Physical Applications of Stochastic Processes

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Lecture - 15
Langevin Dynamics (Part 4)

Right, so we I promised last time that we would talk about multidimensional Langevin equation for a particle in a magnetic field and then I will talk about the properties of the Wiener process itself namely Brownian motion itself. But before that just to complete what I was saying in the last time, if you looked at a Brownian oscillator, a particle moving on the x axis, bound harmonically to the origin by this potential the spring force minus half whatever omega not square minus omega not square x then we wrote down the phase space distribution.

We wrote down a differential equation for the phase space distribution joint distribution in both x and v the conditional density in both these variables and then I said you can solve this, you get a bivariate Gaussian in x and v and if integrate over x you would end up with a distribution in v. If you integrate over v you would end up with a distributions in x.

Uh we also saw that if you have the overdamped oscillator, can be the one for which the gamma is much bigger than omega not the free frequency, then it turns out that the position itself satisfies a Langevin type equation and you get a solution similar to the Ornstein-Uhlenbeck solution except that the relaxation time is not gamma inverse but omega not square over gamma the whole inverse right.

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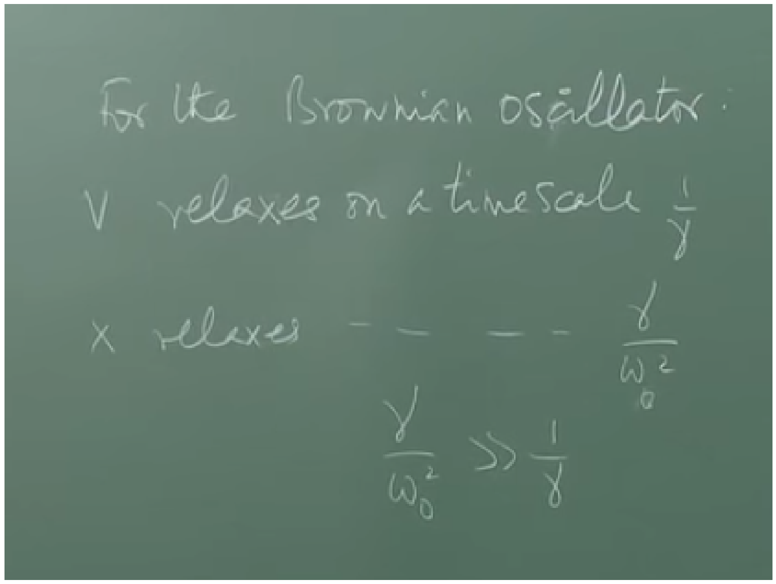

So it turns out that v relaxes on a time scale 1 over gamma but x for the oscillator, for the Brownian for the Brownian oscillator, x relaxes on a time scale which is gamma over omega not square okay. We saw this from the overdamped oscillator thing. The fact that this is a Ornstein-Uhlenbeck process and so is this an Ornstein-Uhlenbeck process is responsible for the statement in many books that the Ornstein-Uhlenbeck process itself is called the oscillator process.

What they mean is that it is the position of the harmonic oscillator when you have the overdamped case okay. Now it is immediately clear from this that in the overdamped case gamma over omega not square is much bigger than 1 over gamma which is exactly what overdamping means. Gamma is much bigger than omega not okay. So the velocity thermalizes if you like much more rapidly than the position does, it is much more sluggish.

But there is no long range diffusion in this problem. Both x and v go to equilibrium distributions which are given by the equipartion theorem if you like in statistical mechanics okay. So much for this. Now for a multidimensional motion, three-dimensional motion for example of a Langevin particle, if the particle is free then you would end up with an equation which says

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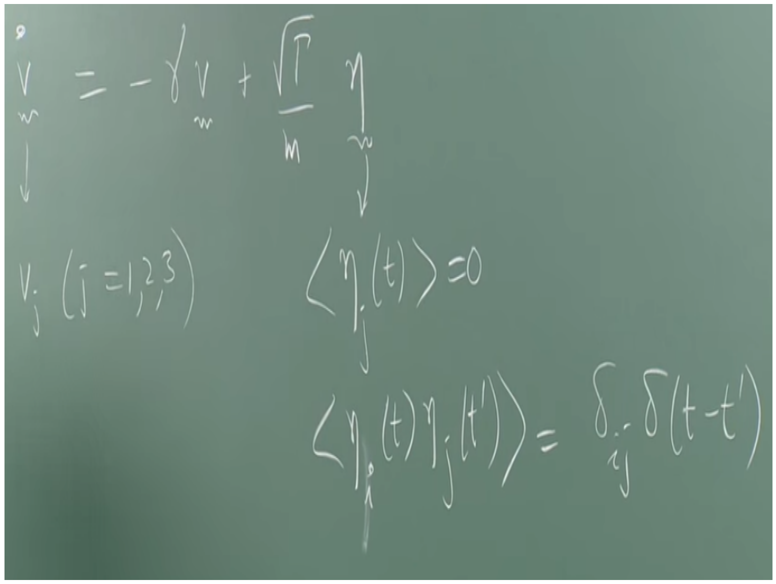

V dot, now it is a vector. So let me put a wiggle underneath to show that it is a vector. This is equal to - gamma times v + square root of gamma over m, a vector valued force eta noise eta here and this is the free particle Langevin equation where each of these is a three-dimensional vector so v for example has components v j, j equal to 1, 2, 3 and this guy also has components eta j running over 1, 2, 3.

And it is a three-dimensional delta correlated stationary white noise which means that this satisfies an equation of the form eta j of t = 0 for every j and moreover eta j i of t eta j of t prime equal to a delta function i j delta of t - t prime. So the different Cartesian components of this noise are uncorrelated to each other immediately.

Now once you have that information by the way we can solve once again, you can solve for the Langevin solve for the velocity and find all its properties from those of eta or you could write down an equivalent three-dimensional Fokker-Planck equation for v and solve this equation to get a generalized Gaussian solution. It will be the three-dimensional analog of the Ornstein-Uhlenbeck process okay. Now we know that that process is exponentially correlated.

We know the velocity correlation time is gamma inverse okay. So we could ask what is the correlation between different components of the velocity okay. And we can do that in many ways. We can solve this for v and then compute the velocity autocorrelation function.

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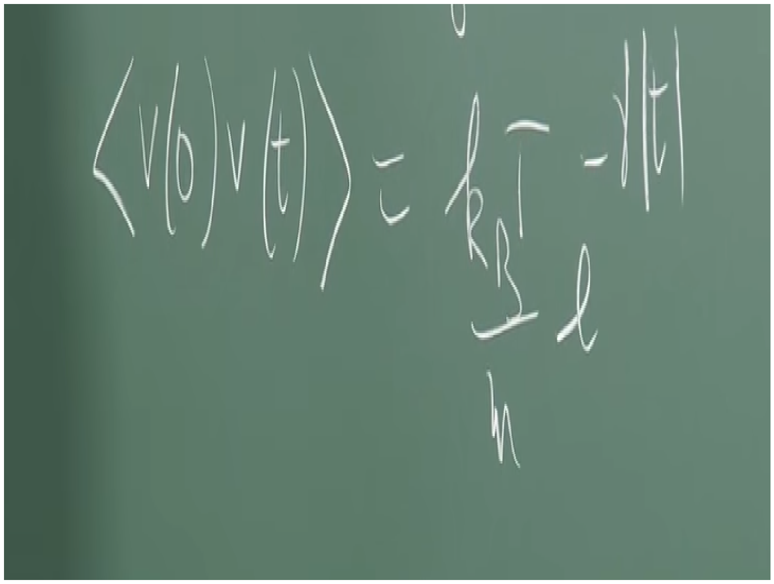

But we already know from the one-dimensional example we saw that in that case v was a stationary process and v of 0 v of t was equal to k Boltzmann T over m e to the - gamma mod t. This is what we had for a single component here. Now it is possible to get this equation directly, get this solution directly if you know that v is a stationary process. We can do this by a very cheap trick and that is to simply write the Langevin equation down. So let us do that.

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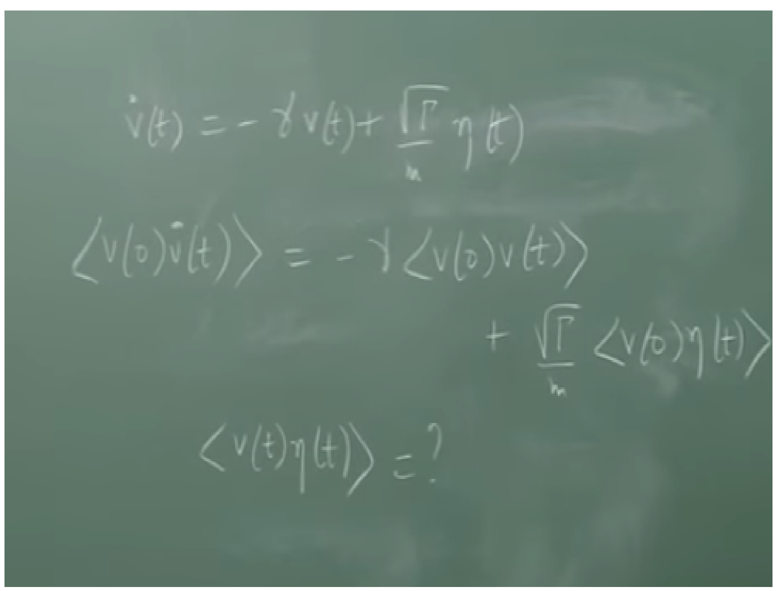

So I have v dot equal to - gamma v + root gamma over m eta of t in the one-dimensional case and this is the function of t. So this is v dot of t equal to v of t + eta of t and I want to find out what is v of 0, v of t in this case. So what I do is to multiply both sides by v of 0 on the left hand side and then average. So v of 0 v of t v dot of t average equal to - gamma v of 0 v of t + root gamma over m the average value of v of 0 with eta of t okay.

And what do you think this average is? What is the correlation between the velocity at time 0 and the force at t greater than equal to 0. By causality you would expect that the force at a later time does not affect the velocity, the output variable at an earlier time. That is causality right. So this is actually 0. This quantity is 0. Now you might say ha what about equal times what about at t = 0. So at equal times what do you think this correlation would be? V of t eta of t.

What do you think this is going to be? That is a more delicate question. But again I argue the same way. Eta of t from this equation here controls the acceleration. So the acceleration is determined by the force. Not the value of the velocity. This is like an initial condition for this force at equal times right. So again it is completely uncorrelated to each other. So that is 0 as well.

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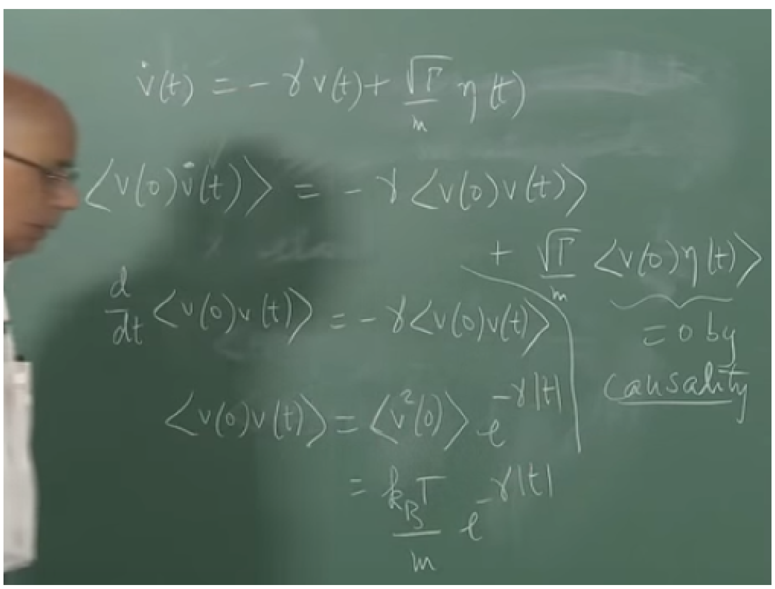

So causality says this quantity is 0. That is a principle we have not yet invoked okay. It says that the cause cannot precede, the effect cannot precede the cause. It is not anticipatory right. Because of that you can set this equal to 0 and this is some function of t, but then d over dt v of 0 v of t is precisely this quantity. There is no time argument here at all.

Or if you like put a t 0 here and a t 0 + t here and it is independent of t 0 and the time derivative with respect to t acts only on this thing here and it is therefore equal to the time derivative of the correlation function and this is equal to minus gamma the same thing okay which immediately says that this should be equal to.

Therefore v of 0 v of t should be equal to v of 0 if you like square sorry square inside v square of 0, that is the initial value, e to the - gamma modulus t because you have written this equation for t greater than 0 okay and this now this quantity is now determined from equilibrium the fact that in stationary in equilibrium in the Maxwellian distribution this quantity is just kt over m right. So this immediately tells you. So even without explicitly solving the Langevin equation you can actually find what the correlation function is okay.

What do you think is going to happen there if I did the same thing? Well I write this out for each Cartesian component and notice exactly the same property as before. I multiply this, for v j I write v j dot, there is a v j here and an eta j and I multiply by v j of 0 on the left hand side and solve the differential equation, use the same causality argument and you get exactly the same answer for each Cartesian component right. So therefore what would this be?

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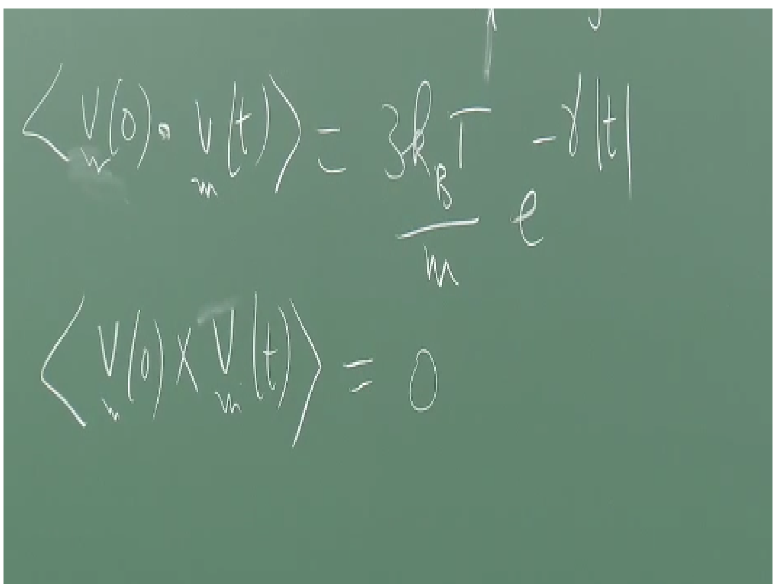

As a vector v of 0 dotted with v of t in equilibrium what would this turn out to be? Ya, it is kt over m times e to the - gamma t for each of these components and then you are taking a dot product so this is equal to 3. What would this be? What would v of 0 cross product, v of t. So now I want correlations of v 1 with v 2, v 1 at an earlier time, v 2 at a later time and so on, unequal components. What do you think is going to happen?

So I write the ith component of this is epsilon ijk vj and vk out here and I do the same trick as before and again compute I can solve for this thing. What do you think it will be? It will turn out to be 0 because the different Cartesian components are completely uncorrelated with each other because they are driven by noises which are uncorrelated with each other. This property here, there is a delta function here.

So the correlation of one component of eta with another component of eta even at the same time is 0. So this is identically 0 okay. What would happen if I switched on a magnetic field? What is going to be the equation of motion? Yes, because now we have exactly the same Langevin equation as before but now there is a term.

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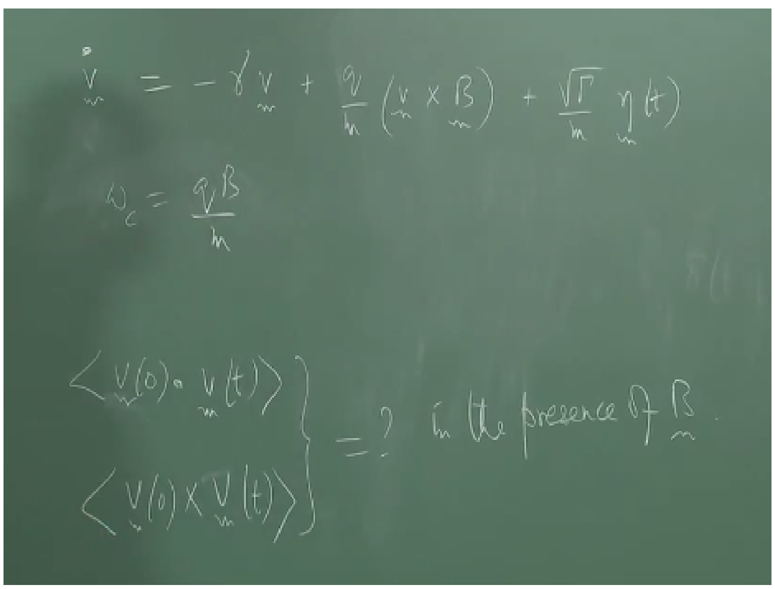

let us say the charge of the particle is q or something like that plus q over m v cross B. So there is this term plus root gamma over m eta of t, that is not affected. Now all kinds of correlations will happen because it is clear that this velocity dependent force, the acceleration in the x direction depends on the velocity in the y direction and so on and so forth, so there is correlations between the different components.

Even though this noise is uncorrelated, the Cartesian components are uncorrelated because of this term here we are going to get correlations between the different velocities and you can solve this problem for a constant magnetic field exactly the same way as before. Once again we know from physical arguments that the energy of the particle is not changed in a magnetic field, that the Maxwellian distribution is not disturbed at all in equilibrium.

There is a stationary distribution, that is the Maxwellian distribution. All that will happen is that the diffusion coefficients in the different directions are going to be different because in the direction of the magnetic field it will be the unperturbed one but in the direction of the field there will be this cyclotron motion inhibiting long range diffusion by changing the diffusion coefficient. Nothing else is going to happen.

So I leave you to compute these quantities in this case, this case here. So, in the presence of B. it is a simple exercise but a very instructive one and you can find out what are the cross correlations in this case okay. This is a very special kind of force here. It is velocity dependent but it is linear in v once again and this drift of course is linear in v. Now we know that anything that is linear in a variable you can solve the problem completely.

You can write the Fokker-Planck equation down and solve it. You can find the Green function, you can solve the differential equation, the Langevin equation itself etc. So in exactly the same way you can extend whatever we did earlier in the absence of a magnetic field to this case here. We already also saw what happens in the case of the oscillator where you have a term that is linear but that is linear in x instead of v and that did not bother us either. We were able to solve it.

So this problem can be explicitly solved, can look at it exactly. In fact as the particle does cyclotron motion with a characteristic frequency what is the characteristic frequency for the cyclotron motion in a magnetic field B? There is a quantity of dimensions, once you have a thing like this what is the cyclotron frequency where the force is q over m v cross B right. So there is a time scale here. This is v dot so there is a 1 over t here times velocity.

There is a velocity here already. So it is clear that whatever is of inverse time scale must be q B over m right. That is the cyclotron frequency okay, is from dimensional arguments. It turns out to be exactly that. The numerical factor is 1. So if you go to a rotating frame of reference, it is rotating about the direction of the magnetic field with this frequency then it is as if you do not have this field.

So in that frame so you must transform from v to a variable u which takes into account that rotation and then of course you can solve the Fokker-Planck equation much more easily. Or you can leave it as it is and work it out including this v okay. So I leave this, computing these correlations as an exercise okay. Now let us get back and examine this Brownian motion itself a little more carefully.

We have to define what is meant by Brownian motion and by this I now do not mean the physical Brownian motion which was what was discovered first when they had these Robert Brown observed among other people. He first understood what the nature of this motion was to some extent when you crush pollen grains and put it in water and then you look at it under a microscope you see these irregular jagged motions jerky motion of these particles and that is what Brown described and it was called Brownian motion okay.

That is physical Brownian motion. We know what it is due to. We know that it is due to molecular collisions and so. But mathematical Brownian motion is an idealization okay. This is defined by several ways and I am going to define in one particular way today. But this is exactly what the x which occurs in the diffusion equation the simple diffusion equation is supposed to describe.

So the process x whose probability density obeys the ordinary diffusion equation in any number of dimensions is called mathematical Brownian motion and let us go back and ask what that diffusion equation was.

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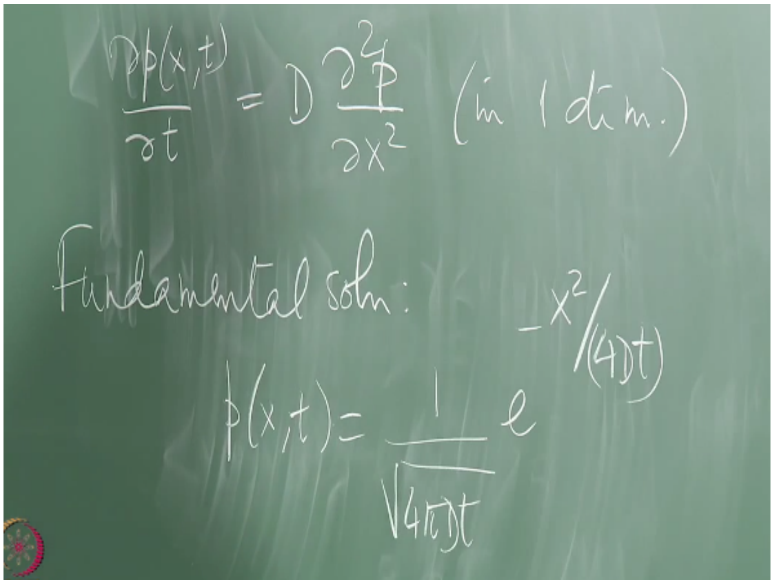

So we had delta p of x t over delta t equal to D d 2 p over dx 2 in one dimension. Otherwise you got a del square out here okay, with some given constant D okay and we know this comes from the Langevin equation for a free particle in one dimension in the presence of this random force eta in the diffusion limit, the limit in which t is much bigger than gamma inverse okay and we also saw that this D is related to that gamma by kt over m gamma was equal to this D.

Right now we are not worried about it. We just ask what does this say? What does this equation say okay and it is useful to write down it is fundamental solution of this particular equation. The solution which says that p and its derivatives vanish at plus minus infinity in x and we assume also that the particles start at the origin at t = 0 okay. So you are really solving for the Green function of this differential operator. So the fundamental solution is p(x, t).

The normalized solution is square root of 4 pi Dt e to the - x square over 4 Dt okay. That is the basic solution. Now of course you started this process at some time t prime and t gets replaced by t - t prime and if you start at some point x not then x gets replaced by x - x not okay. Now we ask what is the stochastic differential equation corresponding satisfied by x corresponding to this equation. We have this correspondence between diffusion processes a Fokker-Planck like equations and stochastic differential equations which we call the Langevin equation right.

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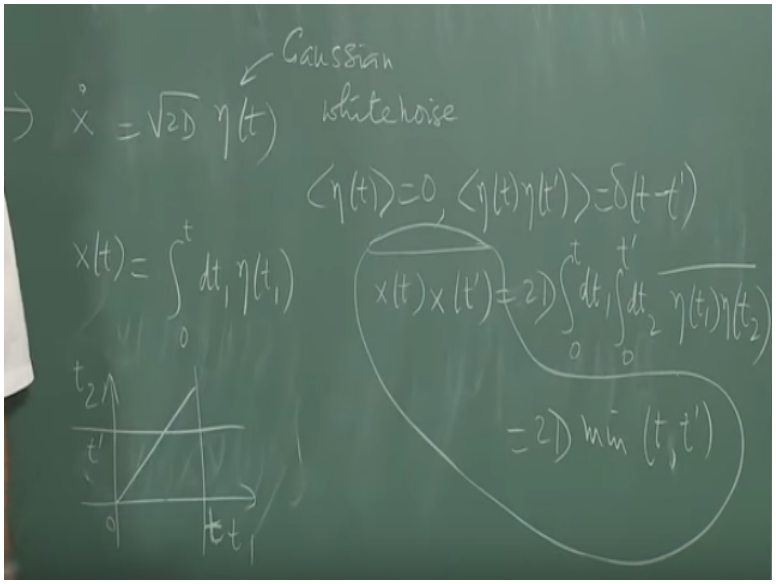

So by that correspondence this thing here is entirely equivalent is saying x dot equal to square root of 2D times eta of t where this is a Gaussian white noise. By that I mean eta equal to 0 and eta of t eta t prime equal to a delta function okay. So this process x I have not distinguished here for ease of writing between the random variable itself and the values it takes which is what appears here in this thing here. It is a matter of notation.

One should be a little careful but no confusion should arise. So this process x is the integral of white noise if you like because you could write x(t) equal to an integral from 0 to t dt 1, eta of t 1 formally, formally one can write it in this fashion. For particles which start from the origin at t = 0 this is the formal solution to this equation. Is this a stationary process? That is a stationary process. It is statistical properties are independent of the origin of time but this is not.

Remember the velocity was a stationary process. When I wrote the Langevin equation down it turned out to be a stationary random process. This one is not a stationary random process here and that is easily seen because all you have to do is to look at x of t, x of t prime and take its average. Now this average we are now taking for all those particles which start at 0 at the origin at t equal to 0 right. So it is a conditional average.

There is no equilibrium distribution corresponding to this because this guy vanishes as t tends to infinity. There is no stationary distribution at all for this process here unlike the velocity process where you got the Maxwellian as the stationary distribution. Now what is this going to be? So let me denote by this overhead bar this conditional average that its fore particles which start at t = 0 at x = 0 okay. So what is this guy going to be equal to.

It is equal to integral 0 to t dt 1 0 to t dt 2 t prime dt 2 and then eta of t 1 eta of t 2 average. Whether I put this average or I put angular brackets here does not matter because our philosophy is that the random noise which comes from all the other particles does not is not affected by the motion of this single test particle okay. So that acts like some kind of heat bath or reservoir and affects the particle we are looking at, the tag particle, but there is no effect of the tag particle on the noise itself okay.

All we have to do is to put this in here and then you immediately see that if you integrate over t 1 and t 2 and t 2 is up to t prime and t 1 is up to some number t. The integration has support over this 45 degree line. Therefore it gives a nonzero contribution only as long as t1 is runs up to t prime and after that it is 0 and if it runs up to t prime you can get rid of the delta function integral delta function to get rid of the t 2 integration.

And then t 1 runs from 0 to t prime right and had it been the other way about had t prime been bigger than t what I have shown here is t prime less than t it would run up to the lesser of the two always. So this immediately is equal to oh there is a 2D also, 2 D equal to 2D the lesser of t and t prime. So that is the point. It is not stationary. This is not a function of t - t prime as you would expect. Had this been stationary you would have ended up with t - t prime.

It is not true okay. However, the increments are stationary because eta itself is stationary. Therefore I could write this as dx if you like the increment in x is eta of t dt and then those increments are stationary because eta is stationary. Or another way of saying it is if I computed x of t - x of 0 and x of t prime - x of 0 and took the average over that you would end up with 2t modulus of t - t prime okay. So this thing here is a very crucial result.

This is a very crucial result and in fact you could say that if you have a process whose average is 0 and whose correlation, autocorrelation satisfies this thing here and you are given that it is a

Gaussian process if you are given that then all the other properties of Brownian motion follow including the fact that it is a Markov process, everything else follows.

So that is one way of defining mathematical Brownian motion namely it is a Gaussian process whose mean is 0, x of t average is 0 at any time t and whose autocorrelation is this function here, some constant times the lesser of t and t prime and this suffices to define the process and it has got remarkable properties. But you already see that x is smoother in some sense than this white noise because it is the integral of white noise but it is going to be rough as we will see as is bad enough as it stands. It is very different from what the velocity was.

That was the Ornstein-Uhlenbeck process and that had an autocorrelation which was exponentially dying down okay. This looks very different altogether and is non-stationary as it stands. Now some statements about the paths, the possible trajectories if you like in a typical realization of this x of t and I am not going to prove this. This requires now a little more mathematical machinery which I am not going to use here but here are some facts.

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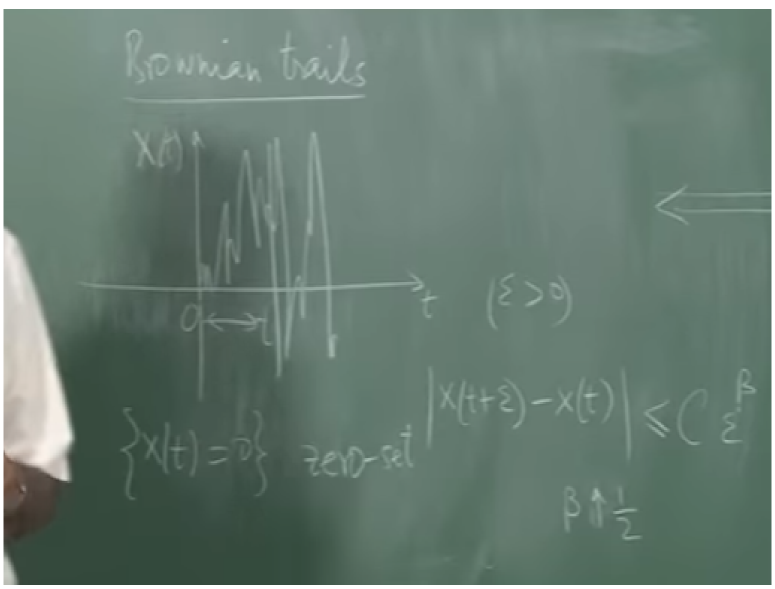

So let us call it Brownian trails. If I plotted, here is t and I plotted this process x of t Brownian motion then a lot of interesting facts emerge. In a typical realization it is hard to plot. There is no way that I can actually draw a realistic realization. But if I started here and did this and like this then the statement is that this x does not have any bound. In any short interval however short of time you can attain arbitrarily large values of x on either side of the axis okay.

The second property is that while it is continuous the curve is continuous it is not differentiable anywhere in the sense that this is a very singular object. This guy is a very peculiar singular object okay although we have written it in this fashion it is really a very singular object. This curve is kinky on all time scales. Does not have a derivative almost everywhere. What should it be in order for it to have a derivative?

It is clear that you must have x of t + epsilon - x of t the modulus of this guy here must be of order epsilon. So that if I divide over epsilon and take the limit, I have a finite number right. But it turns out that this is less than equal to some constant times epsilon taking epsilon to be positive to the power beta where beta is less than half. The fact that you have a finite a positive beta shows that the curve is continuous and it is said to be holder continuous with exponent beta if this is satisfied okay. But it turns out that beta is less than half almost everywhere.

But it can be made arbitrarily close to half from below almost everywhere okay. So this quantity beta tends to half, from below okay. There is an set of points where beta is exactly a half but that is a set of measure 0. On the other hand beta can become arbitrarily close to half from below. So it is almost certainly not differentiable because you need beta equal to 1 for differentiability and this is less than that. So that is the first property.

The other property, the next property is that these points where it crosses the origin they form the 0 set of Brownian motion. So you start from 0 and you ask when does the particle come back to 0 etc. This 0 set, the set of points such that $x(t) = 0$ has a fractal dimension which is half in this case. That set is not countable and its box counting or fractal dimension happens to be half. This is a reflection of the fact that in this Brownian motion x square, the square of the length scales like the time diverse power.

That property appears over and over again in this business. You could also ask suppose you start at 0 and it moves up to the positive side rather than the negative side, in a given time interval what is the fraction of the time that it spends without crossing okay. You can ask for the distribution of that fraction okay.

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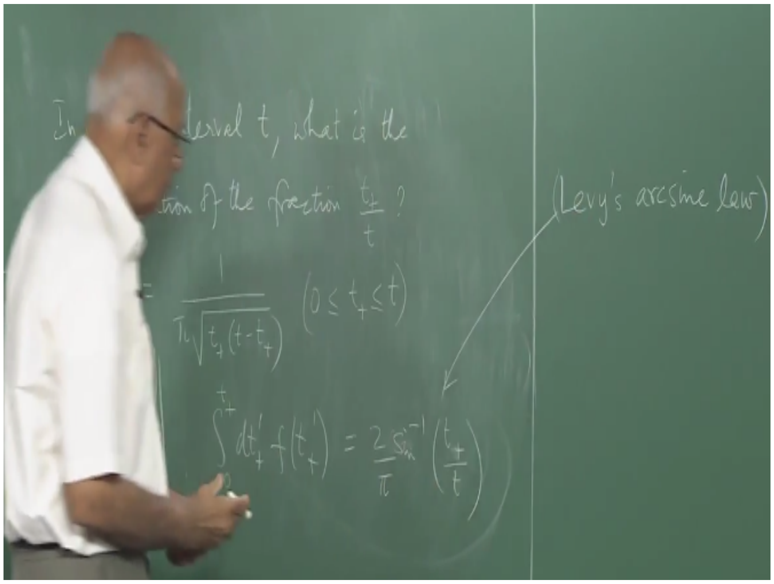

So let us suppose that in a time interval t what is the statistics, what is the distribution of the fraction. Let us suppose that in a given time interval t here and this is 0, we could ask for the fraction of the time. Let us call that t plus over t that it spends in the positive side without crossing 0 or negative side by symmetry okay. So, now this fraction here has an interesting distribution. It must have a probability density function first of all.

And it turns out that that density function, let me call it f, the PDF of this random variable for a given t, that is a random variable t plus this PDF f of t plus normalized PDF when you integrate this from 0 to t you must get 1 of course. So this is equal to 1 over okay and that can be rigorously established. What does that graph look like? What would this look like? Well, becomes unbounded at both 0 as well as so t plus is sitting here, ya it looks like an inverted u of some kind, with some mean value at half a t over 2 okay.

But that is the least probable. That is the least probable value. You would normally expect that in a given time interval t the fraction of the time the probability density function of the fraction of the time it spends on either in the positive side or in the negative side of the x axis is you know the most probable value you would think is a half but it is a least probable value okay.

So it is clear that most of the time this guy is spending is this particle is spending it is time either on the positive side or on the negative side of the axis and yet in any finite interval of time it crosses this axis an arbitrary number of times okay. So it is a very weird kind of motion. The cumulative distribution function 0 to some number t + dt + prime f of t + prime is the actual distribution function of this fraction.

And when t plus is equal to t it should be equal to 1 right. This is equal to sin, sin inverse t plus over t and sin inverse 1 is pi over 2. So this is 2 over pi. That is easy to derive. Complete the square and integrate that is it. This is called a Levy's okay. There is an even more exotic law which says where is this particle likely to be most of the time okay.

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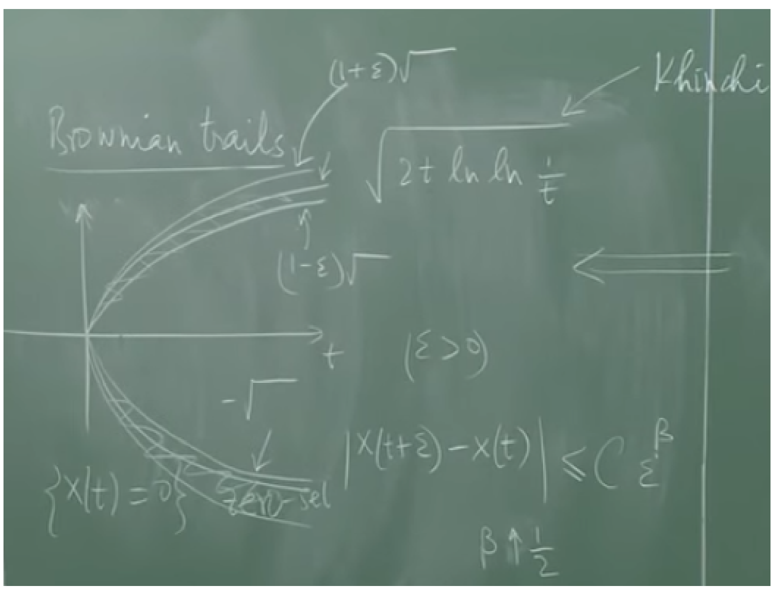

And it turns out that if you plot a time against the following function, it starts at the origin. So if you plot it this function 2t 2Dt well let me let us work in dimensionless units put d equal to 1, 2 t log log 1 over t square root and plot this function for sufficiently small t for t less than 1 this guy is positive goes up there and this is minus that function. This guy is minus square root of the same thing here.

And you multiply this by 1 plus epsilon you would get a curve like that and this is 1 - epsilon and the square root and this function is 1 plus epsilon times this square root okay. Then the particle is almost surely going to be in this range, either there or here, almost surely for every t sufficiently small t okay and for sufficiently large t the 1 over t in the log is replaced by t okay. So look at what is happening.

I mean we sort of know that the mean square root mean square displacement is proportional to square root of t. The mean square displacement proportional to t so the root mean square is proportional to root of t. So we kind of know there is a square root of t is sitting here. That will be like the mean square displacement but now we are making a statement about the path itself. This is called the law of the iterated logarithm.

It is Khinchin's law okay and it is characteristic of Brownian motion. So all these properties and more emerge from the very simple fact that we define mathematical Brownian motion as a Gaussian process with 0 mean and with correlation which look like this, x of t x of t prime correlation is the lesser of t and t prime okay. Now you could ask is there any relation between this process and the Ornstein-Uhlenbeck process.

Because in some sense we said look the Ornstein-Uhlenbeck process came about has an exponential correlation and it came about when we treated the motion of this Brownian particle a little more carefully keeping track of the velocity correlation time and so on. We also said that that process is exponentially correlated and I made a statement that the only continuous stationary one-dimensional continuous stationary Gaussian Markov process is the Ornstein-Uhlenbeck process and it has an exponential correlation right, made that statement.

So in that sense that is a fundamental Gauss-Markov process. This is Gauss and this is Markov, it is not stationary though. Is there a connection between this and that and the answer turns out to be very interesting. It turns out that every Gaussian Markov process continuous Gaussian Markov, continuous of course because it is Gaussian Markov process is some kind of Brownian motion of some kind.

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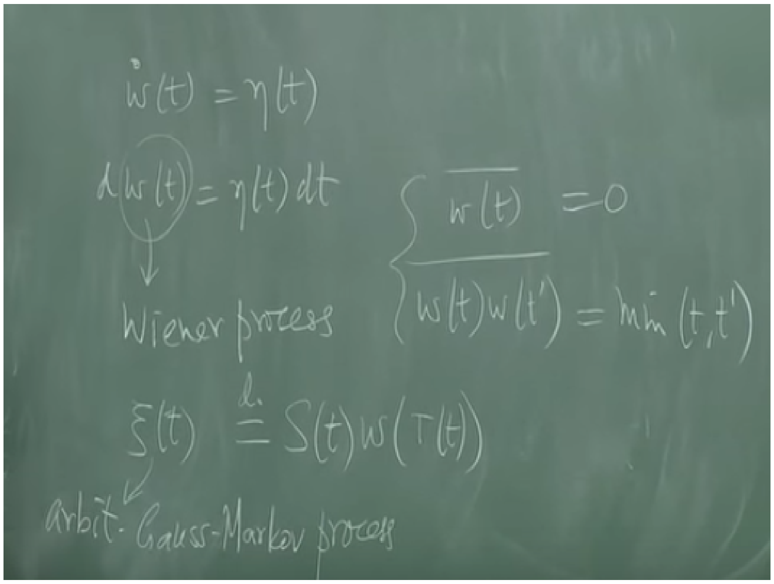

So let me call this Brownian motion, mathematical Brownian motion this quantity dw, let us put this 2d, let us get rid of that 2d and let me call w of t a process such that w dot of t is eta of t. So I do not have these constants or if you like w of t, d w of t is eta of t dt. This quantity this w of t is called a Wiener process or mathematical Brownian motion and it satisfies w of t, w of t prime average equal to the lesser of t and t prime and w of t average equal to 0.

It is a Gaussian process which has got 0 mean and whose autocorrelation is given by this quantity. In loose terms it is the integral of white noise okay and it is a process with stationary increments because this guy is stationary okay and that is an increment in the process okay. Now what I was going to say was I lost my train of thought yes. Now the statement is that every Gaussian Markov process okay it has to be Markov, it has to be Gaussian is some kind of Brownian motion, a Wiener process in rescaled variables.

So if you give me an arbitrary Gaussian Markov process, let us call that process psi of t in distribution as far as the probability distributions of this process are concerned this thing would be equal in distribution. So let me write a d her to show that the probability distributions are the same can be written as some Wiener process in some rescaled time times some other function of t. So this is an arbitrary Gauss-Markov and you can always write it as a Brownian motion in some other variable okay.

We will see an explicit illustration of it. I want to connect up the velocity process, the Ornstein-Uhlenbeck process to Brownian motion of this kind. Now you might wonder how that is going to happen. After all, this fellow here has a correlation which is very different. It is not a function of t - t prime and yet this fellow here if it is the Ornstein-Uhlenbeck is a function of t - t prime right. How did this happen. Well, the mapping is as follows. You can kind of guess what the mapping is going to be.

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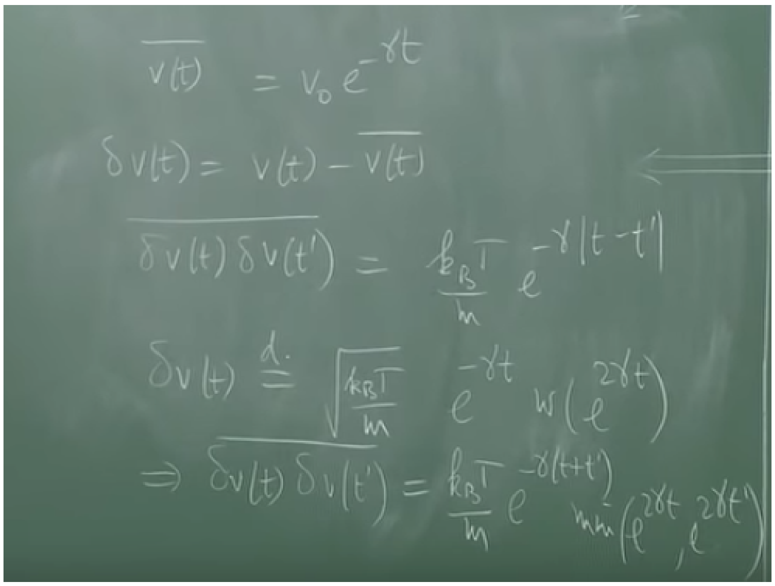

let us recall what the velocity process did. So we had a v of t such that the average value of v of t for a given initial condition was equal to v not e to the - gamma t if you recall from the Langevin equation and we found that the correlation time so if I define delta v of t equal to v of t - v of t bar and if I subtract out this average conditional average here then delta v of t delta v of t prime this quantity conditional average we discovered this guy here was equal to k Boltzmann T over m e to the - gamma mod t - t prime.

This is what we discovered right. This is what we saw from the Langevin equation and it follows from e to the Langevin equation directly or from the solution to the Fokker-Planck equation okay, agree? Now, how is this going to be related to that? Well, the claim is the following. Delta v of t is equal in distribution apart from this constant so let us get rid of it by writing square root of k B T over m.

This guy is t - t prime if t is greater than t prime and t prime - t if t is less than t prime right. So let us write this as equal to e to the - gamma t a Wiener process of e to the twice gamma t. So this scale factor outside is e to the - gamma t and the time is getting replaced by e to the t okay. Then we apply this rule.

So it immediately follows this thing will immediately follow that delta v of t delta v of t prime average is k Boltzmann T over m e to the - gamma into t + t prime, these 2 factors and then the smaller of e to the 2 gamma t, e to the 2 gamma t prime okay. Of course this is a monotonically increasing function of t. So if t is bigger than t prime it is this guy and that immediately kills this factor and makes it a plus sign so you end up with this result here.

So this is how magically this exponential where the exponents add up. It immediately once you take the exponents and write e to the t instead of t or e to the 2 gamma t immediately this correlation looks like this but in a rescaled variable. So you could say that Brownian motion is the Ornstein-Uhlenbeck process in logarithmic time if you like or the Wiener process in exponential time gives you the Ornstein-Uhlenbeck process okay.

But the statement is every Gaussian Markov process can be converted to Brownian motion. So you could now ask what about the path sample paths of the velocity process itself. What would those look like? After all the x process had this very jagged property and I said it is a fractal it has got 0 sets and so on which are fractals and not differential; what would happen to the v process?

Well, the crude answer is whenever you have this kind of white noise somewhere in there you are going to see this weird property always. Whenever there is some driving force which is a Wiener process, you are going to have which is the derivative of Wiener process you are going to have this problem always of this very irregular paths okay. So even for the velocity process that is still true.

But is really smoothed out in many other ways because of this finite correlation time and so on okay. One could ask what happens in other dimensions, higher dimensions and so on. So that the next thing we are going to talk about is a kind of generalization. I already mentioned there is a connection between these diffusion processes and the Fokker-Planck equation here. We would now like to go backwards and ask what about the Fokker-Planck equations obeyed by other random variables connected to Brownian motion. For instance here is one.

If you have diffusion in d dimensions right, so you have a probability density which satisfies the diffusion equation in d dimension. So it is like a whole lot of Brownian motions in higher dimensions right. What sort of properties do those trails have? That is one question. The other questions is what kind of Fokker-Planck or stochastic differential equation is satisfied by other functions of these variables which are Brownian motion.

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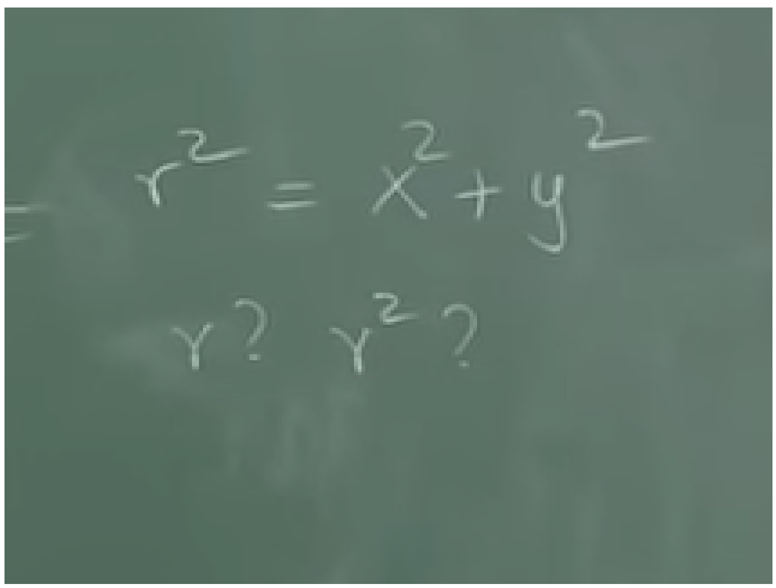

For instance in 2 dimensions if you had x and y you could ask what about the random variable x square + y square. What kind of distribution will that have? What about it is probability density function and so on. What kind of Langevin equation would that satisfy? So what we will do is to write down from the origin differential equation we write down what the solution is or what the Fokker-Planck equation is corresponding diffusion equation is for these random variables.

And then use this connection between the stochastic differential equation and the master equation in reverse to write down the stochastic differential equations and see what they new information we get and that is an interesting exercise. So one of them would be to ask if I call this equal to r square what about the diffusion equation satisfied by r square, what about the Langevin equation satisfied by r square okay. How does that look.

What are its features and so on. You could also ask what about r what about r itself, r square. These are different random variables if you like. What about the square of a Brownian motion. What about the nth power of Brownian motion in one dimension say. What about x square or x cubed or x to the n. What does that look like? What about e to the x. What does that look like and so on. We will talk about that. E to x is a very weird property. It is also a kind of Brownian motion.

It is called geometric Brownian motion and that is the one that is used in the analysis of these financial markets. So this famous Black-Scholes equation which you have which people use in stochastic differential equations is applied to stock market prices is essentially geometric Brownian motion also called exponential Brownian motion. So we will try to write down the stochastic equation for that and see what their properties are. Well, some properties will become, they are not intuitively obvious.

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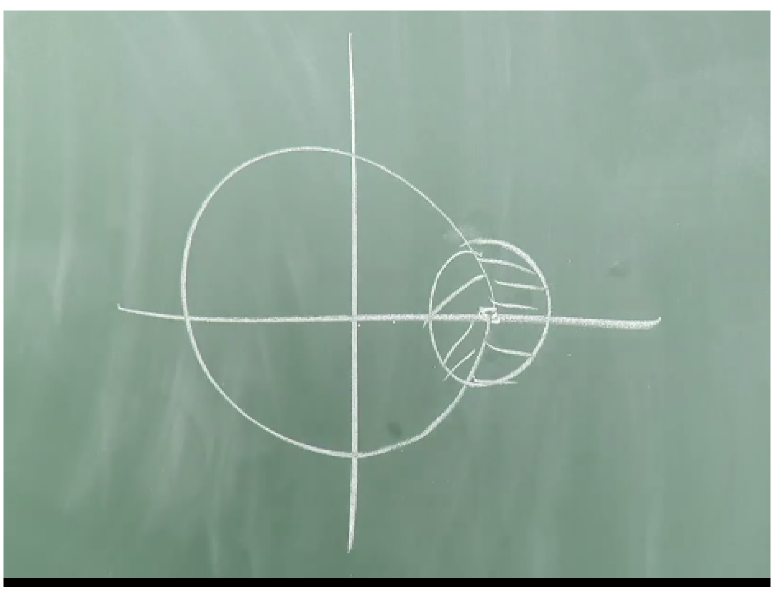

For instance if you ask in two dimensions for instance if you say okay the particle is moving on this plane what kind of trail does it have? It turns out that this particle also has a probability 1 of returning to the starting point whatever it be, in other words intersecting itself arbitrarily an arbitrary number of times. In fact the trail if you wait long enough will fill the plane. It is space filling. It has a fractal dimension of 2.

Almost every point is visited an infinite number of times, revisited an infinite number of times. That is one of the properties and you could also ask is it completely unbiased. It is because it says the x and y directions these motions are not correlated to each other at all. But if you write it in terms of the radial variable then you see there is going to be a bias immediately.

Because it says that although there is no drift term when you write the Langevin equation in the Cartesian components if I now look at it at some point here at some instant of time right and ask what about the radial variable what is that going to do then you see without loss or generality we will take that point here just for ease of illustration. Anything that pushes it outside this circle is going to increase the radial distance. Anything inside is going to decreases it right.

Now, assuming that you have equal kicks in both the x and y directions in an unbiased way in arbitrary directions if you draw if you say in a given kick it cannot go further than that although that is not true it can go arbitrarily far. You immediately realize that if all these points are equally probable the measure of this set is bigger than the measure of that set. So you now realize that the tendency will be to increase r rather than decrease it. There is more tendency to increase r.

And of course as you get close to the origin, if you are at the origin then any perturbations is going to increase r automatically. But it is true at every point in this fashion. So this means that when you write the stochastic differential equation for r in addition to the diffusion term there will be a drift term as well and this is a real effect. So we need to see what that term looks like. Intuitively I would expect that the closer you are to the origin, the greater this drift will be.

So I expect it will go like 1 over r maybe or something like that. We will see that it indeed is so and it will also be dimension dependent. So for every d greater than equal to 2 this effect is going to show up and we will write this down, these equations down and look at it. So we will do that next time.