Physical Applications of Stochastic Processes
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Lecture - 11
Continuous Markov Processes

Today and in the next few lectures we are going to study a very important class of continuous Markov processes called diffusion processes and there is a very large body of literature on these processes and very large number of applications both in physics and in other subjects, chemical physics for instance, chemistry and so on and so forth but what we are going to focus on is not so much the detailed technical mathematical rigorously mathematical aspects of the subject as the possible applications to various physical situations.

Let me begin by recalling to you that we just started defining continuous Markov processes and in particular I said we will talk about stationary continuous Markov processes. We can relax this assumption of stationarity a little bit and talk about stationary or non-stationary processes and as you will see as we go along the most important non-stationary random process is in fact the diffusion of a particle.

That itself it is position is a non-stationary process. The velocity turns out to be stationary but the position is non-stationary as you will see when we go along.

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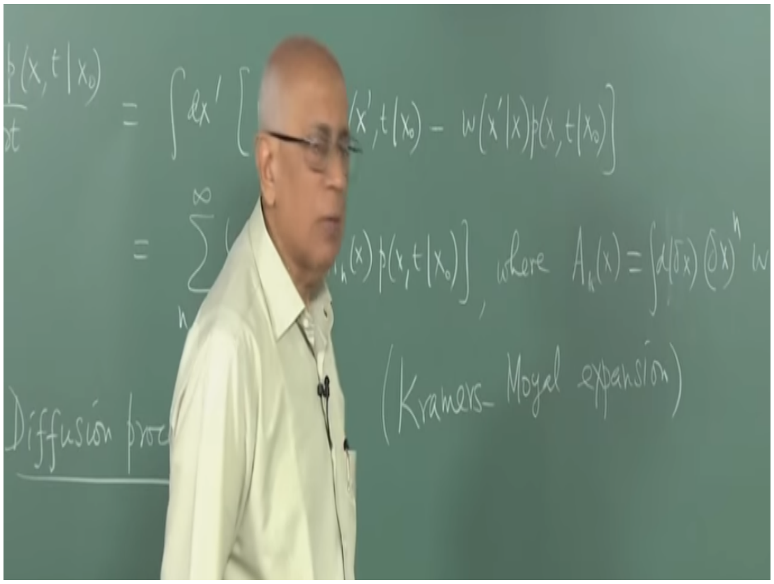

So to start with recall that we said that these process are described by a conditional density which satisfies an equation of this kind in the stationary case okay; x is the random variable and a set of value is taken by the random variable and the probability density function the conditional density function conditioned on this initial value satisfies an equation of the form an integral over all possible x primes the x primes and then inside you have a w of x x prime p of x prime t at x not minus the lost term which is w of x prime x p of x t x not.

This is the continuum analog of the discrete equation that we wrote down in the case of a process which took on a discrete set of values or attained the discrete set of states okay. Now as in the other case this is an integrodifferential equation although it is linear in this p and therefore is not a very trivial equation to solve. Certainly if it had been a matrix equation we could have written the solution as the exponential of a matrix multiplied by t and then try to look for methods of exponentiating this matrix but here that is not true.

This is some kernel so it is an integrodifferential equation. This is some kernel function of x and x prime ditto here and it is not at all so obvious what the solution is here. One approach would be to try to convert this to a differential equation but because it is an integral equation and you are integrating on all values of this here and no conditions have been put on these transition probabilities at all these transition rates.

It is immediately sort of intuitively clear that the order of the differential equation in the x variable will tend to become infinite in this case and in fact that is so and I will write that down without going through the intermediate steps except to indicate how to do it. What you have to do is to treat this as a function of x prime and you put x for instance is x - x prime you put it equal to some delta x or something like that.

Does not have to be small and then you will do a taylor expansion in terms of this delta x and whenever you get derivative operators you try to put it on the p by integration by parts. And the result is that this becomes equal to also equal to and I am going to skip these steps as summation from n equal to 1 to infinity - 1 to the n over n! delta n over delta x n A n of x p of x t for given x not. So it becomes equal to that formally.

These 2 are equal to each other where these coefficients A n of x equal to an integral of moments of this guy x + delta x starting with x for example and then delta x to the power n and d delta x. This difference and the nth moment of this increment with this as the weight factor is the definition of A n of x here. And delta x is over the range, allowed range here. So it is a definite integral. This thing is a definite integral and it is a function of where you start namely x out here.

So this is the exact formal equivalence and now of course it immediately raises the question of when is this valid when is it convergent and so on and so forth. I am not going to talk about those technical issues at the moment except to say that you can make specific conditions put specific conditions under which this equation reduces to this infinite order partial differential equation and this is called the Kramers-Moyal expansion.

Does not serve much purpose except for formal purposes because it is an infinity order differential equation. So it is not any easier to solve than this but it gives a little bit of physical insight as to what are the terms that are contributing out here and how do you interpret them and so on and so forth okay. But there is one great simplification that occurs for a specific class of processes called diffusion processes.

And by this I do not mean I mean diffusion in a technical sense which is not restricted to the physical diffusion of a particle in space or anything like that but a mathematical term which says there is a class of processes called diffusion processes for which this equation simplifies enormously okay.

And there is a theorem a rigorous theorem which says remarkably enough if these moments of this increment delta x the amount by which x jumps to a new value if these moments vanish for any n greater than equal to 3 if so happens that A n is identically 0 for any n greater than equal to 3 then A n is equal to 0 for all n greater than equal to 3 okay. It is called Pawula's theorem and it is a remarkable theorem. It is not magic. It is possible to derive it fairly straightforwardly.

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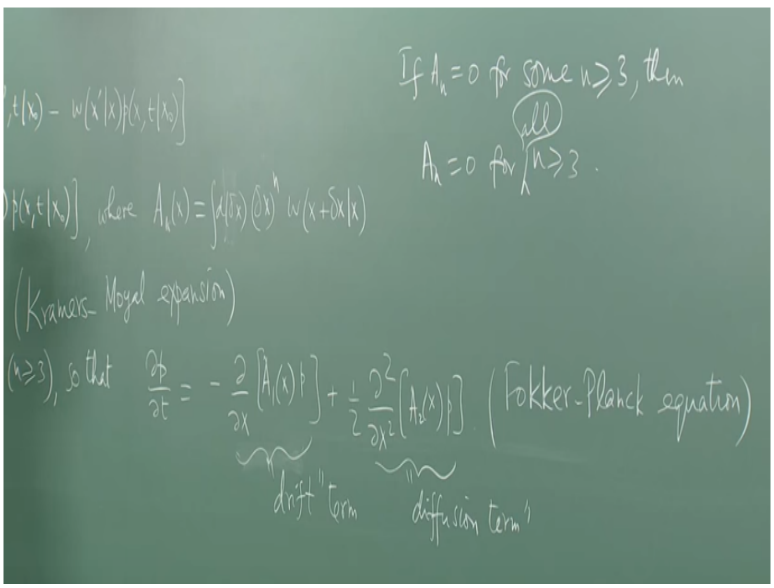

But the statement is if A n = 0 for some n greater than equal to 3 for all okay. And of course that immediately simplifies matters enormously and processes for which this happens are called diffusion processes okay because that would immediately imply so that this equation becomes the following. Delta p over delta, let us leave out all the arguments of this p equal to - delta over delta x A 1 of x p +s one half delta 2 over delta x 2 A 2 of x because the higher moments are 0 out here and this two factorial I have put in here as a half there okay.

This is called the Fokker-Planck equation. They originally derived it under different context altogether in a related context but in a physical context of a particle diffusing in space and this was to the velocity of this particle but today we call it a Fokker-Planck equation in general for any diffusion process and I will use the same terminology. And you can go ahead and interpret what this means and it will turn out we will see the specific examples that this term represents the effect of noise on this variable x.

That is what causes x to fluctuate randomly whereas this term very often describes the effect of deterministic evolution in this x as we will see from the examples. So this part is what x would do, its distribution would do in the absence of any noise and this is what makes it random. So very often this is called the drift term and that is called the diffusion term and we will use these terms in general even though they come from the physical application I am going to talk about.

So this is if you like this is the drift and this portion is the diffusion. It is still not a trivial equation as you can see because it is got this term here. It is a second order in the position variable. I will frequently call this a special variable because for want of a better term although it need not be that at all. In fact in the original context of the Fokker-Planck equation it was a velocity variable. It is first order in time but second order in the other variable.

So this is technically not as simple an equation as say Laplace's equation or Poisson's equation because of this inhomogeneity. You'll recognize special cases of it. For example if this A 1 had not been present and if that A 2 were a constant this looks like the diffusion equation the ordinary diffusion equation for particles diffusing on a line which would be delta p over delta t is d times d2p over dx 2.

We will see how that comes about. Now of course one could ask what happens, can I solve this equation in general and so on. The answer is no. In general for arbitrary coefficients A 1 and A 2 it is not so trivial to solve at all. Can I incorporate non-stationary process in this? Yes, indeed. That has nothing to do with the vanishing of the moments or anything like that. It is an independent statement.

If these were time dependent explicitly then of course you have a non-stationary Markov process and then you have to be careful. You have to write x not comma t not etc. keep track of that and then this would become time dependent here and it would be time dependent here because w would be time dependent explicitly. So it is possible to incorporate non-stationarity into this business by looking at time dependent coefficients. We are not going to do that.

All the cases we look at would be stationary in that sense but we will come across a non-stationary process. We will see what happens in that case where you do not have explicit time dependence and yet you will have a non-stationary random process which will be the position for instance. We will see how it comes about okay. Now to make it familiar with the diffusion equation one possibility is to derive this diffusion equation independently altogether.

We already looked at diffusion on a linear lattice in the presence of a bias. We looked at a random walk and the question was can the random walk be made into an equation of this kind. Then of course you begin to see immediately the connection between these two. So we will do that now. But one thing I want to point out is that there is a special case of even this Fokker-Planck equation and that is very important. It is a very basic process.

In fact it is the most basic Gaussian stationary Markov process and it is just one of them it turns out and everything else can be mapped on to that and it is called the Ornstein-Uhlenbeck process. It corresponds to the case in which you have the following, extra simplification.

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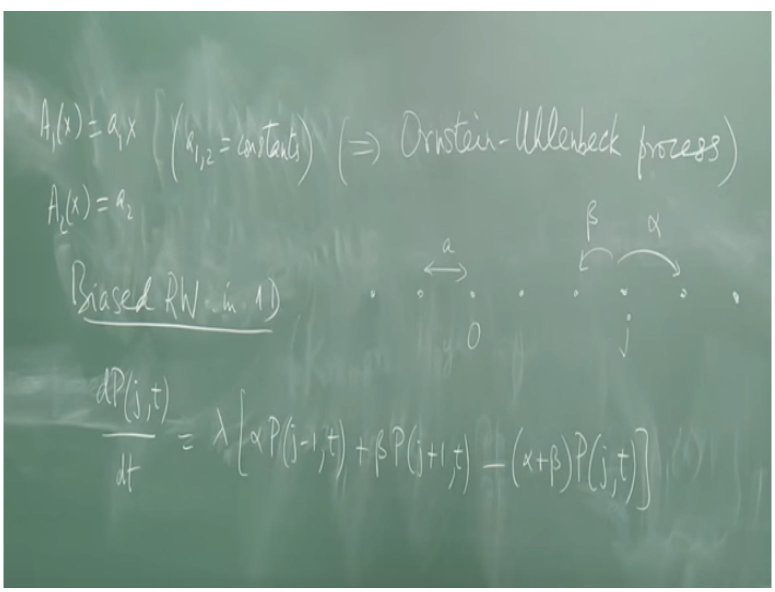

So the case in which A 1 of x equal to a 1 x and A 2 of x equal to a 2; a 1, 2 equal to constants. So case in which the drift is linear in x and the diffusion term is just a constant, this coefficient A 2 is a constant. This is called, this particular process then is called an Ornstein-Uhlenbeck process. So let us write it down, a very important special case and we will spent some time solving this, for the density function of this Ornstein-Uhlenbeck process along with a physical example.

But in the meantime let us go back and see whether we can derive this kind of equation from the random walk model altogether. So let us go back to the case of a biased random walk on a linear lattice in one dimension. So if you go back and recall what the statement of this problem was. We had a linear lattice, an infinite lattice say labeled by the site index j, some arbitrary site was the origin and then you had a probability if you are at the site j of jumping to the right with a probability alpha and to the left with a probability beta and this was true at every site.

You toss this unfair coin and either jump to the right or to the left okay. Now we did that in the discrete time case but we also did it in the continuum. We said the time was continuous and the steps were given by a Poisson process with some mean rate lambda in which case the process that corresponded to right step, steps to the right, had a mean rate lambda alpha and those to the left had mean rate lambda beta and if you recall the master equation in that case was dP j, t.

I suppress the fact that we started at the origin. We keep that going so that just a matter of simplifying the notation. This was equal to lambda times alpha P of j - 1 t + beta times P of j + 1 t - alpha + beta P of j, t. Alpha + beta is equal to 1 but I have put that back here as alpha + beta because I am going to recombine terms okay. So do you recall? This this was the master equation for this probability P of j, t okay.

And the initial condition was P of j, 0 is delta of j, 0 the Kronecker delta. We solved this and we discovered the distribution was a modified Bessel function ij of 2 lambda t square root of alpha beta etc. Right now we are not interested in that solution but we want to see what happens in the continuum limit when this j becomes a continuous index. So what we do is to introduce a lattice constant this spacing a.

I am going to let a go to 0 and correspondingly let the rate of jumps lambda become infinite because the distance you have to jump is going to go to 0 and the rate becomes infinite in a specific manner so as to derive a finite limit for the right hand side, a proper limit for the right hand side.

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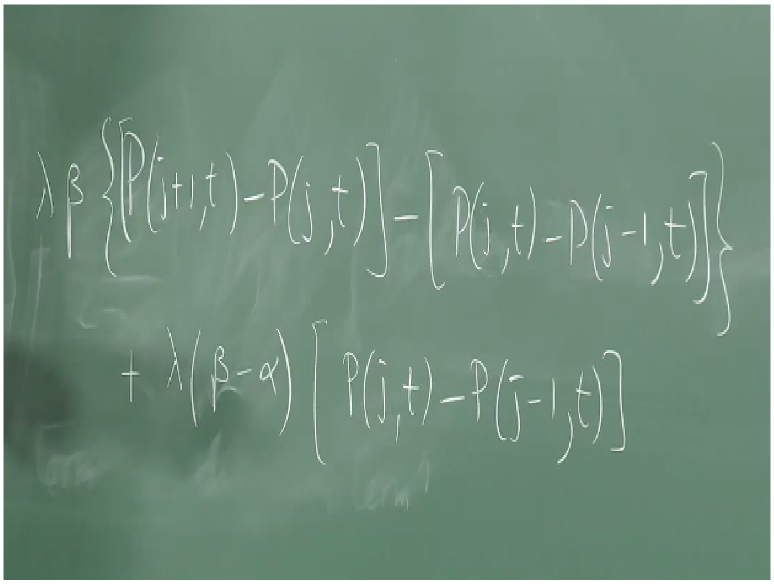

So the first step is to write this as equal to lambda times let us choose this first beta of P j + 1, t - P of j, t - let us subtract the other difference also P of j, t - P of j - 1 t. So I have taken care of this term and this beta here in this portion here and then I added I subtracted that too so I need to add that back so this becomes + lambda times beta - alpha times P of j, t - P j - 1, t. So that is the other term.

Did we go through this continuum approximation earlier? Have we explicitly done that? Okay, so it is worth looking at it carefully to see what exactly is involved. So what I have done is to add and subtract this thing here and I get this thing here. Now it is clear what you should do in order to get the continuum limit because this looks like the second difference. This looks like P j + 1 - twice P j - P j + P j - 1.

So this looks like the double difference, the second derivative, and this looks the first derivative if j were a continuous variable right. So what we need to do is to multiply and divide by the lattice constant and take limits. So out here I need, I can rewrite this.

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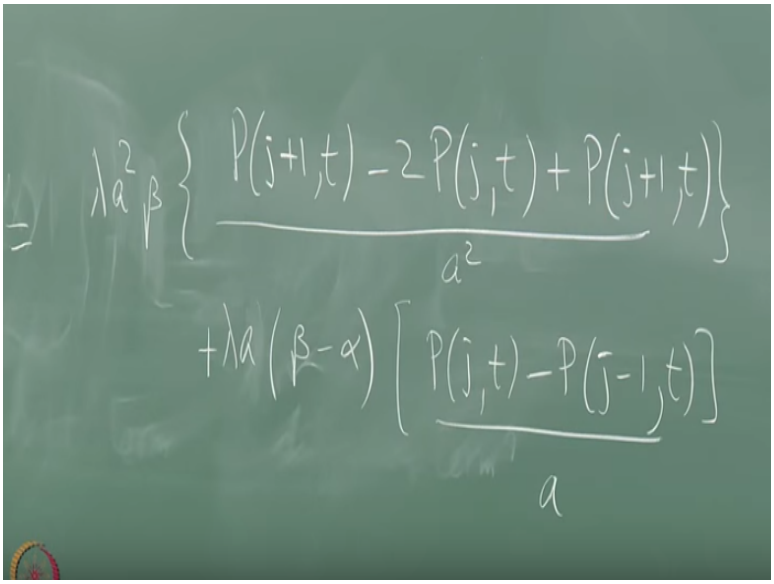

This thing can be rewritten as P of j + 1, t - twice P of j, t + P of j + 1, t. I divide the whole thing by a square because it is a double difference here and multiply by a square. I multiply this by a and divide this by a okay and take the limit.

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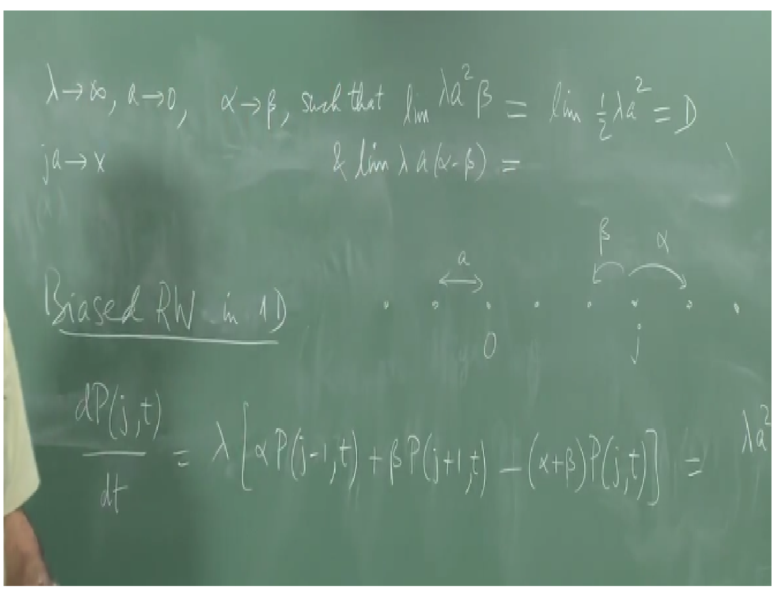

So the correct limit that we need to take the continuum limit is lambda tending to z infinity a tending to 0 and I want lambda a square to become finite. That can only happen if beta - alpha also tends to 0 so that I get something which goes like an a square here. So we need a times alpha tends to beta such that and j tending to infinity because what I am going to do is to put j a tends to x so j also becomes infinite such that j times a is my x coordinate just like we went to the continuum limit in time by saying the time step n multiplied by tau the unit time step was such that n tends to infinity tau goes to 0 such that n tau went to t the continuous variable.

And exactly the same way j tends to infinity a tends to 0 such that j a goes to the variable x such that what we need here is lambda a square beta limit lambda a square beta is finite equal to some number d. By the way if alpha tends to beta this is the same as the limit half lambda a square equal to d because beta is also going to go to alpha and a times alpha - beta times lambda tends to limit equal to what would be the physical dimensions of this limit of this quantity?

That the length and that is the rate velocity just a velocity. So let us call it alpha - beta equal to c okay. Then then this quantity P of j, t tends to the probability density P of x, t but you got to pay attention to the fact that there is a dimensional change here because this is dimensionless probability that is a density probability density. It has dimensions 1 over a, 1 over length right. So you have to be careful about it. There is an extra a factor there which you can put in but it is not so serious because it will appear on both sides out here.

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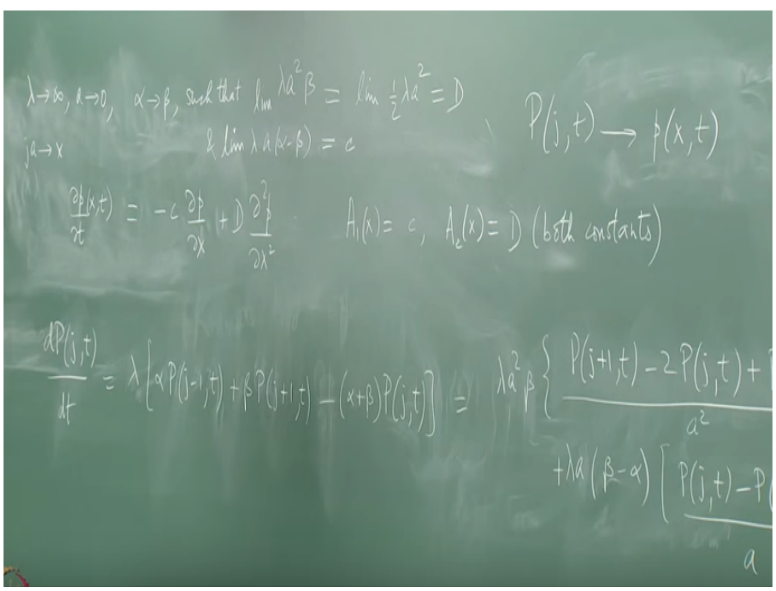

And when you take that limit you end up with this equation becoming delta p over delta t of x, t starting from some x not we do not care we do not put it here equal to this quantity is d and now here we have - c delta p over delta x because that is the first derivative plus D okay and that is exactly in the form of this Fokker-Planck equation that we have written down. So in this problem A 1 of x equal to c, A 2 of x equal to D both of which are constants; a very trivial example in which these coefficients have actually become constants here.

So the position in the case of biased diffusion, the position variable looks like it is the Markov process with a base of Fokker-Planck equation with constant coefficients, both the drift and the diffusion terms are constant, looks exactly like that right, agree?

This kind of equation for the positional probability density when you have diffusion in the presence of an external field, this fellow here looks like it is a drift caused by some external field because you are saying systematically either alpha is bigger than beta or smaller than beta it drifts to one side whichever is larger and that is exactly what happens when you have a constant force on the particle.

So this looks like this equation is describing the diffusion of a position of a particle positional probability density of a particle subject to diffusion but under a constant external force of some kind and indeed it is so. It is indeed so because if you recall the problem of sedimentation that we talked about this is exactly what happened. You had an extra term exactly of this kind. We even saw the solution. We wrote the steady state solution I think in that case and what did we get in that case?

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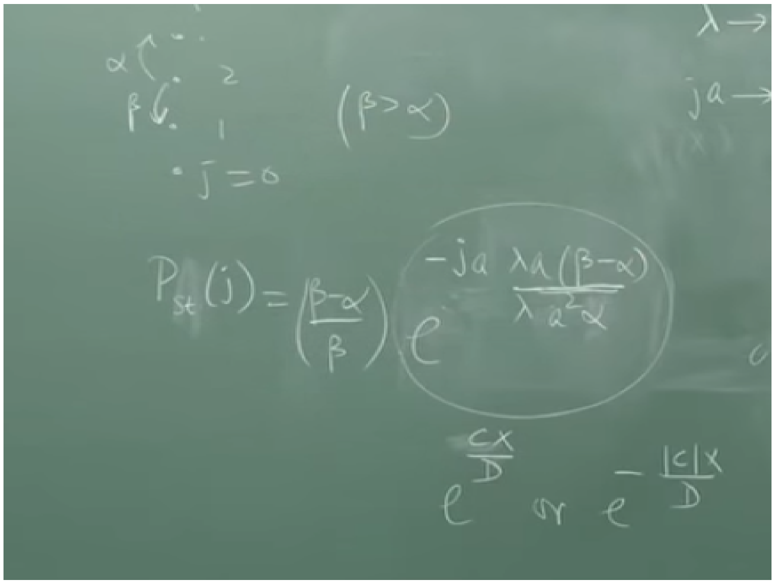

The problem we looked at was I said j = 0 here 1 here 2 here and we looked at a case where this part was bounded and then it turned out that P of j P stationary of j was proportional to the bias alpha over beta. So you have bias such that these rates are alpha and these rates are beta downwards and this was proportional to alpha over beta to the power j. This is what we had. We imposed a boundary condition on this.

We said it cannot go below 0 out here. So the rate alpha - 1 to j = 0 was 0 and the rate from j to -1 the beta 0 was also equal to 0. Then we immediately got this as a steady state solution and we need to normalize this. We need to normalize this guy over j from 0 to infinity should be equal to 1 and that of course if you sum this geometric series is 1 over 1 - alpha over beta which is beta over alpha – beta, correct.

So this whole thing is proportional to this guy so equal to whatever is normalization times this. In this problem beta was greater than alpha right. This is the steady state solution we got but we will now let us try and take the continuum limit of it here and see what you are going to get. So I need to put all these guys in. I need to put in all these fellows here back again. So let us do that. Alpha is going to go to beta but I can write this fellow here.

In this problem beta is bigger than alpha in this case. So we got to be a little careful here about the sign. I define my drift velocity c as alpha - beta. So if c is positive it says alpha is bigger than beta otherwise c is going to be negative. We have to remember that sign here. So let us write this as e to the power j log alpha or beta or log beta over alpha with a minus sign and I am going to take the limit in which alpha is equal to beta.

So this becomes the limit. That guy becomes 1 the log of 1 it is going to get. So what should I write? I write this as log beta - alpha or alpha + alpha over alpha. You can write it like that surely which is 1 + beta - alpha over alpha. I mean we can do this very rigorously but can see what is happening and this is going to go to 0 beta - alpha. What is log 1 + z as z goes to 0 the leading term?

Z itself right. So this is can be replaced as beta - alpha by alpha in this form apart from some normalization. We will worry about that later and I want to make this j into x. So I multiply by an a divide by an a but a times beta - alpha is going to c right. So let us multiply this by another a. This is what I had and I multiply this by another a. So I got to put another a here and let us put a lambda here and a lambda here.

We are all set to take this limit because what does this whole thing go to? This fellow goes to x, that guy goes to c - c and this fellow goes to d. Where did the x come from? Alpha this was an alpha which is the same as beta in the limit right. So you are going to get something like e to the power cx over d where c is negative, agree? I probably use the symbol c for the downward velocity limiting velocity.

But I have defined this c as an upward positive in the upward direction so increasing j so that is why I change the sign here for this okay. But this is exactly what we got earlier. We interpreted this as the Peclet number and so on but that is exactly what this gives you, this equation gives you because if you go back to this equation and ask what is the stationary solution what is it going to be?

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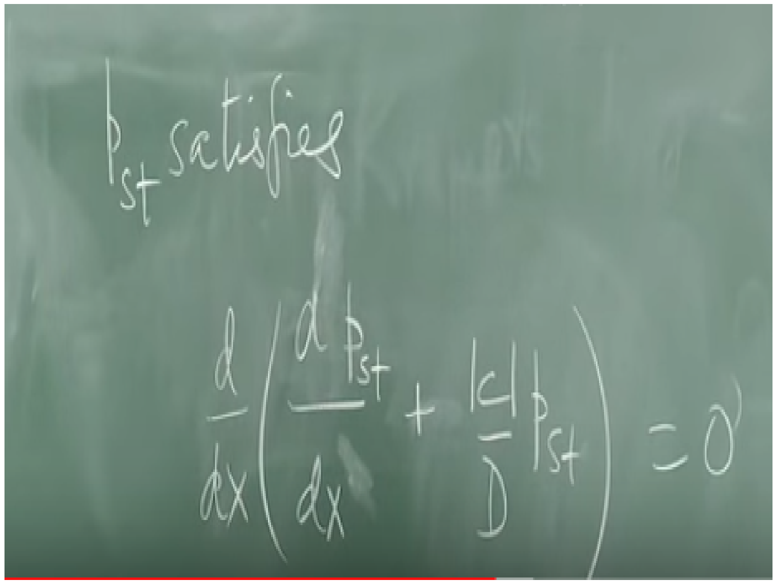

P stationary satisfies the equation D d 2 p over d x 2 stationary - c dp stationary over dx = 0. That is what this tells you right. But I can pull out a d, d over dx from here and write this as d over dx dp stationary over dx - e times p c over D equal to 0 and c by the way is minus this is plus modulus okay. That is the equation and what is the solution? What is the solution to this equation?

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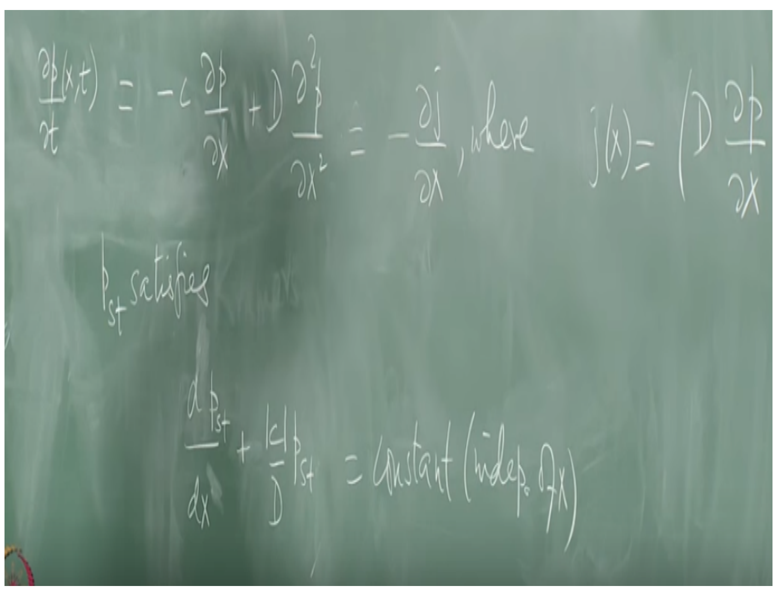

Not quite, not quite because what you can what you can say from this is that this quantity is independent of x so this guy must be equal to some constant independent of x whereas here we did not have any such problem. We did the random walk problem and we immediately got the answer right away. But here we are getting an equation which says this fellow is actually independent of x. Nothing more than that.

What would you have to do to match that to this and make sure that that constant is 0. You have to put a boundary condition somewhere. We already put a boundary condition on the floor. We said it cannot go below the floor. We already did that here in this case. We have not yet imposed that condition there. We need to impose that condition which will be precisely that this quantity is 0 at x = 0 because this is the flux.

Remember that this equation can be written in the form of an equation of continuity because I can write this as equal to - delta j over delta x where j of x equal to d times dp over dx, in this case + mod c p. So it is in the form of a continuity equation in this case and that is the flux at any point because it is precisely a continuity equation for this probability density. And then it says you cannot go through the floor.

So it means this quantity this current here must vanish at x = 0 but in the stationary case and only in the stationary case this quantity is independent of x completely and since it vanishes at x = 0 it must vanish for all x because it is independent of x okay. So I emphasize again. This quantity is not 0 for x not equal to 0 in general. There is a current otherwise you would have you would not have any dynamics at all certainly.

P of x, t in general is a function of t okay but when you go to the stationary distribution there is no t dependence any more okay. So the statement is that the boundary condition says that the current as a function of t vanishes at x = 0 for all t. You got a partial differential equation. I have to give you an initial condition and I have to give you a boundary condition. The boundary condition must be valid for all t.

The initial condition is valid for all x for a given t right where t = 0. So in this case this acts as a boundary condition and it says this quantity here vanishes at x = 0 and that same boundary condition applies even in this stationary distribution. But in the stationary distribution you discover that this quantity must be independent of x and since it is 0 at x = 0 it is 0 everywhere identically and once you put that in, this is the solution. So you see our discrete model went exactly into that.

This is just a verification that these limits are all right that all these factors were right, just right and it gives you this equation here. In the special case in which you have this particle diffusing under a constant force field here you can apply to other cases. It could be an electric field causing a steady drift or whatever but this is the exact continuum limit. So the lesson is that the biased random walk with a constant bias the same bias at all sites is equivalent to the diffusion of a particle under a constant force field in the continuum limit okay.

But now we are approaching the whole thing from the continuous Markov process angle okay. So we are going to write down although we did not have any differential equation for the position of the particle in the random walk problem but only difference equations for the probability density not that we have a continuous Markov process we could go back and ask one more thing which is to ask okay it is a random variable but does the variable itself satisfy a differential equation or not.

This is not a differential equation for the variable, it is a differential equation for the probability density of this variable and that is a nice object. But the variable itself will be very irregular, will be random, because it is being driven by some fluctuations in this case. We are going to find out that this will satisfy a differential equation but it is what is called a stochastic differential equation, a random differential equation and it should correspond and be consistent with the fact that the probability density satisfies this master equation here.

The general name for a particle which is for the positional probability density of a particle diffusing under an external force field it is called a Smoluchowski equation.

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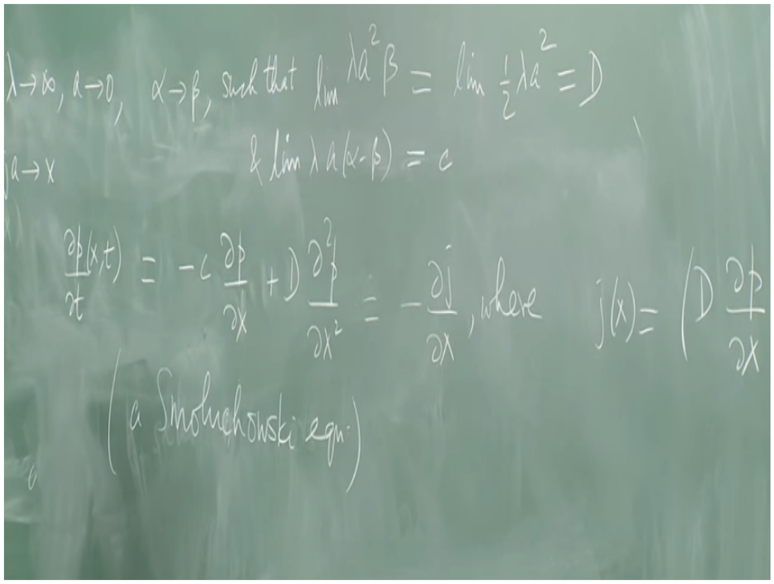

So this is an example of the Smoluchowski equation. I will call it a Smoluchowski equation because it is much more general than this. You already saw that this has the effect of constant force field. What would you say happens would happen if there was a force here explicit force which was position dependent. How do you think this equation would change? It would not be a constant. This a 1 of x would not be a constant right.

A 1 of x in some sense would be the force. So if the force were due to a potential v of x I would expect that something like - v prime of x appears in this drift term okay and then the diffusion, the scattering would come from the D part here. So this is something to keep in mind that the first term will be a drift due to deterministic forces and the second term would be the diffusion due to random forces okay. We will systematize that.

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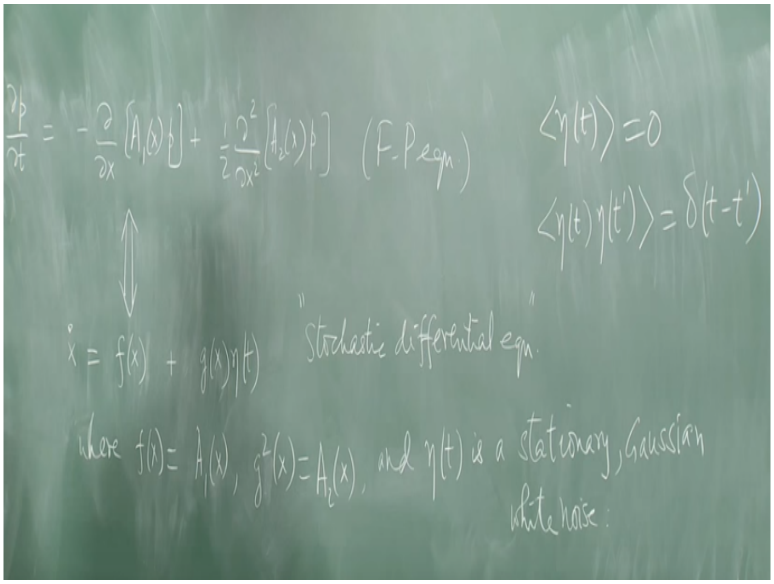

So let us go back to this Fokker-Planck equation and ask is there a correspondence between an equation of the form delta p over delta t, I will continue to use the variable x which does not have necessarily the connotation of a position but a random, continuous random variable, Markov process; so delta p over delta t equal to - delta over delta x A 1 of x p + one half. This is the Fokker-Planck equation okay.

And it turns out that this Fokker-Planck equation is entirely equivalent to a certain stochastic differential equation for this variable x, random variable x which is now called a diffusion process in the mathematical sense and that equation is the following. I will write it down but I am not going to prove this and this is entirely equivalent to a certain differential equation for x itself which reads in sort of physics notation it is not the most rigorous notation.

It reads x dot equal to some function of x, perhaps even a function of t if this is a function of t but we are looking at stationary processes. So let us just call it f of x times + g of x times a white noise and let me call it by eta t and explain what this eta is and this is a stochastic differential equation where f and g are prescribed functions and they are related to A 1 and A 2 as I will write it down in an instant.

But this eta of t is called a Gaussian white noise, a stationary Gaussian white noise and I will explain what that is separately where f of x is essentially A 1 of x and g of x g square of x is A 2 of x and eta of t is a stationary Gaussian white noise. I have to say what this means okay. Eta of t is a random process in time such that all its probability distributions multiple time probability distributions are all Gaussian in shape. So that is why it is called a Gaussian noise.

It is stationary. So all its statistical its statistical average and higher moments are all time independent. Correlation functions are functions only of the time difference etc. and it is a white noise. In other words it is delta correlated in the following sense. Equal to 0, 0 mean and it has got a delta correlation delta correlated delta function as an autocorrelation.

It is clearly the limit, the mathematical limit of some physical noise whose correlation time would not be 0 because this implies the correlation time is 0 whereas I would expect for a stationary process if t is bigger than t prime I would expect this correlation to look like this as a function of mod t - t prime I would expect this correlation to come down in this fashion and this characteristic time scale would be the correlation time of this noise.

But that is now going to 0 and the amplitude is going to infinity such that in the limit it becomes a delta function, this guy okay. So it is a mathematical idealization clearly. It would have to be justified on physical grounds each time okay. For instance in the problem of the collisions of in a gas of particles in a fluid for instance, this noise would be caused by all the other molecules colliding against some particular tagged particle molecule.

Then this eta of t would be the correlation time of that force the random force caused by the collisions, all these other guys and the scale on which the particle's motion itself is tracked, the time scale would be much longer. It would remember its memory for much longer than what the noise does. So the correlation time of the noise typically would be of the order of nanosecond or a picosecond for instance whereas the correlation time of the velocity of the particle that is being tracked that could be of the order of microseconds.

So as far as the microsecond is concerned a nanosecond or smaller intervals are essentially 0 intervals. So in that sense one can justify this approximation okay. But each time in any problem when you model this you have to ask whether there is a clear separation of time scales of this kind or not. But at the moment from a mathematical point of view, the formal point of view this is what this equivalence is. So the statement is that a stochastic differential equation of this kind is entirely equivalent to this Fokker-Planck equation for the probability density of this random variable.

So please take this as a theorem. I'm not going to prove it here. But take this as a theorem. We are going to exploit it over and over again. Now you can see why I call this a drift term because if you did not have this noise at all this is deterministic evolution of this variable under some prescribed function f of x here. It may be - dv over dx. We do not care what it is or anything else.

So this term is indeed describing deterministic dynamics and the noise is entirely here in this thing here and that showing up in the second term here okay. What is interesting in this problem as supposed to even simpler problems is that this g has x dependence in general. So it says given a current value of x of this random variable the way the noise affects it the amplitude of that noise depends on this random variable on this x.

On the other hand and that that is why it shows up here inside here but in the example we looked at in the diffusion problem as a diffusion equation remember this A 2 turned out to be a constant. So in that case this would have been square root of 2 d and that is it.

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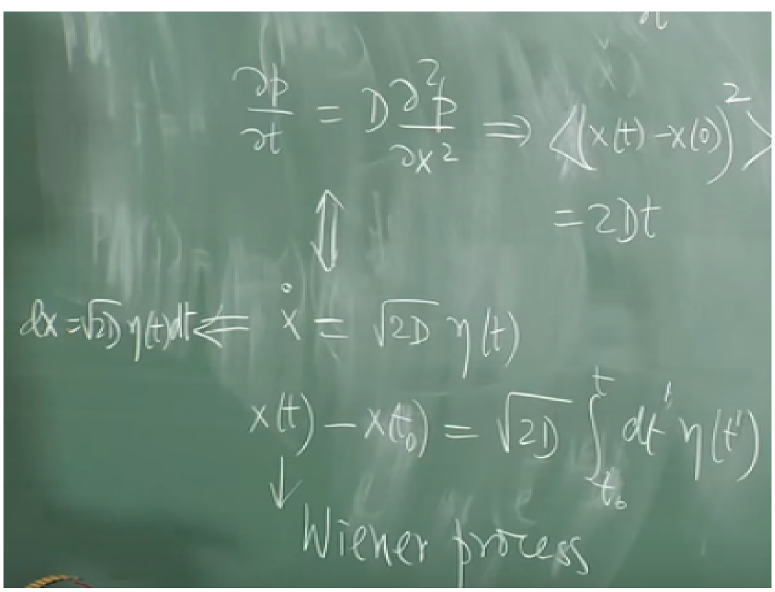

So now we can kind of identify what would be the stochastic equation corresponding to delta p over delta t equal to D D 2p over delta x 2. This would be equivalent to a stochastic differential equation for x which would be of the form x dot equal to in this case there is no A 1. So clearly there is no external force or anything of that kind, no drift at all. A 1 is 0 identically and A 2, well this d half D is a 2.

So a 2 is square root of is 2D and therefore g is square root of 2D and that is it. This is the stochastic differential equation corresponding to for the position corresponding to the simple diffusion equation in one dimension okay. One can write a formal solution for this guy and that formal solution would be x of t, you have to define these integrals - x of 0 equal to square root of 2D times integral 0 to t dt prime eta of t prime, agree?

We can call this x of p not if you like and integrate from t not to t. So in that sense a plain diffusing particle doing free diffusion the x variable corresponds to the integral of white noise. This guy corresponds to the integral of white noise and it is called a Wiener process. Is it a stationary random process. It is Markov because we wrote the master equation down. I said it satisfies the Fokker-Planck equation and so on. So it is clearly a Markov process.

But is it a stationary Markov. By the way it is Gaussian. That is something else you have to recognize because we know by hindsight we know the solution of this guy although we did not derive it here explicitly. We know the fundamental solution of this is that Gaussian e to the - x square over 4dt which I will come back to talk about. So what it is telling us is you are going to hit the particle with a Gaussian white noise that means the distribution of this eta is Gaussian and it is delta correlated stationary and Markov.

Then what is the output variable the driven variable x after this integration. What properties does it have? Well, it remains Gaussian because its probability density is Gaussian. So the shape remains Gaussian, that is robust. What else happens? It is Markov. It is certainly Markov. It obeys this Fokker-Planck equation but is it stationary. Well the stationarity remain. Does this look like, if I had x of p not here and this is t not to t does this guy look like a function of t - t not in general, no. No, certainly not.

It is not stationary and you already know this because given this diffusion equation what does it imply for this quantity x of t - plus x of 0 whole square. What does this become? It is the mean square displacement from some given origin and what is that equal to. It is diffusing and therefore what is it equal to 2Dt exactly, exactly it is 2Dt. It is a function of t.

So it cannot be stationary. Because if it is a stationary random process all these moment should be independent of t but here right away it tells you it is not stationary immediately. We have not computed what the correlation function of x is. We have not found what is x of t prime, x of t double x of t x of t prime average. We have not found that yet. But certainly we found what is x of t x square of t average and that is 2Dt just a Gaussian integral. So it is not stationary.

It has stationary increments because you can write this guy obviously as dx equal to square root of 2D eta of t dt. You can write it like that and then of course this is stationary. These are stationary increments but it is not a stationary random process by itself or more less rigorously its derivative is stationary but this function is not, the variable itself is not and that you can see directly when you take something which has got stationarity but you integrate it in this fashion, integration makes it nonlocal in some sense so it is not stationary.

So in general that is a lesson that when you integrate white noise you may not retain the stationarity property but we are going to see that if you put a proper drift you will be able to do this. That is what the velocity would do and then it would attain an equilibrium distribution and so on. There is another way of saying that this guy is not stationary because its probability density P of x, t given an x not this fellow is decreasing as a function of time and it does not tend to any stationary distribution. As t tends to infinity it goes to 0.

This Gaussian broadens out over an infinite range. The total area under the curve remains 1 but the value at any point is turning to 0. So there is no stationary distribution in this problem. The variable is not a stationary random variable either okay. So we will get back to all these things. But at the moment I want you to simply remember the fact that the most general definition of diffusion process could be either this or this it does not matter either way.

Now mathematicians do not like to work with these delta correlated noises. They would rather work with the differential something which smoothes it out by integrating. So Wiener process can be handled more rigorously than this singular object here. So you integrate it once to make it smooth and so on but we are not going to pay attention to these niceties. We will be careful not to make any mistakes but at the same time we will do this rather heuristically here provided we know there are certain rules which we have to obey.

And one of them is this that there is this equivalence between the Fokker-Planck equation on the one hand and this on the other and we will see what happens with this. Now already you can begin to see whether there is going to be a stationary distribution or not by saying if so this thing must have a solution when you make this an ordinary differential with respect to x and if this guy has a solution which is normalizable and so on then you know there is a stationary distribution okay.

And if you do not have this, if you do not have this and A is constant then the stationary distribution is triviality itself.

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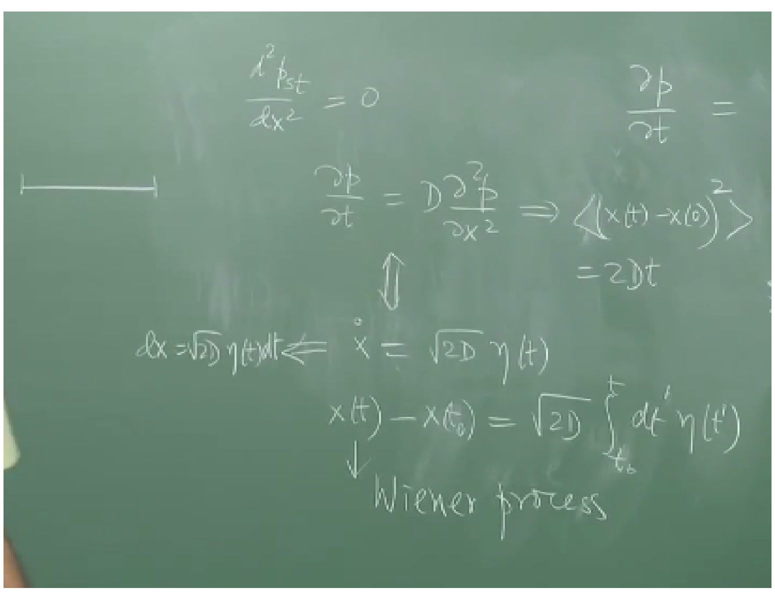

In this case if at all there is p stationary it must satisfy d 2 p stationary over dx 2 = 0 but what does that say about p stationary. It must be a linear function. That is certainly not normalizable. Right away you are wrong. It is finished; in an infinite range, in an infinite range or even in a semi-infinite range it is still not normalizable. What would happen if you had a finite range. Yes, then indeed it can be. So then you have you do not require integration up to infinity.

Then yes, indeed it is true. Suppose you are told that p stationary is between these 2 points and nothing more than that. I put a diffusing particle here. It is like putting a drop of ink inside a beaker of water. The ink does not go anywhere but it becomes uniform everywhere intuitively you know this by diffusion. So that is what will happen. This particle's probability density will be uniform, will be constant in this boundary provided there is no escape from the ends. There is no leakage from the ends.

So in that case yes indeed because then I would say p stationary equal to Ax + b and I would put boundary conditions at the ends and you discover finally that A is 0 and you have just b which is normalized okay. So the boundary conditions also play a crucial role in the whole thing okay. We will talk about these aspects next time. We will start with the Fokker-Planck equation and see where we can go from there.