplace transforms rather than Fourier transforms but what you have here is this guy here out here. So, you could formally say that this generalized susceptibility is the Laplace transform of the response function which after all is defined for its argument from t = 0 upwards analytically continued to S = - i Omega.

So, you could say that this is also = the Laplace transform of Phi of t evaluated at S = - i omega, so that technical difficulties with convergence and so on can be overcome. So, this is just to make contact with cases where you start applying the force from t = 0 onwards etcetera then you get exactly the same answer if you took the Laplace transform and replace the S with -i only analytically continue to that point okay.

So, much for the general case this is what it is we still do not know anything about this Phi of t on the other hand the kind of problem we have been looking at we need to motivate the fact that this Phi of t has a very special form it turns out to be an autocorrelation function in the absence of the external force and the question is where does that come from right here there is no such no mention of any extra anything at all.

You saying you are applying a force F and you are saying there is a response here so what happens in that case is the following and that is where the formalism of linear response theory comes in but let me say it in simpler terms okay.

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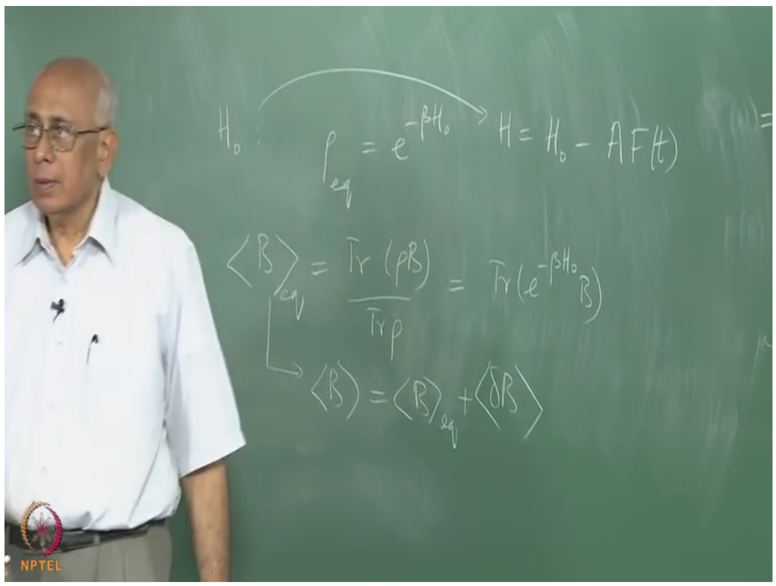

What really happens is that you start by saying here is a system in thermal equilibrium at some temperature t and there is an equilibrium density matrix which is e to the - beta times the Hamiltonian of the system. So, you have a system with Hamiltonian H naught and it is in thermal equilibrium so the density matrix in thermal equilibrium Rho equilibrium = e to the - beta H naught and then you can find the average value of any given quantity by the prescription of equilibrium statistical mechanics.

So, if you have some observable B and this guy is some observable the average value of B is Trace Rho times B divided trace B we will normalize things so that trace Rho is also always = 1 we can always do that so the denominator goes away otherwise you have to keep this thing. So, this fellow here is = trace e to the - beta H naught times B and we can compute its variance and so on and so forth.

Now I perturb the system by applying an external force on it of some kind this force always couples to some physical observable of the system and let us without loss of generality say that it couples to some observable a so that this Hamiltonian H naught goes to edge = H naught - this observable a time some coupling strength let me call it F of t okay. I have in mind the problem of the particle in which I am going to apply an external force.

And then if it is a constant force for example then the potential energy corresponding to this constant force you need the force is - dv over dx, so I set V = - X times f of T and if F is a constant for instance you would get - d over dx - X F is in fact F, so that is the reason for the - sign will purely a matter of convention it is just to tell you that in the case when F turns out to be a constant force and I put a = X I would in fact get the derivative of that potential is = F before - the derivative is = the force.

So, the question is what happens to the expectation value of V in the first order in this force F and that is going to be of the form B goes from here we this is an equilibrium goes to B = B equilibrium + Delta B that is the effect of this external force this is first order in this small quantity F okay and I need to compute this average.

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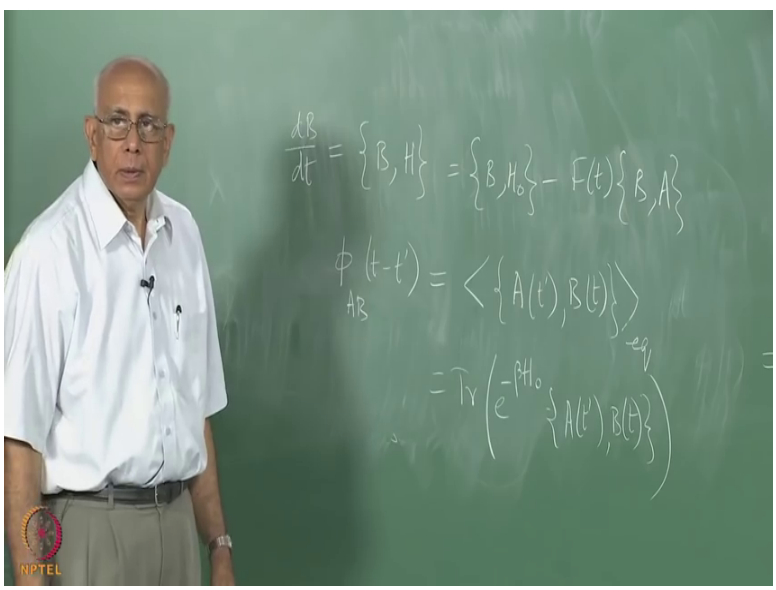

Now the way to do that is straightforward because when you have any classical variable and we are doing everything in the classical Hamiltonian context for any observable whatsoever you can write db over dt if this does not explicitly involve time if this quantity does not explicitly no observable does not explicitly involve time but involves only the canonical coordinates say since we are doing the Hamiltonian framework.

This is given by the analog of the Heisenberg equation of motion in classical mechanics right and what is that = the Poisson bracket of B with H which is = the Poisson bracket of B with H naught - F of t times the price of bracket of B with A and we have to solve this equation this is the differential equation that you have to solve and then compute averages and so on and so forth. So, I am not going to do that except to write the answer down.

And it turns out that if you do this then Delta B turns out to be = ok first a word on how this is done you should explain how this is done well the response function in this case Phi and now I need to remember that A is the perturbation and B is the observable so let us call this Phi Ab of t - t prime turns out to be in this case the expectation value of the Poisson bracket of A of t Prime with be in equilibrium that means in the absence of this perturbation.

So, what does that mean that means this quantity this average say steak trace of this quantity inside with respect to the density matrix into the - beta H naught that is the meaning of this average here and the reason is simple because what you have to do is to pretend this is kept to first order. So, in some sense to solve such an equation you would have to exponentiate whatever is on the right hand side.

And keep this to first order that means own it comes down anything on the other hand this fellow remains to all orders up there okay. So, at the end of a little bit of manipulation this is the answer that you get here but it still hasn't put it in the form of a correlation function. Now, that will depend on the following very simple observation this is = a trace e to the - beta H naught Poisson bracket of A of t prime B of t.

And now exploit the fact that there is cyclic invariance of the trace and then it is not hard to show by the way you can tell what is B of t at any time t you can write it in terms of B of 0 by the analog of whatever you did in quantum mechanics when you went from the Schrodinger to the

Heisenberg picture with e to the H naught and so on, on the left. So, a little bit of manipulation gets you to the following.

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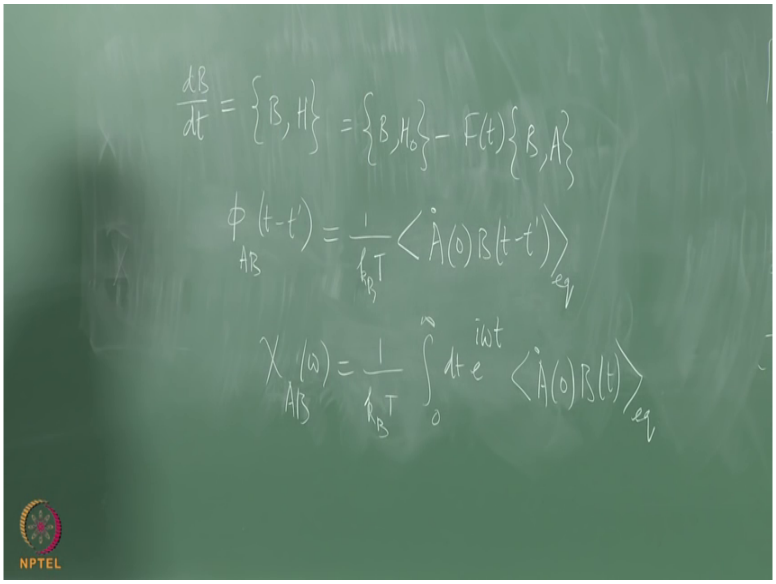

So, it takes you to this thing here becomes trace or some bracket of e to the - beta H naught with A of 0 B of t - t prime this becomes = that by the cyclic invariance of the trace okay. So, notice first that you have got this function of t - t prime emerging that comes about by putting the time dependences here in terms of A of 0 B of 0 etcetera and using the cyclic problem in variance of the trace the next step is to compute this quantity.

But look at what this is this is = if you had q's and p's as your degrees of freedom for example it would be Delta e to the - beta H naught over Delta q Delta A of 0 by Delta p - Delta e to the - beta H naught Delta p Delta A of 0 over Delta q summed over degrees of freedom and so on. So, I am assuming there are n degrees of freedom and I put a qi pi it is offensive. But if I differentiate this it is = delta H naught or delta q with A - beta e to the - beta H naught outside.

So, you get A - beta e to the - beta H naught and then this is replaced by delta H naught but this is = - beta times e to the - beta H naught oh by the way after you do this you have to take a trace you got to multiply by this into a trace. So, right now all we are doing is to simplify this fellow all the way down times Poisson bracket of H naught with A of 0 but that is = beta times e to the - beta H naught Poisson bracket of A of 0 with H naught.

But we now take recourse to this any operator it is time derivative is the Poisson bracket of the operator with the Hamiltonian in the absence of the external force it is the free Hamiltonian and the operators assumed mean have no explicit time dependence that is what we put into F of t. So, this guy is therefore = beta e to the - beta H naught times A dot of 0 because that is the definition of dA over dt and then you set t = 0 after you differentiate.

So, this becomes = trace beta times trace e to the - beta H naught A dot of 0 B of t - t prime trace of this whole time which is nothing but 1 over kBT time the average of A of 0 B of t A dot of 0 in this in equilibrium. So, that is how the correlation appears live the autocorrelation appears but notice the perturbation is in A the operator or dynamical variable that appears is in A but what is appearing here is A dot.

So, we have a formula that tells us this response function classically this fellow here is 1 over k Boltzmann T times the equilibrium autocorrelation that is the response function. So, it immediately gives us a formula for what the generalized susceptibility is because the susceptibility therefore Chi AB of Omega must be 1 over k Boltzmann T an integral from 0 to infinity dt e to the i Omega t A dot of 0 B of t in equilibrium.

So, it follows at once in general that this is what the susceptibility is what we need to do is to see whether our Langevin model for which we had an explicit stochastic differential equation for B of t will tally with this if we write it in the proper language.

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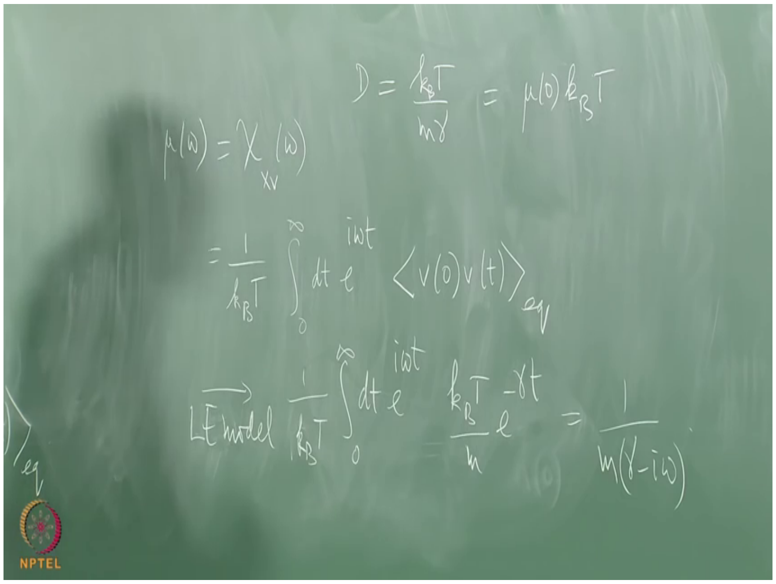

Now what is it we are doing when we are measuring the mobility you are measuring the velocity response average velocity response so what is Mu of Omega = it is a generalized susceptibility but what is A in that case and what is B in that case well A has to; B is clearly the velocity we are measuring the average velocity that is what the measurement of the mobility implies. And what is A = you are applying a mechanical force right. So, A is X the position right so by definition this guy = Chi Xv of Omega.

Position velocity cross whatever its susceptibility but that your advice here must be = 1 over k Boltzmann T times an integral from 0 to infinity dt e to the i Omega t and then A dot of 0 but A is X A dot is V right. So, this is what brings in the way here expectation V of 0 V of t inequality independent of the Langevin model we did not do anything we did not bring in any stochastic differential equation at all.

That is the general formula for the dynamic mobility in this one component system. But the Langevin model gives you a formula for this order correlation because you now have a detailed stochastic differential equation which is giving something about some information about the dissipation in the system etc and it is a model it is still a model right.

And in that model in the Langevin model this is 1 over k Boltzmann T integral 0 to infinity dt e to the i Omega t e to the - gamma k Boltzmann T over m e to the – Mu t in that model this is what we got and now it is a simple step to see the kt cancels and it gives 1 over m gamma - i Omega which is what we know already. So, this is derived from the Langevin model directly. We did not play around with the stochastic differential equation in particular we did not put in an external force a random force.

We did not talk about its correlations we did not do anything like that we just took linear response theory directly and use this formula and you get exactly the same answer. So, this is consistent the Langevin model is consistent with linear response field. But response theory gives you a general sort of formula in fact it will tell you what to do in the quantum case when these are when this is the Heisenberg equation of motion.

This is I H db over dt is a commutator here and when the Poisson brackets were replaced with commutators and things do not commute with each other and so on then you get a slightly more general formula here you actually get a not a Poisson bracket of A with B but a commutator of unequal time commutator A at time t, B at time t prime the other way about A at time t prime, B at time t and then from that you play around and you do not quite get this.

You get a more complicated formula for the generalized susceptibility but once again it will involve equilibrium correlations okay. So, notice that something fairly non trivial has been done we started with the response function which involved an unequal time for some bracket or in the quantum Langevin commutator and you are able to evaluate it and finally write it in a simplified form in terms of an autocorrelation of some kind.

So, there is a general relationship between the power spectrum of the spontaneous fluctuations in the output variable for instance and the corresponding dynamic susceptibility average response, if now what about the other relation what about the second fluctuation dissipation theorem that depended directly on writing a stochastic differential equation putting in an external force of some kind etc putting in explicitly a random noise making some assumptions about this noise etc.

But I said that we should like to write the power spectrum of that force also in this form. So, by the way we can we can write down in this formula here notice how t appears quite naturally appears here, incidentally what was the diffusion coefficient = in this problem it was just this integral here as it stood right. So, the diffusion coefficient is related to the susceptibility the mobility at 0 frequency and what was the relation an m gamma was the mobility at 0 frequency right.

So, we could write I am not happy with that relation sorry 1 over m okay that is A side outcome of the fact that at 0 frequency the mobility essentially measures a diffusion coefficient. Now, one could go a step further and ask be after all had a very simple model the Langevin model is there a more general way of writing this formula down this response down paying attention to the fact that this thing susceptibility turns out to be too trivial it just got by the way a couple of comments about the susceptibility.

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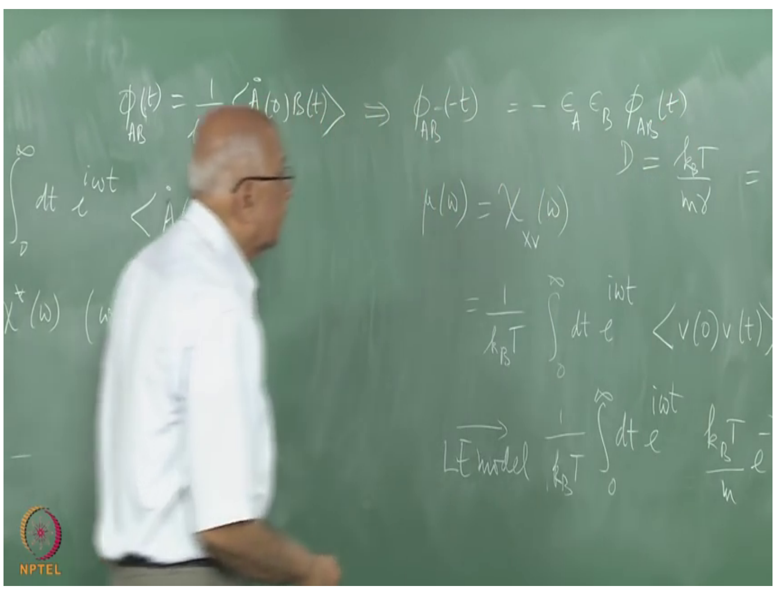

Let me make those two as well here say that properly so look at the generalized mobility Omega is 1 over k Boltzmann T integral 0 to infinity dt e to the i Omega t now we will assume that A and B are real observables or Hermitian operators in the quantum case. Then it immediately follows from this that for real frequencies Chi of - omega Chi star of Omega, so there is a symmetry property we saw a similar symmetry property for the power spectrum.

We saw that this function for a single component object it had to be positive had to be a symmetric function. You know now everything depends on what the time reversal property of this quantity is wasting here and in general we cannot make any statement at all because A and B need not have definite time reversal properties right. On the other hand in the simple example we looked at this guy this fellow was e to the - gamma modulus t.

So, it was a symmetric function ok we can ask what is the general statement what can we say in general well what we have to note is that this guy here if you write this as 1 over k Boltzmann T no sorry what did we do what did I right here sorry I should not write it like this like this is explicitly A dot of 0 that times 1 over kBT is = the response function Phi. Now if you look at what this Phi AB is it is = 1 over k Boltzmann T times A dot of 0 B of T.

And you ask this will imply that Phi AB of - t + Phi AB of T what will this B well it depends on the time reversal properties of these operators are observables there is an A dot of 0. So, if A for example is a velocity it will change sign under time reversal. If it is a position it does not change sign and so on. So, it is got some time reversal property let us call that epsilon A which is + 1 if is does not change sign under time reversal and – 1 if it changes sign.

And then there is an epsilon B which is also sitting here and there is an A dot, so there is a d over dt sitting there and that is going to change a sign so - and then Phi AB so in the most general case you tell me what is a what is me and I tell you what this Phi will do and in general it need not have a time parity need not have a specific epsilon it could be mixed but in the cases where it has a definite symmetry one way or the other even or odd you can write down what it is.

So, this whole number is either + 1 or - 1 we can therefore assign a definite even or odd nature as a function of time to this response function not necessary in general. So, the question is what can we conclude from this formula here. Well the first thing we see is that if this integral exists at 0 frequency it certainly exists for complex frequencies provided the frequencies are in the upper half plane provided the imaginary part is positive because that provides a damping factor okay.

So, this says this function here if it exists for real Omega will certainly exist for complex Omega in the upper half plane and will be an analytic function of Omega. So, you can write dispersion relations for it in real and imaginary parts are related by Hilbert transforms okay. In particular there are no singularities in the upper half plane.

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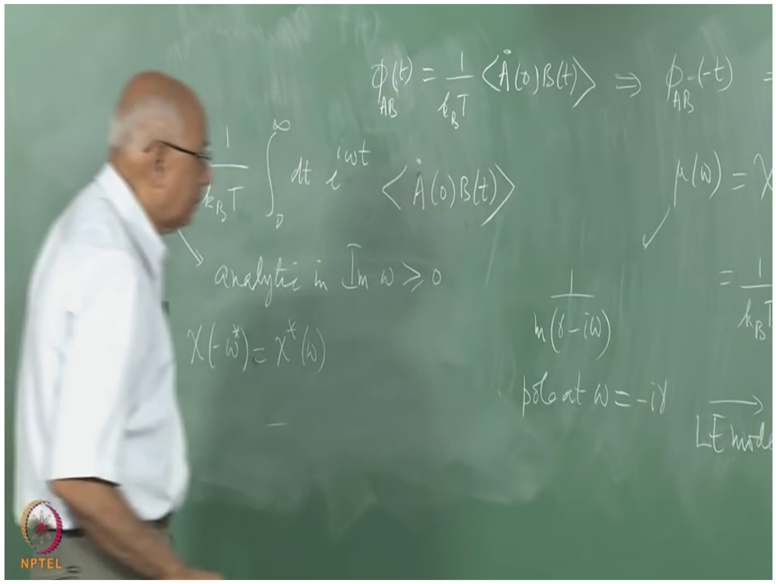

So, this function is analytic imaginary Omega greater than = 0 as a function of the complex frequency Omega its analytic then this symmetry property that I wrote down will be shifted to Chi of - omega = Chi star - Omega star is = pi star that is easily very fine if Omega is in the upper half plane Omega star is in the lower half plane and you put a - sign it goes back to the upper half so this is a reflection property in the upper half plane and does not refer to what it does in the lower half plane at all because we have no information on it as it stands.

Now you know that if you have an analytic function of a complex variable it cannot be analytic everywhere if it is then it is a constant including at infinity then it is a constant. So, this thing here must definitely have singularities in the lower half plane one or more singularities in general lower half plane because of the Fourier transform convention have chosen I chose + signs etc etc.

I stuck to that convention then it is analytic in this so the point is a causal linear retarded response will lead to a susceptibility which is analytic in one of the half planes either upper or lower half. The example we looked at mu of Omega this fellow here was one over n gamma - I

Omega it has a pole at Omega = - I am so it is in the lower half plane. What happens if you had a slightly more general system than this LR circuit.

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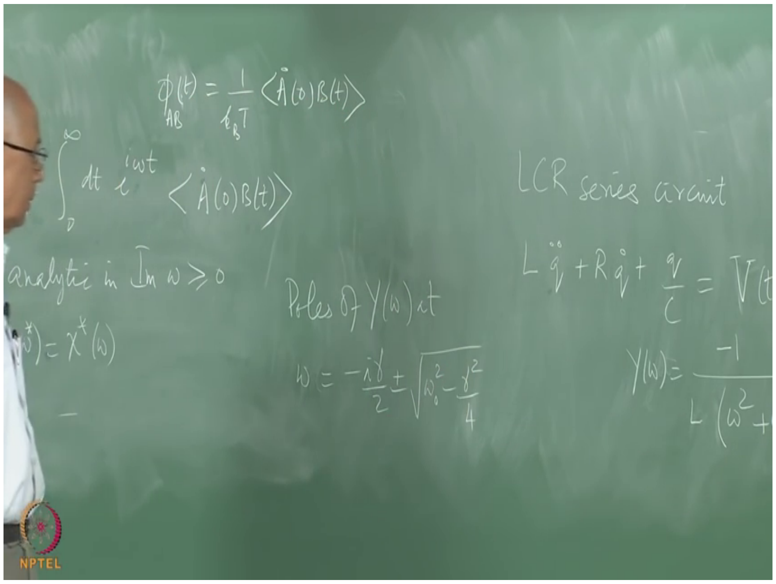

Let us put an LCR circuit and see what happens what is the admittance for an LCR series circuit. What is the admittance well the equation of motion is l times the charge which is q double dot + R times q dot + q over C = the voltage applied voltage or whatever V of t right. So, if I take the Fourier transform on both sides then they are related to each other by the complex admittance. So, what is Y of Omega in this problem = 1 over q double dot.

So that produces - i Omega whole square so it is - L Omega squared and this produces - i Omega R + 1 over C right. So, let us take the - sign here and write it as L Omega squared + this guy - the guy let us divide by L, so 1 over L times R over L - 1 over LC in this fashion sorry R over L and R over L is what we call gamma the inverse characteristic time constant so this fellow here is = - 1 over L times Omega squared + i omega gamma - Omega naught squared.

That is the square of the frequency of the purely reactive circuit without any resistance and where are the poles of this guy at Omega = - i gamma over 2 + or - square root of - so that is = Omega naught squared - gamma squared over 4, I took out the two and divided this okay. So, that is where the poles are. Now if it is an under damped circuit then of course this is bigger than that and then both poles are in the lower half plane right.

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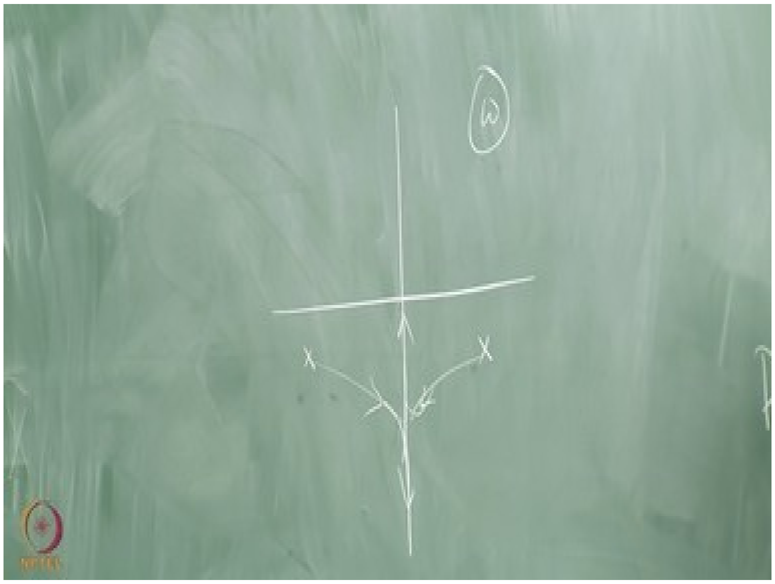

So, indeed it satisfies this causality this analyticity condition and where are these poles by the way in the Omega plane to start with one of them is at - i gamma over 2 + that square root and the other fellow is it - the square root we are out here right. So, this is the fellow that corresponds to the + 1 corresponds here negative 1 curse to this. What happens if I now change this frequency or increase the friction a little bit?

Such that finally it becomes critically damped I go on increasing gamma or a decrease in omega not till it becomes over damped what will happen to these poles they cannot go up they cannot go up because it is got to be analytic in the upper half plane right. When will they coincide well they will coincide when they become both these poles will start moving in this fashion and they coincide at one point at critical damping right.

So, this goes away and you have just one of them at - some number here and then what happens to the polls well one of them will go up like this and the other one will go down like this because now this fellow becomes pure imaginary right. So, one of them as you increase the parameter gamma towards infinity one of the moves towards 0 and the other one moves out to - infinity but remains in the lower half plane throughout.

This is a matter of convention we have said that our fully transform convention is such that the generalized susceptibility is the cause they have an the retarded causal retarded susceptibilities have an analytic in the upper half plane the frequency and that is borne out in these simple cases. But still this is not general enough especially in the case of even in the case of this simple Langevin model this is not good enough.

Because we have assumed a friction constant but we did not say that this friction constant depends on time at all a much more general thing would be to say that this dissipation will itself be time dependent okay. We need to put a memory kernel, so we will do that next time you see quickly how the fluctuation dissipation theorems will continue to hold good but this time the second fluctuation dissipation theorem will become a non trivial statement we will see you.